Computing the Colorability of a Knot

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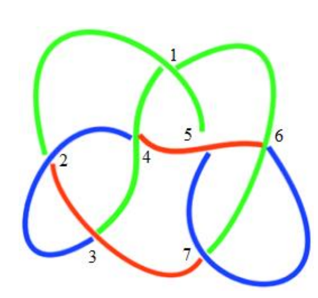
Introduction

A knot is formally defined as continuous closed curve in 3-space that does not intersect itself anywhere, but can be thought of as a tangled rope with the ends connected. Knot colorability is a useful method of distinguishing knots. Different methods of computing a knot's colorability include Linear Algebra and Graph Theory.

Colorability

A projection of a knot is **tricolorable** if each of the strands in a projection can be colored one of 3 different colors so that at each crossing either 3 different colors come together or all the same color comes together.



A knot diagram can be **labeled mod p** if each edge can be labeled with an integer from 0 to p-1 such that

i. At each crossing the relation 2x-y-z=0(mod p) holds, where x is the label on the overcrossing and y and z are the other two labels.

ii. At least two labels are distinct.

Labeling Theorem

If some diagram for a knot can be labeled mod p then every diagram for that knot can be labeled mod p. (Livingston)

Mod p Matrix

Method:

Given a knot diagram, label each strand of the diagram with a variable, say x_i. At each crossing a relation between the variables is defined: If strand x_i crosses over strands x_j and x_k, then $2x_i - x_j - x_k = 0 \pmod p$.

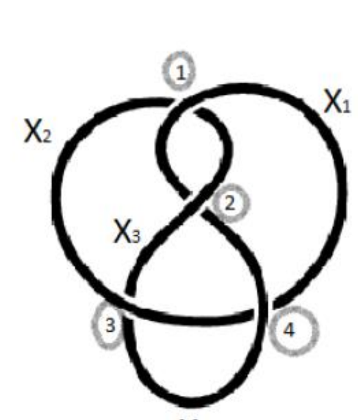
A knot can be labeled mod p if there is a mod p solution to this system of equations with not all x_i being equal.

Theorem

There is an n x n matrix corresponding to a knot diagram with n strands. Deleting any one column and any one row yields a new matrix. The knot can labeled mod p if and only if the corresponding set of equations has a nontrivial mod p solution. (Livingston)

Example

Consider the Figure-Eight Knot shown to the right. We construct the following equations using the method described above:



$2x_1 - x_2 - x_3 \equiv 0 \pmod p$

$2x_3 - x_1 - x_4 \equiv 0 \pmod p$

$2x_2 - x_3 - x_4 \equiv 0 \pmod p$

$2x_4 - x_1 - x_1 \equiv 0 \pmod p$

These equations correspond to the following matrix and reduced matrix determinant:

$$\begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 0 & 2 & -1 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 0 & 2 \end{pmatrix} \qquad \begin{vmatrix} 2 & -1 & -1 \\ -1 & 0 & 2 \\ 0 & 2 & -1 \end{vmatrix} = 5$$

Deleting the last row and column then calculating the determinant shows that p = 5. Thus the Figure-Eight Knot is mod 5 colorable!

Spanning Trees

Theorem

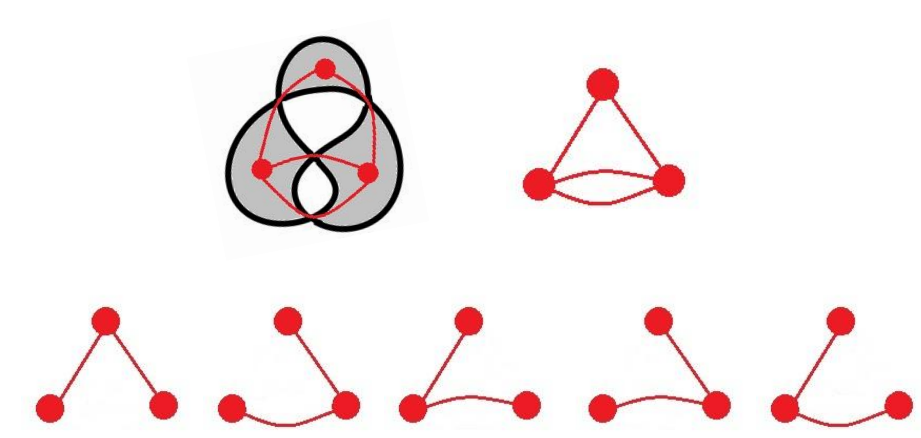
For an alternating knot K with colorability mod p, the number of spanning trees of the planar graph of K is p. (Harary, Kauffman)

Method:

Shade a knot K in such a way that no two adjacent regions are shaded, making sure that the infinite outermost region is not shaded. Put a vertex at the center of each shaded region. With an edge, connect the vertices through the crossings. This results in the planar graph of knot K.

Example

Again, consider the Figure-Eight Knot with planar graph and spanning trees shown below:



Since there are five spanning trees, we see that the Figure-Eight Knot is indeed mod 5 colorable.

Kirchhoff's Method

The **valence** of a vertex v, denoted $\mathrm{val}_G(v)$, is the number of edges having v as an endpoint. Also, for two vertices $v_i \neq v_j$ let us write ϵ_{ij} for the number of edges joining i and j.

We now define the **Laplacian matrix** of G to be the n x n matrix L(G) given by

$$[L(G)]_{ij} = \begin{cases} \mathrm{val}_G(v_i) & \text{if } i = j, \\ -\epsilon_{ij} & \text{if } i \neq j. \end{cases}$$

Kirchhoff's Matrix-Tree Theorem

Let v_i be a vertex of graph G and let L'(G) be the matrix obtained from L(G) by deleting the row and column corresponding to v_i. Then the determinant of L'(G) gives the number of spanning trees of G. (Kirchhoff)

Example

As we have seen, the Figure-Eight Knot is mod 5 colorable. Let us verify this one last time with the method described by Kirchhoff's Theorem. L(G) and |L'(G)| are shown below.

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 3 & -2 \\ -1 & -2 & 3 \end{pmatrix} \qquad \begin{vmatrix} 2 & -1 \\ -1 & 3 \end{vmatrix} = 5$$

Thus, using Kirchhoff's Matrix-Tree Theorem, we have again verified the mod p coloring number of the Figure-Eight Knot.