Semigroups and the Characteristics of their Interassociates

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Basic Definitions

- A **semigroup** is a set S with an associative binary operation.
- A semigroup, $(S,\cdot)$, has an **interassociate**, $(S,\star)$, if for all $a,b,c\in S$, $a\cdot(b\star c)=(a\cdot b)\star c$ and $a\star(b\cdot c)=(a\star b)\cdot c$.

Examples

Here are three semigroups.

$\cdot$	0	a
0	0	0
a	0	0

Example 1

$\cdot$	0	a
0	0	0
a	0	a

Example 2

$\cdot$	a	a^2	a^3
a	a^2	a^3	a^2
a^2	a^3	a^2	a^3
a^3	a^2	a^3	a^2

Example 3

Any group is also a semigroup, but not every semigroup is a group, as seen in these examples.

Objective

In 2004 Gould, Linton, and Nelson discussed monogenic semigroups and their interassociates.There has not been a discussion of what properties different types of semigroups must share with their interassociates. In this project, we investigated what properties of semigroups are shared with their interassociates. If a semigroup had a property such as commutativity, we tried to prove that any interassociate would also have that property or to find an example of an interassociate without that property.

Properties That Fail

- A semigroup, $(S,\cdot)$, is **null** if $0\in S$ and $a\cdot b=0$ for all $a,b\in S$. Example 1 above is clearly null and is an example of a null semigroup with an interassociate that is not null.
- A semigroup, $(S,\cdot)$, is a **band** if $x^2=x$ for all $x\in S$. Example 2 is a band since $0^2=0$ and $a^2=a$.
- A semigroup, $(S,\cdot)$, is a **semilattice** if $x^2=x$ and $xy=yx$ for all $x,y\in S$. Example 2 is a semilattice since it is a band and it is commutative.

Lemma: All of the above properties fail to be preserved by interassociates.

Counterexample: For example, let $(S,\cdot)$ and $(S,\star)$ have the cayley tables shown in Examples 1 and 2. Then $(S,\cdot)$ and $(S,\star)$ are interassociates because $(a\cdot b)\star c=0\star c=0=a\cdot(b\star c)$. But for each property above, exactly one of Examples 1 and 2 has the property.

References

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- Gould, M., K.A. Linton, and A.W. Nelson, *Interassociates of Monogenic Semigroups*, Semigroup Forum **68** (2004), pp. 186-201.
- Howie, J. M., "Fundamentals of Semigroup Theory", Oxford University Press, 1995.

Properties That Survive

- **Property:** A **group** is a set G with an associative binary operation, usually $+$ or $\cdot$, which has an identity element and is such that each element has a unique inverse. If $(S,\cdot)$ is a group, any interassociate $(S,\star)$ is a group that is isomorphic to $(S,\cdot)$.

  Proof: Let $(S,\cdot)$ be a group and $(S,\star)$ an interassociate of $(S,\cdot)$. Then, since $(S,\cdot)$ is a monoid, $a\star b=a\cdot c\cdot b$ for some fixed $c\in S$. Let $e=c^{-1}$. Then we have

  $$b\star e=bce=bcc^{-1}=be=b$$
  $$e\star b=ecb=c^{-1}cb=eb=b$$

  Therefore $(S,\star)$ has an identity $e=c^{-1}$.
  Now Let $b=a^{-1}$, and then $b=ea^{-1}e=c^{-1}a^{-1}c^{-1}$. This gives us

  $$a\star b=acb=acc^{-1}a^{-1}c^{-1}=aea^{-1}c^{-1}=aa^{-1}c^{-1}=ec^{-1}=c^{-1}=e$$
  $$b\star a=bca=c^{-1}a^{-1}c^{-1}ca=c^{-1}a^{-1}ea=c^{-1}a^{-1}a=c^{-1}e=c^{-1}=e.$$

  Hence every element of $(S,\star)$ has an inverse. Thus $(S,\star)$ is a group. □

- **Property:** A semigroup, $(S,\cdot)$, has a **zero element** z if for all $a\in S$, $za=z$ and $az=z$. If $(S,\cdot)$ has a zero element, then every interassociate $(S,\star)$ has a zero element.

  Proof: Let $(S,\cdot)$ have a zero element, z. Also let $(S,\star)$ be an interassociate of $(S,\cdot)$. Then from $z\cdot(a\cdot b)\star c=(z\cdot a)\cdot(b\star c)$ we can see that $z\star c=z$. Similarly, $c\star z=z$. Therefore $(S,\star)$ also has a zero element. □

Properties That Survive With Certain Limits

- A semigroup, $(S,\cdot)$, is **periodic** if for all $x\in S$, there are $n,m\in\mathbb{Z}$ such that $x^n=x^m$.
- **Conjecture:** Assume $(S,\cdot)$ is periodic and that $(S,\star)$ has the form $x\star y=xky$ for some fixed $k\in S$. Then $(S,\star)$ is periodic.

  Proof: Assume $(S,\cdot)$ is periodic and $(S,\star)$ is an interassociate with the form $x\star y=xky$ for some $k\in S$.
  Then

  $$x^{\star 2}=x\star x=xkx$$
  $$x^{\star 3}=x\star x\star x=xkxkx=(xk)^2x$$
  $$x^{\star 4}=x\star x\star x\star x=xkxkxkx=(xk)^3x$$
  $$x^{\star 5}=x\star x\star x\star x\star x=xkxkxkxkx=(xk)^4x.$$

  Clearly $x^{\star n}=(xk)^{n-1}x$, for $n\in\mathbb{Z}$. Since $xk\in S$, we know $(xk)^w=(xk)^z$ for some $w,z\in\mathbb{Z}$. Hence $(xk)^wx=(xk)^zx$ and $x^{\star n}=x^{\star m}$ for $n=w+1$ and $m=z+1$. Therefore $(S,\star)$ is periodic. □

  It is well-known that any finite semigroup is periodic, so any interassociate of a finite periodic semigroup must also be periodic. It is not yet known if any interassociate of an infinite periodic semigroup is periodic.

- **Property:** If $(S,\cdot)$ is commutative and cancellative, then any interassociate $(S,\star)$ is also commutative and cancellative.

Unsolved Properties

Some of the properties we attempted to solve but were unable to without certain restrictions include **commutativity**, **cancellativity**, **infinite periodic** semigroups and **orthodox** semigroups, where an orthodox semigroup has the property that if $e^2=e$ and $f^2=f$, then $efef=ef$. For example. we found commutativity and cancellativity will survive together, but not separately.