Non-Abelian Wilczek-Zee Geometric Phase for Particles in Uniform External Magnetic Fields

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Introduction

In nonrelativistic quantum mechanics, a single spin-0 particle in a uniform external magnetic field can be described by a Hamiltonian

$$H_0 = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 + V(\mathbf{q})$$

so that the wave functions associated with the energy eigenstates of the particle satisfy

$$H_0 \left|\mathbf{n}\right\rangle = E_{\mathbf{n}} \left|\mathbf{n}\right\rangle$$

The Adiabatic Theorem states that if the parameters of a system undergo adiabatic variation along a closed path in parameter space, the particles wave function will accumulate phase:

$$\left|\mathbf{n}\right\rangle \longrightarrow e^{i\Gamma} \left|\mathbf{n}\right\rangle$$

This phase consists of two parts: the dynamic phase and the geometric phase. The dynamic phase is given by the exact motion

$$\theta_{\mathbf{n}}(T) = -\frac{1}{\hbar}\int_0^T E_{\mathbf{n}}(t')\,dt'$$

whereas the geometric phase is given by the line integral along the path C

$$\gamma_{\mathbf{n}}(C) = \oint_C \left\langle \mathbf{n}\right|\nabla_{\mathbf{B}}\left|\mathbf{n}\right\rangle \cdot d\mathbf{B}$$

The integrand in the above equation behaves like a gauge potential, and by Stokes' Theorem there exists a gauge field

$$\mathcal{F}_{\mathbf{n}} = \nabla \times \mathcal{A}_{\mathbf{n}} = \nabla_{\mathbf{B}} \times \left\langle \mathbf{n}\right|\nabla_{\mathbf{B}}\left|\mathbf{n}\right\rangle$$

so that

$$\gamma_{\mathbf{n}}(C) = \oint_C \mathcal{A}_{\mathbf{n}} \cdot d\mathbf{B} = \int_\Sigma \mathcal{F}_{\mathbf{n}} \cdot d\mathbf{A}$$

where the rightmost integral is taken over the surface Σ enclosed by the curve C.

Using the language of differential geometry, the gauge potential and field can be recast as differential forms; namely the connection 1-form and the curvature 2-form, respectively, on an appropriately defined fiber bundle. In this frame, the geometric phase is defined to be

$$\gamma_{\mathbf{n}}(C) = \int_C \omega_{\mathbf{n}} = \int_\Sigma \Omega_{\mathbf{n}}$$

where ω and Ω are given through exterior derivatives:

$$\omega_{\mathbf{n}} = i\left\langle \mathbf{n}\right|d\left|\mathbf{n}\right\rangle\ , \quad \Omega_{\mathbf{n}} = d\omega_{\mathbf{n}}$$

The geometric phase considered so far is called the Berry phase, which is abelian, so that under the Adiabatic Theorem particles initial in an energy eigenstate will remain in an energy eigenstate. By this construction, the Berry phase results from a connection on a $U(1)$ fiber bundle over the parameter space of the Hamiltonian, $\mathbb{R}^3$.

Spin and the Wilczek-Zee Phase

Now consider the nonrelativistic particle in a uniform external magnetic field. If the particle has nonzero spin (in general the particle has spin-j), the magnetic moment of the particle will interact with the external field, adding a term to the particle's Hamiltonian:

$$H = H_0 + g\boldsymbol{\sigma}\cdot\mathbf{B}$$

where the spin vector

$$\boldsymbol{\sigma} = \sigma_1\mathbf{e}_1 + \sigma_2\mathbf{e}_2 + \sigma_3\mathbf{e}_3$$

is given by the $2j+1 \times 2j+1$ traceless Hermitian Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & b_j & 0 & \cdots & 0 & 0 \\ b_j & 0 & b_{j-1} & \cdots & 0 & 0 \\ 0 & b_{j-1} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & b_{-j} \\ 0 & 0 & 0 & \cdots & b_{-j} & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -ib_j & 0 & \cdots & 0 & 0 \\ ib_j & 0 & -ib_{j-1} & \cdots & 0 & 0 \\ 0 & ib_{j-1} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -ib_{-j} \\ 0 & 0 & 0 & \cdots & ib_{-j} & 0 \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} j & 0 & 0 & \cdots & 0 & 0 \\ 0 & j-1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & j-2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -j+1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -j \end{pmatrix}$$

with the constants defined to be

$$b_m = \frac{1}{2}\sqrt{(j+m)(j+1-m)}$$

The eigenstates of this new Hamiltonian can be constructed through the tensor product of the spin-0 Hamiltonian and the spin-m eigenstates along the direction of the external magnetic field:

$$\left|\mathbf{n}, m\right\rangle = \left|\mathbf{n}\right\rangle \left|m\right\rangle$$

These eigenstates will satisfy the equation

$$H\left|\mathbf{n}, m\right\rangle = \left(E_{\mathbf{n}} + mgB\right)\left|\mathbf{n}, m\right\rangle$$

For most particles, the value of g will be small compared to the energy of the system, so that transitions between spin states will be much more probable than transitions between the larger energy levels $E_{\mathbf{n}}$. To incorporate the possibility of spin transitions, we introduce the connection and curvature forms on a $U(2j+1)$ fiber bundle over the sphere S^2 (space of possible directions of the external magnetic field) governing the non-abelian Wilczek-Zee phase.

The connection and curvature forms becomes a $2j+1 \times 2j+1$ matrices

$$\left(\omega_{\mathbf{n}}\right)_{kl} = i\left\langle \mathbf{n}, m_k\right|d\left|\mathbf{n}, m_l\right\rangle$$

and

$$\Omega_{\mathbf{n}} = d\omega_{\mathbf{n}} + i\omega_{\mathbf{n}} \wedge \omega_{\mathbf{n}}$$

By the product rule of the exterior derivative and the orthogonality of the eigenstates

$$i\left\langle \mathbf{n}, m_k\right|d\left|\mathbf{n}, m_l\right\rangle = i\delta_{kl}\left\langle \mathbf{n}\right|d\left|\mathbf{n}\right\rangle + i\left\langle m_k\right|d\left|m_l\right\rangle \Longrightarrow \omega_{\mathbf{n}} = \omega_0 I_{2j+1} + \omega_{\text{spin}}$$

Furthermore, through the linearity of the exterior derivative, the bilinearity of the wedge product, and the anti-commutitivity of the wedge product (for 1-forms),

$$\omega = \omega_1 + \omega_2 \Longrightarrow \Omega = \Omega_1 + \Omega_2$$

so that we can separately consider the spin-0 and spin-j contributions to the overall geometric phase. We now consider the spin-j contribution (for ease of notation we drop the subscripts).

For ease of derivation, it is desirable to express the spin-m eigenstate in the direction of the magnetic field (characterized by polar angle θ and azimuthal angle ϕ) as a spinor rotation of the spin-m eigenstate in the $\mathbf{e}_3$ direction:

$$\left|m\right\rangle = U(\theta,\phi)\left|m(\mathbf{e}_3)\right\rangle$$

with the spinor rotation operator given by

$$U(\theta,\phi) = U_3(\phi)U_2(\theta) = e^{-i\phi\sigma_3}e^{-i\theta\sigma_2}$$

Writing the connection and curvature forms in terms of their components

$$\omega = \omega_\mu dx^\mu \qquad \Omega = \frac{1}{2}\Omega_{\mu\nu}dx^\mu \wedge dx^\nu$$

it can be seen that the components of the forms are related by

$$\Omega_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu + i\left[\omega_\mu, \omega_\nu\right]$$

Making use of the Hadamard Lemma

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \ldots$$

the components of the connection are found to be

$$\omega_\theta = iU^\dagger(\theta,\phi)\,\partial_\theta\,U(\theta,\phi) = \sigma_2$$

$$\omega_\phi = iU^\dagger(\theta,\phi)\,\partial_\phi\,U(\theta,\phi) = -\sigma_1\sin\theta + \sigma_3\cos\theta$$

and the connection and curvature, along with the corresponding gauge potential and field (see Fig. 1), are given by

$$\omega = \sigma_2\,d\theta + \left(-\sigma_1\sin\theta + \sigma_3\cos\theta\right)d\phi \longleftrightarrow \mathcal{A} = \frac{1}{B}\sigma_2\hat{\theta} + \frac{1}{B}\left(-\sigma_1 + \sigma_3\cot\theta\right)\hat{\phi}$$

$$\Omega = -\left(\sigma_1\cos\theta + \sigma_3\sin\theta\right)d\theta \wedge d\phi \longleftrightarrow \mathcal{F} = -\frac{\hat{B}}{B^2}\left(\sigma_1\cot\theta + \sigma_3\right)$$

As a specific case, if the external magnetic field precesses at a constant angle θ with respect to the $\mathbf{e}_3$ axis (see Fig. 2), the phase matrix for a spin- $^1/_2$ particle will be

$$\begin{pmatrix} a(\theta) & b(\theta) \\ b(\theta) & \overline{a}(\theta) \end{pmatrix}$$

where coefficient functions are shown in Fig. 3.

Example: A Quantum Dot

The electron in a quantum dot can be modeled as a spin- $^1/_2$ particle in an anisotropic harmonic oscillator

$$V(\mathbf{q}) = \frac{1}{m}\left(k_1^2q_1^2 + k_2^2q_2^2 + k_3^2q_3^2\right)$$

The Hamiltonian for this electron can be symplectically transformed into canonical position and momentum

$$\begin{pmatrix} p \\ q \end{pmatrix} = L\begin{pmatrix} P \\ Q \end{pmatrix} \qquad L = \begin{pmatrix} L_1 & L_3 \\ L_2 & L_4 \end{pmatrix}$$

from the eigenvalues κ obtained by solving the following cubic equation, where ω_i are the cyclotron frequencies:

$$\left(k_1^2+\kappa^2\right)\left(k_2^2+\kappa^2\right)\left(k_3^2+\kappa^2\right)+\omega_1^2\left(k_1^2+\kappa^2\right)\kappa^2+\omega_2^2\left(k_2^2+\kappa^2\right)\kappa^2+\omega_3^2\left(k_3^2+\kappa^2\right)\kappa^2 = 0$$

This transformation yields a diagonal Hamiltonian whose solutions are well known

$$H(\mathbf{P},\mathbf{Q}) = \frac{1}{2m}\left(P_1^2 + P_2^2 + P_3^2\right) + \frac{1}{m}\left(\kappa_1^2Q_1^2 + \kappa_2^2Q_2^2 + \kappa_3^2Q_3^2\right)$$

Carrying the transformation through the calculation of the Berry phase using the solutions to the above Hamiltonian, the gauge potential is found to be

$$\mathcal{A}_{\mathbf{n}} = \sum_{i=1}^{3}\left(n_i + \frac{1}{2}\right)\left[\kappa_i F_{ii} + \frac{G_{ii}}{\kappa_i}\right]$$

where

$$F = \frac{1}{2}\nabla\left(L_2^{-1}L_1\right) - \left(\nabla L_2^{-1}\right)L_1 + \frac{1}{2}L_1^*\nabla\left(L_4L_2^{-1}\right)L_1$$

$$G = \frac{1}{2}L_2^*\nabla\left(L_4L_2^{-1}\right)L_2$$

The total geometric phase for the quantum dot will then be given by

$$\gamma_n(C) = I_2\oint_C \mathcal{A}_n \cdot d\mathbf{B} - \int_\Sigma\left(\sigma_1\cos\theta + \sigma_3\sin\theta\right)d\theta \wedge d\phi$$

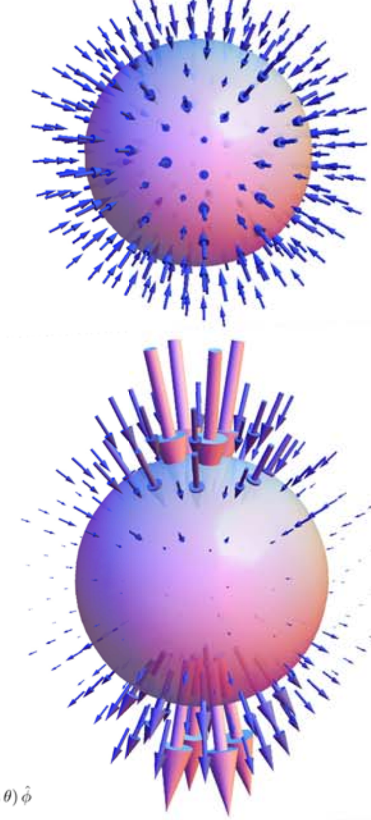

Fig. 1: Gauge fields associated with the spin-j curvature. The top field corresponds to σ_3 and the bottom field corresponds to σ_1. The phase accumulated along a closed path on S^2 is dependent upon the flux of these fields through the interior of these paths.

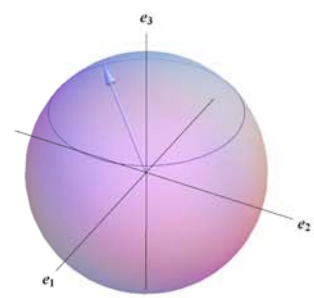

Fig. 2: An adiabatically precessing uniform external magnetic field at an angle θ with respect to the $\mathbf{e}_3$ axis.

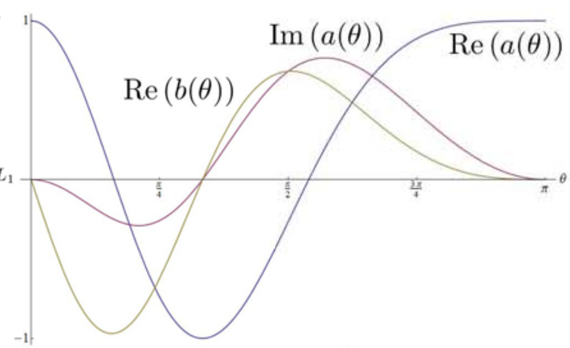

Fig. 3: Real and imaginary parts of the phase matrix for a precessing external field.