The Minimum and Maximum Number of Stupid Dice Allowed in an Equal Relabeling

Crystal Faqih, Civil Engineering   Mentor: Dr. Amber Rosin

Kellogg Honors College Capstone Project

Background

- Given a pair of standard 6-sided dice, we can roll the following sums with the given probabilities:

sums:	2	3	4	5	6	7	8	9	10	11	12
probabilities:	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

- Can the dice be relabeled so that the sums 2 through 12 are equally likely? It is well known that this is impossible. (There are 11 sums and 36 possible outcomes and 11 does not divide 36.)
- However, seven dice result in the sums 7 through 42 (again with unequal probabilities), but in this case, a relabeling with equal probabilities is possible. (See the blue dice for such a labeling.)
- In [1], it is shown that m 6-sided dice can be relabeled in such a way that the standard sums (i.e. the sums for m standard 6-sided dice) are equally likely if and only if m has the form

$$m = \frac{6^s 2^{4t} - 1}{5} \quad \text{or} \quad m = \frac{6^s 3^{4t} - 1}{5}.$$

- It is also shown in [1] that such relabelings always require "stupid dice," which are dice that have a 1 on every face.

How many stupid dice do we need for 7 dice?

- For 7 dice the standard sums are $7, 8, 9, \ldots, 42$, so the number of standard sums is $36 = 2^2 3^2$.
- Since the dice are 6-sided, the number of possible outcomes is $6^7 = 2^7 3^7$.
- For the sums to be equally likely, each sum must appear $\frac{\text{number of outcomes}}{\text{number of sums}} = \frac{6^7}{6^2} = 6^5 = 2^5 3^5$ times.
- To roll a 7, we must have a 1 on each of the 7 dice. Let a_i be the number of 1's on die i.
- The number of ways to roll a sum of 7 is $a_1 a_2 a_3 a_4 a_5 a_6 a_7$ which must equal 6^5.
- A stupid die results whenever $a_i = 6$.
- We need 5 2's and 5 3's in the product $a_1 a_2 a_3 a_4 a_5 a_6 a_7$. In other words, there are 7 slots for the 5 2's and the 5 3's, so some of the 2's and 3's will have to overlap. (This is why we have to have stupid dice.)
- To get the least number of stupid dice, we want the least amount of overlap for the 5 2's and 5 3's. This can be done as follows:

die 1	die 2	die 3	die 4	die 5	die 6	die 7
2	2	2	2	2		
		3	3	3	3	3

  So 3 dice have overlap meaning we have 3 stupid dice.
- The least number of stupid dice is 3. The labeling above is an equal labeling with the least number of stupid dice.
- To get the greatest number of stupid dice, we want the most amount of overlap for the 5 2's and 5 3's. This can be done as follows:

die 1	die 2	die 3	die 4	die 5	die 6	die 7
2	2	2	2	2		
3	3	3	3	3		

  So 5 dice have overlap meaning we have 5 stupid dice.
- The greatest number of stupid dice is 5.

The general case

- We consider the case when $m = \frac{6^s 3^{4t} - 1}{5}$.
- For the sums to be equally likely, each sum must appear $\frac{\text{number of outcomes}}{\text{number of sums}} = \frac{6^m}{6^s 3^{4t}} = \frac{6^{m-s}}{3^{4t}} = \frac{3^{m-s} 2^{m-s}}{3^{4t}} = 2^{m\ s} 3^{m\ s\ 4t}$ times.
- Let $a_i =$ the number of times 1 appears on die i.
- Then the number of ways to roll the sum m must be

$$a_1 a_2 \cdots a_m = 2^{m-s} 3^{m-s-4t}.$$

- What are the possible values for $a_1, a_2, \ldots, a_m$?
- Since 2 and 3 are prime, each a_i can only be made out of 1's, 2's, and 3's.
- Clearly since our dice have six sides, $a_i \leqslant 6$.
- So we know that a_i can only be 1, 2, 3, 4, or 6.
- Since m and $6m$ have to occur the same number of times, and m can only be rolled by using all 1's and $6m$ can only be rolled using the largest number that appears on every die, the largest number on die i must also appear a_i times.
- This actually means that a_i can only be 1, 2, 3, or 6.

The general case - continued

- Die i is stupid if $a_i = 6$.
- We need $m - s$ of the dice to have a 2. We also need $m - s - 4t$ of the dice to have a 3.
- To get the most amount of stupid dice, we want as much overlap as possible, so we make $m - s - 4t$ of the dice have both a 2 and a 3. So the greatest possible number of stupid dice is $m - s - 4t$.
- To get the least amount of stupid dice, we want as little overlap as possible. We need $m - s - 4t$ 2's and $m - s$ 3's. If there were no overlap, this would require $m - s - 4t + m - s = 2m - 2s - 4t$ dice. However, we only have m dice, so $2m - 2s - 4t - m = m - 2s - 4t$ must overlap. So the least possible number of stupid dice is $m - 2s - 4t$.

A Dice Labelling

- In [1] a equal relabelling of m is given using $m - s - 4t$ stupid dice, which shows that the it is actually possible to get the most amount of stupid dice found earlier.
- We will illustrate a relabelling of m dice using $m - 2s - 4t$ stupid dice to show that the least number of stupid dice is also possible.

Label the first s dice as follows:

$$\begin{array}{llllll}
1, & 1, & 1+3^0, & 1+3^0, & 1+2\cdot 3^0, & 1+2\cdot 3^0 \\
1, & 1, & 1+3^1, & 1+3^1, & 1+2\cdot 3^1, & 1+2\cdot 3^1 \\
\vdots & & & \vdots & & \vdots \\
1, & 1, & 1+3^{s-1}, & 1+3^{s-1}, & 1+2\cdot 3^{s-1}, & 1+2\cdot 3^{s-1}
\end{array}$$

Label the next s dice as follows:

$$\begin{array}{llllll}
1, & 1, & 1, & 1+3^s\cdot 2^0, & 1+3^s\cdot 2^0, & 1+3^s\cdot 2^0 \\
1, & 1, & 1, & 1+3^s\cdot 2^1, & 1+3^s\cdot 2^1, & 1+3^s\cdot s^1 \\
\vdots & & & \vdots & & \vdots \\
1, & 1, & 1, & 1+3^s\cdot 2^{s-1}, & 1+3^s\cdot 2^{s-1}, & 1+3^s\cdot 2^{s-1}
\end{array}$$

Label the next $4t$ dice as follows:

$$\begin{array}{llllll}
1, & 1, & 1+2^s\cdot 3^s, & 1+2^s\cdot 3^s, & 1+2^{s+1}\cdot 3^s, & 1+2^{s+1}\cdot 3^s \\
1, & 1, & 1+2^s\cdot 3^{s+1}, & 1+2^s\cdot 3^{s+1}, & 1+2^{s+1}\cdot 3^{s+1}, & 1+2^{s+1}\cdot 3^{s+1} \\
\vdots & & & \vdots & & \vdots \\
1, & 1, & 1+2^s\cdot 3^{4t+s-1}, & 1+2^s\cdot 3^{4t+s-1}, & 1+2^{s+1}\cdot 3^{4t+s-1}, & 1+2^{s+1}\cdot 3^{4t+s-1}
\end{array}$$

Label the remaining $m - 2s - 4t$ dice with 1's on every face.

Further Questions

- Is it possible to find equal relabelings using a number of stupid dice that is between the least and greatest possible number?
- Is the above the only equal relabeling using the least number of stupid dice?

References

[1] Rosin, A., Swift, R.J., & Sharobiem, M. (2008), Dice Sums, *The Mathematical Scientist*, Vo. 33, 2, 99-109.