Deformation Quantization of the Harmonic Oscillator

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Background

Classical Mechanics:

In general, Hamiltonian mechanics is performed on a symplectic manifold (phase space) (X, Λ) with X^* functions (observables). Here $X = T^*Q$, the cotangent bundle of the system's configuration space, and Λ is a symplectic form. Using the symplectic form, we may derive the Poisson bracket for the manifold, a skew-symmetric bilinear operator $\{\cdot,\cdot\} : X^* \times X^* \to X^*$. We use the bracket in combination with the Hamiltonian of the system to determine the time evolution of observables acting on the system,

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{f, H\} + \frac{\partial f}{\partial t}, \tag{1}$$

where f is an observable and H is the Hamiltonian. Equation (1) reduces to Hamilton's equations of motion when we use canonical coordinates and the canonical Poisson bracket.

Canonical Quantization:

A goal of quantization is to change a classical system consisting of phase space and smooth, continuous functions into a quantum system consisting of a Hilbert space and self-adjoint differential operators. In canonical quantization:

Classical		Quantum
q	$\mapsto$	Q
p	$\mapsto$	P

with $Q(\psi) \equiv q\psi$ and $P(\psi) \equiv -i\hbar\frac{\partial}{\partial q}\psi$, and $[Q, P] = i\hbar$. The last relation is the commutator, a quantum analog to the Poisson bracket. But more choices are required to associate $f(p, q)$ with $F(P, Q)$ since p and q commute while P and Q do not commute.

Deformation Quantization:

The above problem of choice creates a need for a more well-defined theory. For example

$$qp^2 \mapsto QP^2, \qquad pqp \mapsto PQP = QP^2 - i\hbar P, \qquad p^2q \mapsto P^2Q = QP^2 - 2i\hbar P$$

with the LHS's are equivalent to one another while the RHS's are not. Deformation quantization addresses this issue by changing the multiplication to a star product. For example choosing the order such that Q is always to the left of P determines the product

$$q \star p = qp, \qquad p \star q = qp + i\hbar,$$

in the language of deformation quantization.

Star Products

We now formally introduce star products. They satisfy the following properties:

1. $f \star g = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} P^k(f, g)$, where P^k is a bidifferential operator and $\lambda = i\hbar/2$.
2. $f \star g = fg + \mathcal{O}(\hbar)$.
3. $[f, g]_\star = f \star g - g \star f = i\hbar\{f, g\} + \mathcal{O}(\hbar^2)$.
4. $f \star 1 = 1 \star f = f$.
5. $\overline{f \star g} = \overline{g} \star \overline{f}$.

When starting from the Poisson manifold (X, Λ), we may define the bidifferential operators in Property 1 by

$$P^k(f, g) = \Lambda^{i_1 j_1} \cdots \Lambda^{i_k j_k} \nabla_{i_1 \cdots i_k} f \, \nabla_{j_1 \cdots j_k} g, \tag{2}$$

where ∇ is the covariant derivative. We also write the star product of two functions f and g as

$$f \star g = f \, e^{\lambda \overleftrightarrow{P}} g. \tag{3}$$

Using Property 3, the Moyal equations of motion,

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{1}{i\hbar}[f, H]_\star, \tag{4}$$

analogous to (1) if we think of the observables as evolving in time. One solution to (4) is $f_t = \mathrm{Exp}(-\frac{tH}{i\hbar}) \star f \star \mathrm{Exp}(\frac{tH}{i\hbar})$ where

$$\mathrm{Exp}\left(\frac{tH}{i\hbar}\right) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{t}{i\hbar}\right)^k (H\star)^k, \tag{5}$$

and $H^{\star k} = H \star \cdots \star H$ (k times). If we can find a Fourier-Dirichlet expansion of (5),

$$\mathrm{Exp}\left(\frac{tH}{i\hbar}\right) = \sum_{k=0}^{\infty} \pi_k e^{E_k t/i\hbar}, \tag{6}$$

then E_k are the eigenvalues of H and π_k are the orthonormal eigenstates states.

Harmonic Oscillator

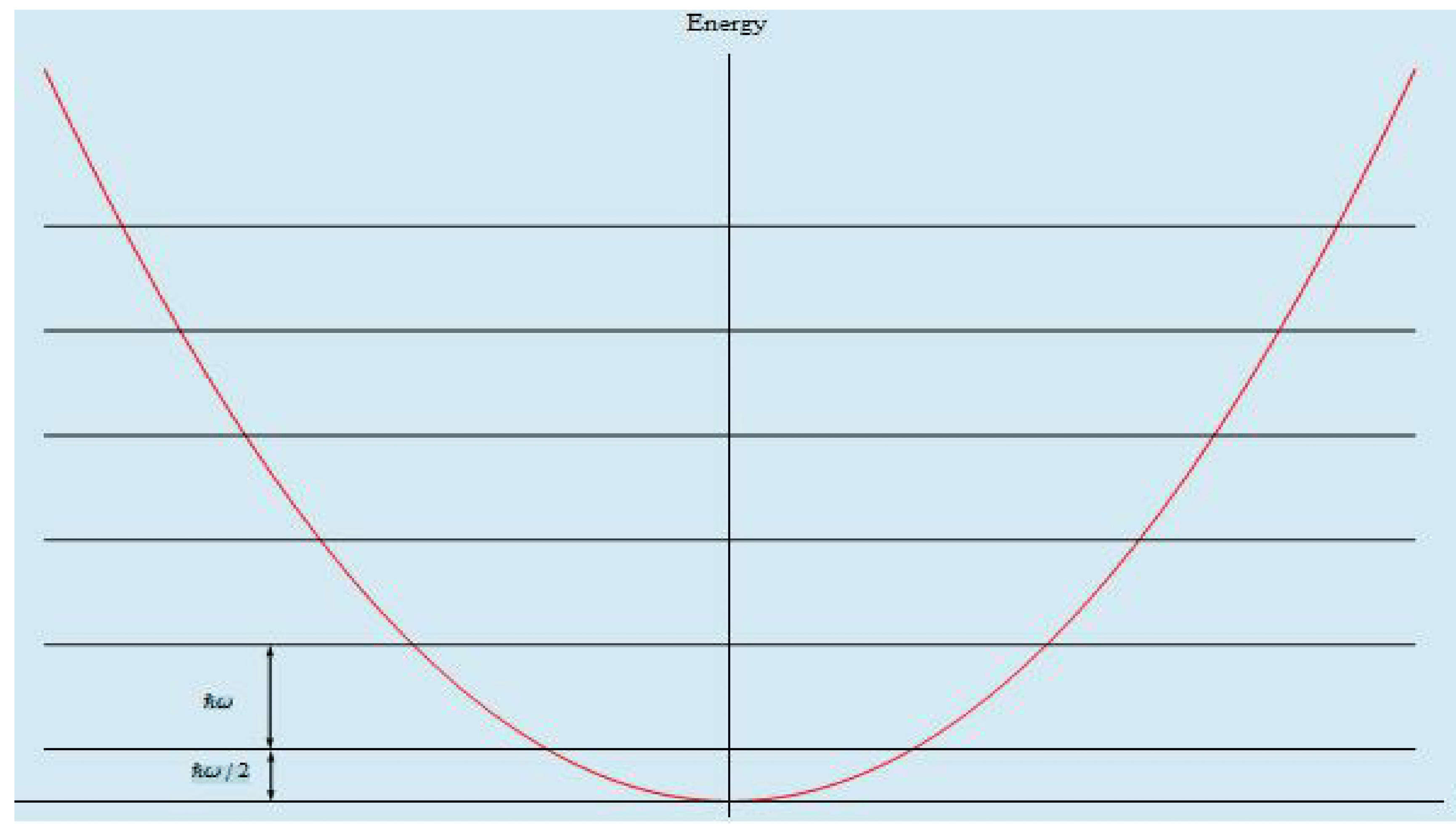

Fig. 1: Graph of the first six energy levels of the quantum harmonic oscillator.

We now perform the deformation quantization of the harmonic oscillator. Our symplectic manifold is $\mathbb{R}^2$ with the symplectic form

$$\Lambda^{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

so (2) becomes $P^k(f, g) = \Lambda^{i_1 j_1} \cdots \Lambda^{i_k j_k} \partial_{i_1 \cdots i_k} f \, \partial_{j_1 \cdots j_k} g$, and the star product, called the Moyal star product, is written as,

$$f \star g = f \, e^{\lambda\left(\overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p} \cdot - \overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q} \cdot\right)} g. \tag{7}$$

Letting pesky factors $m = \omega = 1$, we write the Hamiltonian as $H = \frac{1}{2}(p^2 + q^2)$. Since (5) is cumbersome, we want to find a closed form for $\mathrm{Exp}(tH/i\hbar)$. First we find the recursion relation $K_n(H) = (H\star)^n = HK_{n-1}(H) - (\hbar^2/4)K'_{n-1}(H) - (\hbar^2/4)HK''_{n-1}(H)$ and prove the following propositions:

Proposition 1

For any fixed $(p, q) \in \mathbb{R}^2$ the power series in t:

$$\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{Ht}{i\hbar}\star\right)^n \Big|_{p,q} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{t}{i\hbar}\right)^n K_n(H(p, q)) \tag{8}$$

has a radius of convergence equal to π. For $|t| < \pi$, (8) has the closed form

$$\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{t}{i\hbar}\right)^n K_n(H(p, q)) = (\cos(t/2))^{-1} \exp\left(\frac{2H}{i\hbar}\tan(t/2)\right). \tag{9}$$

Proposition 2

For fixed $t \in (-\pi, \pi)$ the series (8) converges in $\mathcal{D}'(\mathbb{R}^2)$ for the weak topology to

$$(\cos(t/2))^{-1} \exp\left(\frac{(p^2 + q^2)}{i\hbar}\tan(t/2)\right). \tag{10}$$

Then with the closed form (9)/(10) in hand we find the Fourier expansion as in (6):

Proposition 3

For fixed $(p, q) \in \mathbb{R}^2 - \{0\}$, (10) defines a periodic distribution $S \in \mathcal{D}'(\mathbb{R})$. It has a Fourier expansion,

$$S = \sum_{n=0}^{\infty} \pi_n(p, q) e^{-i(n+1/2)t}, \tag{11}$$

with

$$\pi_n(p, q) = 2\exp\left(-\frac{2H(p, q)}{\hbar}\right)(-1)^n L_n\left(\frac{4H(p, q)}{\hbar}\right), \tag{12}$$

where $L_n = L_n^0$ is the usual Laguerre polynomial of degree n.

Proposition 4

For fixed $t \in \mathbb{C}$ with $\mathrm{Im}\, t \leq 0$ and $t \neq (2k+1)\pi$, $k \in \mathbb{Z}$, we may use (9) to write

$$\mathrm{Exp}\left(\frac{tH}{i\hbar}\right) = \sum_{n=0}^{\infty} \pi_n e^{-i(n+1/2)t}, \tag{13}$$

which converges in $\mathcal{S}'(\mathbb{R}^2)$ for the weak topology.

Thus we see that the energy levels are $E_n = (n + 1/2)\hbar$, or $(n + 1/2)\hbar\omega$ if we reinsert ω, as in Fig. 1. This is identical to the result found by performing the calculation with the usual methods.