A new series for the density of abundant numbers

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Abstract

A natural number $n$ is called abundant if the sum of proper divisors of $n$ exceeds $n$. In 1933 Behrend determined bounds for the natural density $d\mathcal{A}$ of abundant numbers, with subsequent improvements leading to the 1998 bounds of Deléglise, $0.2474 < d\mathcal{A} < 0.2480$. We present a new infinite series expression for the density of abundant numbers which can be used to calculate an improved lower bound.

1 Introduction

A natural number $n$ is called deficient, perfect, or abundant depending on whether the sum of its proper divisors $s(n)$ is less than, equal to, or greater than $n$, respectively. The distribution of perfect numbers has interested mathematicians since antiquity. More recently, in 1929 Bessel-Hagen asked in [1] whether the set of abundant numbers has a natural density. Letting $S$ denote a set of natural numbers and $S(x)$ the number of members in $S$ not exceeding $x$, we define the natural density $dS$ to be

$$\lim_{x \to \infty} \frac{S(x)}{x},$$

if such exists. Denoting the set of abundant, perfect, and deficient numbers by $\mathcal{A}$, $\mathcal{P}$, and $\mathcal{D}$, respectively, Davenport [5] proved the existence of the densities of each of these sets, and further that $d\mathcal{P} = 0$.

By finding an upper bound for $\limsup \mathcal{A}(x)/x$, Behrend [2, 3] had already shown in 1932 that $d\mathcal{A} < 0.47$, and in the following year that

$$0.241 < d\mathcal{A} < 0.314.$$
Several improvements have been made to these bounds (see for example, [10, 11]). Finally, in 1998, by generalizing Behrend’s upper bound method and implementing it as a computer program, Deléglise [6] found that

$$0.2474 < d_{\mathcal{A}} < 0.2480.$$ 

Here we will consider what can be thought of as a natural generalization of Behrend’s lower bound method.

## 2 Behrend’s Lower Bound Method

Let $\sigma(n)$ denote as usual the sum of all positive divisors of $n$, so that $\sigma(n) = s(n) + n$. We define

$$h(n) = \frac{\sigma(n)}{n},$$

which has the useful interpretation

$$h(n) = \sum_{d|n} \frac{1}{d}.$$ 

Using this function, we can easily prove the following lemma:

**Lemma 1.** For any multiple $mn$ of $n$, $m,n \in \mathbb{N}$, we have $h(mn) \geq h(n)$. In particular, if $n$ is a nondeficient number, then $mn$ is also nondeficient.

**Proof.** Observe that the set of divisors of $mn$ contains the set of divisors of $n$, so that $h(mn) \geq h(n)$. For the second assertion, we use that $n$ is nondeficient if and only if $h(n) \geq 2$. □

This lemma implies that the set of nondeficient numbers can be expressed in terms of multiples of some subset. The members of the minimal such subset are called **primitive nondeficient**, or pnd for short. Namely, a primitive nondeficient number is a number $n$ such that $h(n) \geq 2$, while for each proper divisor $d$ of $n$, $h(d) < 2$. We analogously define for any real number $\alpha \geq 1$ a **primitive $\alpha$-nondeficient number** ($\alpha$-pnd) to be a number $n$ such that $h(n) \geq \alpha$ while for each proper divisor $d$, $h(d) < \alpha$. The set of pnd’s and $\alpha$-pnd’s will be denoted $\mathbb{P}$ and $\mathbb{P}_\alpha$, respectively.
Remark 1. In the literature primitive non-deficient numbers are often called primitive abundant numbers, and are abbreviated $pan$. This definition becomes the natural one provided an abundant number $n$ is redefined so that $h(n) \geq 2$, as is done in many recent works. However, in this paper we will keep the historically established sense of the word abundant.

In what follows, we will find the following definition and notation useful.

**Definition.** Let $S$ and $T$ be subsets of the natural numbers. We define the set

$$ \mathcal{M}(S) := \{ sn : s \in S, n \in \mathbb{N} \}, $$

called the set of multiples of $S$ and say that $S$ generates $T$ if $T = \mathcal{M}(S)$. We call $S$ a primitive generator of $T$ if $S$ generates $T$ and for each $s, t \in S$ such that $s \neq t$, $s \nmid t$.

With this notation, we observe that since $d\mathcal{P} = 0$, we may express the density of abundant numbers as

$$ d\mathcal{A} = \mathcal{M}(\mathcal{P}). $$

We will find it convenient in working with densities and sets of multiples to make use of the following observations: If $S$ is the singleton set $\{s\}$, then $d\mathcal{M}(S) = 1/s$, and in general if $M$ is the set of numbers congruent modulo $n$, $d\mathcal{M} = 1/n$. Densities are finitely additive, namely if $S_i, 1 = 1, \ldots, n$ are pairwise disjoint and $dS_i$ exists, then $d \cup_i S_i = \sum_i dS_i$. It is known that for $S$ finite, say $\{s_i\}_{i=1}^k$,

$$ d\mathcal{M}(S) = \sum_{i=1}^k (-1)^{i-1} \sum_{1 \leq k_1 < k_2 < \cdots < k_i \leq k} \frac{1}{[s_{k_1}, s_{k_2}, \ldots, s_{k_i}]} $$

by the inclusion-exclusion principle. (Here $[s_1, s_2, \ldots, s_i]$ denotes the least common multiple of $s_1, s_2, \ldots, s_i$ for $i > 1$, and $[s] = s$.) Thus, in principle, we could calculate densities of multiple sets using this sum. However, this quickly becomes computationally infeasible due to the number of terms increasing roughly as $2^k$.

Behrend used 22 pnd’s to determine the density of their multiples, but evidently not by directly using the inclusion-exclusion expression, which would involve on the order of $2^{22}$ or about 4 million calculations. Rather, by carefully keeping track of the multiples handled thus far, Behrend calculates which multiples of the next pnd contribute to the density. This method can be generalized in the following way.
3 Inclusion-Exclusion Consolidation

Definition. For a sequence \((s_k)_{k=1}^n\) we write \(\mathcal{M}_k(s_1, s_2, \ldots, s_n)\) for the set of multiples of \(s_k\) that are not multiples of any \(s_j, j < k\). Thus,

\[ \mathcal{M}_k(s_1, s_2, \ldots, s_n) := s_k \mathbb{N} \setminus \bigcup_{j<k} s_j \mathbb{N}. \]

We note that this allows us to partition \(\mathcal{M}(\{s_1, s_2, \ldots, s_n\})\) into the disjoint union of subsets \(\mathcal{M}_k(s_1, s_2, \ldots, s_n), k \leq n\).

We also define an auxiliary set \(C'_k\) associated with each number \(s_k\) which keeps track of the contribution made by \(s_k\) to the density.

Definition. Let \(C_k := \{c_{j,k}\}_{j=1}^{k-1}\) where \(c_{j,k} := c_j / (c_j, c_k)\). We call \(c_{j,k}\) the cofactor of \(s_k\) for \(s_j\), and denote the sequence of cofactors of \(s_k\) by \(C_k\). We define

\[ C'_k := \{c \in C_k : c' \in C_k, c' \neq c \implies c' \nmid c\}. \]

The set \(C'_k\) will be called the reduced cofactor set for \(s_k\).

Note that whereas \(C_k\) is a sequence, \(C'_k\) is defined as a set since we will not be concerned with the order of its members nor with any multiplicity which may occur in the terms of the sequence \(C_k\).

Theorem 1. Let \((s_j)_{j=1}^k\) be a sequence of natural numbers and let \((C'_j)_{j=1}^k\) be the corresponding sequence of reduced cofactor sets. Suppose in addition that for each \(j\) the elements of \(C'_j\) are pairwise coprime. Then

\[ d(\mathcal{M}_j(s_1, \ldots, s_j)) = \frac{1}{s_j} \prod_{c \in C'_j} \left(1 - \frac{1}{c}\right) \] (1)

and

\[ d(\mathcal{M}(\{s_j\}_{j=1}^k)) = \sum_{j=1}^k \frac{1}{s_j} \prod_{c \in C'_j} \left(1 - \frac{1}{c}\right). \] (2)

Proof. It suffices to show that

\[ d\left(s_k \mathbb{N} \setminus \bigcup_{j=1}^{k-1} s_j \mathbb{N}\right) = \frac{1}{s_k} \prod_{c \in C'_k} \left(1 - \frac{1}{c}\right). \] (3)
We use induction on \( k \). When \( k = 1 \) we are done, so assume (3) is true for \( k = r - 1 \). Calling \( S' \) the set of multiples of \( s_r \) not divisible by \( s_j \) for \( j \leq r - 2 \), we have

\[
d S' = d \left( s_r N \setminus \bigcup_{j=1}^{r-2} s_j N \right) = \frac{1}{s_r} \prod_{c \in C''_r} \left( 1 - \frac{1}{c} \right),
\]

where \( C''_r \) is the reduced cofactor set for \( s_r \) with the sequence \((s_1, \ldots, s_{r-2})\).

The set \( T' \) of multiples of \( s_{r-1} \) in \( S' \) is

\[
T' := [s_{r-1}, s_r] N \setminus \bigcup_{j=1}^{r-2} s_j N,
\]

having density

\[
d T' = \frac{1}{[s_r, s_{r-1}]} \prod_{c \in C''_r} \left( 1 - \frac{1}{c} \right) = \frac{1}{s_r s_{r-1}} \prod_{c \in C''_r} \left( 1 - \frac{1}{c} \right),
\]

where \( C''_r \) is the reduced cofactor set for \([s_{r-1}, s_r]\) with the sequence \((s_1, \ldots, s_{r-2})\).

We need to show that \( d T' = 0 \) if and only if there is some \( i \) such that \( c_{i,r} \mid c_{r-1,r} \). Thus, we write

\[
c_{i,r} \mid c_{r-1,r} \iff [s_i, s_r] = c_{i,r} s_r \mid c_{r-1,r} s_r = [s_{r-1}, s_r]
\iff s_i \mid [s_{r-1}, s_r]
\iff (s_i, [s_{r-1}, s_r]) = s_i
\iff (s_i, [s_{r-1}, s_r]) = 1,
\]

so 1 appears as an element of \( C''_r \) and \( d T = 0 \). On the other hand, if 1 does not appear in \( C''_r \), then \( c_{i,r} \nmid c_{r-1,r} \) for any \( i < r - 1 \), so \( c_{r-1,r} \in C''_r \).

Finally, we must show that in the latter case \( C'_r = C''_r \). But we have already shown in the previous proof that the cofactor sets are the same before reduction. Thus, they must be the same after reduction as well. We conclude that the difference \( d S' - d T' \) is

\[
\frac{1}{s_r} \prod_{c \in C''_r} \left( 1 - \frac{1}{c} \right) - \frac{1}{a_c c_{r-1,r}} \prod_{c \in C''_r} \left( 1 - \frac{1}{c} \right) = \frac{1}{s_r} \prod_{c \in C'_r} \left( 1 - \frac{1}{c} \right),
\]

as asserted. \( \square \)
4 A careful ordering of pnd’s

We seek to identify a sequence of pnd’s that satisfies the conditions of Theorem 1. That an arbitrary sequence will not work can be seen in the case of the following sequence of pnd’s:

\[ 2^2 \cdot 5, \ 2^2 \cdot 7, \ 2 \cdot 3 \]

This gives the numbers 10 and 14 as cofactors for 6. However, 10 and 14 are not relatively prime and neither is divisible by the other.

We define an ordering that depends on the contribution that a prime factor of a number \( n \) makes to its abundance \( h(n) \), in the following sense. Let \( h(n) = \sigma(n)/n \). If \( p \) is a prime dividing \( n \), and \( e \) is the multiplicity of \( p \) so that \((n/p^e, p) = 1\), we have

\[
h(n) = h\left( \frac{n}{p^e} \right) \left( 1 - \frac{1}{\sigma(p^e) - 1} \right).
\]

Thus the effect on \( h \)-value that removing \( p \) from \( n \) has depends on \( \sigma(p^e) \), with larger values \( \sigma(p^e) \) having a smaller effect.

We seek to order prime powers according to this effect. However, it may be the case that more than one prime power have the same sigma value. For instance, \( 2^4 \) and \( 5^2 \) both have a \( \sigma \)-value of 31. In such a case we will want to distinguish the two prime powers. We thus make the following definition:

**Definition.** Suppose there are \( k \) prime powers \( p_i^{e_i} \) with equal \( \sigma \)-values and with \( p_1 < \cdots < p_k \). We define the significance of the prime power \( p_i^{e_i} \), \( \text{sig}(p_i^{e_i}) \), to be

\[
\text{sig}(p_i^{e_i}) = \frac{1}{\sigma(p_i^{e_i}) + \frac{1}{k}}.
\]

Thus, for two prime powers \( p^e \) and \( q^f \), if \( \sigma(p^e) < \sigma(q^f) \) then \( \text{sig}(p^e) > \text{sig}(q^f) \), and in the event that \( \sigma(p^e) = \sigma(q^f) \) and \( p < q \), we have \( \text{sig}(p^e) > \text{sig}(q^f) \).

We extend the definition of significance to all natural numbers as follows. For \( n > 1 \), we take \( \text{sig}(n) = \min\{\text{sig}(p^e) : p^e \mid n\} \). Finally, if \( n = 1 \), we take \( \text{sig}(1) = 1 \).

The sequence \( \Sigma = (p_i^{e_i})_{i=1}^{\infty} \) of prime powers ordered by decreasing significance thus begins

\[ 2, 3, 5, 2^2, 7, 11, 3^2, 13, 2^3, 17, 19, 23, 29, 2^4, 5^2, 31, 37, 3^3, 41, \ldots \]
Remark 2. Note that the ordering of prime powers by significance differs from the natural ordering in that prime powers $p^e$ for $e > 1$ show up later than they would otherwise. Note also with the notation for prime powers ordered by significance, $\Sigma = (p_i^e)_{i=1}^\infty$, $p_i$ may be equal to $p_j$ for $i \neq j$. In fact, each prime $p$ is equal to $p_i^e$ for infinitely many $i$’s.

We will now construct a sequence of pnd’s that satisfies the conditions of Theorem 1. For each term $p_i^e$ in $\Sigma$, we consider the set of pnd’s which have $p_i^e$ as the least significant prime power factor, which we will call the $p_i^e$-block. The sequence of blocks found in this way contains all pnd’s since any pnd has a unique least significant prime power factor, and such a pnd will be found in the corresponding block. We will then say that $\mathbb{P}$ is ordered by significance. If we wish, we could further order the members of each block, say using lexicographic ordering by significance, but in fact we will not be concerned with how elements are ordered within each block.

To state the next theorem, we introduce the notation

$$L_k := \text{lcm}\{p_i^e : i \leq k\}.$$  \hfill (4)

**Theorem 2.** The ordering of pnd’s by significance satisfies the conditions of Theorem 1. Moreover, the reduced cofactor set for each pnd consists only of primes, and is given explicitly for a pnd $a$ in block $p_k^e$ as

$$C' = \{p : p \mid L_k/a\}.$$

**Proof.** It suffices to show that the primes $p$ in the cofactor set $C$ for $a$ in block $p_k^e$ are exactly those satisfying $p \mid L_k/a$, and that each composite $c \in C$ is divisible by some prime $p \in C$.

First suppose $p \mid L_k/a$. Then $p \neq p_k$ and $ap/p_k$ is abundant, since if $p^e \parallel a$, then $p^{e+1}$ has greater significance than $p_k^e$. Thus, $ap/p_k$ has a pnd divisor $a'$ appearing before $a$, and

$$\frac{a'}{(a',a)} = p.$$

Thus, all the primes claimed are in $C'$.

Conversely, suppose $a'$ appears before $a$ in the sequence and $p \mid a'/(a',a)$. Then for some $e > 0$, $p^e \parallel a'$ and $p^e \nmid a$. But $p^e \mid L_k$. Thus, $p \mid L_k/a$. 

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So, we have shown that each prime dividing $L_k/a$ is in $C$ and that each prime factor of each $c \in C$ divides $L_k/a$. Thus, $C'$ is the set of primes dividing $L_k/a$.

By this theorem, we now have a compact way of writing the product in (2):

$$\prod_{c \in C'} \left( 1 - \frac{1}{c} \right) = \frac{\varphi(c_i)}{c_i}, \quad c_i = \prod_{c \in C'} c.$$  \hfill (5)

Then we can extract the relevant information for each pnd $a_i$ in the single number $c_i$. We define the sequence

$$C = (c_i)_{i=1}^\infty$$

to be the cofactor sequence for the sequence of pnd’s $P$ ordered by significance, and in general let $C_\alpha$ be the cofactor sequence for the sequence of pnd’s ordered by significance. Thus, we have the following corollary.

**Corollary 1.** Let $\Sigma_1$ be any subsequence of $\Sigma$ ordered by significance such that if $p^e$ is a term of $\Sigma_1$, then so is $p^f$ for $1 \leq f < e$. Let $P_1$ be the sequence of pnd’s formed using $\Sigma_1$ and the procedure described in this section, with $P_1 = (a_i)_{i=1}^r$ ordered by significance. Let $(c_i)_{i=1}^r$ be the cofactor sequence for $P_1$. Then the density of the set of multiples of $P_1$ is given by

$$d.\mathcal{M}(P_1) = \sum_{i=1}^r \frac{\phi(c_i)}{c_i} \cdot \frac{1}{a_i} = \sum_{i=1}^r \frac{\phi(L_k/a_i)}{L_k},$$

where $k$ is the index of the block in which $a_i$ belongs.

This sum allows us to calculate a lower bound for the density of abundant numbers and generalizes Behrend’s calculation for a large class of subsets of pnd’s. We will call this the pnd method. One notable feature of the pnd method is the reduction of the number of operations required to calculate successive terms of the series. As was remarked above, the terms involved in the inclusion-exclusion calculation increases exponentially as the number of pnd’s. By Theorem 1, to calculate the contribution of an $n$th pnd to the density we expect to compare it to the $n - 1$ previous pnd’s, which amounts to a total of $(n - 1)n/2$ operations, or quadratic growth with the number
of pnd’s. However, our final result is even better. The contribution of an \( n \)th pnd need not be compared against any of its predecessors, so the growth is linear with \( n \).

Using the pnd method, we may implement a computer program as follows. Using a sieve, we calculate the prime factorizations of each number. This allows us to identify the pnd’s as well as calculate their cofactors. In fact, since the calculations involving each pnd are independent of each other, we can easily distribute the calculations over multiple computers. A search for pnd’s up to 11 million, which took 13 seconds on a DELL Latitude E6410 laptop, resulted in 9345 pnd’s which were sufficient to match Deléglise’s 1998 result. A more lengthy search on a supercomputer over the course of several weeks found that the pnd’s up to \( 4 \times 10^{10} \) yield the following lower bound for abundant numbers:

**Theorem 3.**

\[
d \mathcal{A} > 0.24760444.
\]

We conclude this section with a table of the first few pnd’s ordered by significance, the block they belong to, their cofactor sequence, their reduced cofactor, and the corresponding reduced cofactor set.

<table>
<thead>
<tr>
<th>( a )</th>
<th>block</th>
<th>cofactor sequence</th>
<th>( L_k/a )</th>
<th>reduced cofactor set</th>
</tr>
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<td>(3)</td>
<td>1</td>
<td>{3}</td>
</tr>
<tr>
<td>( 5 \cdot 2^2 )</td>
<td>( 2^2 )</td>
<td>( 3, 2 )</td>
<td>3 \cdot 2</td>
<td>{3, 2}</td>
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<td>( 3, 5, 5 )</td>
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</tr>
<tr>
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<td>7</td>
<td>( 2^2 \cdot 2^2 )</td>
<td>( 2^2 )</td>
<td>{2}</td>
</tr>
<tr>
<td>( 5 \cdot 7 \cdot 11 \cdot 3^2 )</td>
<td>( 3^2 )</td>
<td>( 2, 2^2 \cdot 2^2 )</td>
<td>( 2^2 )</td>
<td>{2}</td>
</tr>
</tbody>
</table>

## 5 Asymptotics of the pnd method

Let us denote by \( \mathbb{P}[y] \) the set of pnd’s in \( \mathbb{P} \) consisting of pnd’s from \( p^x \)-blocks with \( \sigma(p^x) \leq y \). By the pnd method we can calculate the density \( d \mathcal{M}(\mathbb{P}[y]) \). We now determine a bound on the error \( d \mathcal{A} - d \mathcal{M}(\mathbb{P}[y]) \). A simple bound can be found by taking the reciprocal sum of the elements of the set \( \mathbb{P} \setminus \mathbb{P}[y] \). We first show that this set is contained in \( \mathbb{P} \setminus \mathbb{P}(y) \), so that

\[
\sum_{\substack{a \in \mathbb{P} \setminus \mathbb{P}(y) \atop a > y}} \frac{1}{a} = \sum_{\substack{a \in \mathbb{P} \setminus \mathbb{P}[y] \atop a > y}} \frac{1}{a} \quad (6)
\]
is an upper bound. The containment can be seen by the chain of implications

\[ \sigma(p^e) > y \implies p^e > y/2 \implies 2p^e > y \implies a > y, \]

where the first implication is from the observation that for any prime power \( p^e \) we have \( h(p^e) < 2 \), and the final implication uses that \( p^e \) is a proper divisor of \( a \).

We now show that as \( y \to \infty \), the error bound (6) goes to zero.

**Theorem 4.** The error \( \text{d}\mathcal{A} - \text{d}\mathcal{M}(\mathbb{P}[y]) \) behaves as

\[ \text{d}\mathcal{A} - \text{d}\mathcal{M}(\mathbb{P}[y]) = O\left( \left( \frac{\log y}{\log \log y} \right)^{1/2} \exp\left( -\frac{1}{25} (\log y \log \log y)^{1/2} \right) \right) \]

for sufficiently large \( y \).

**Proof.** By partial summation we can write

\[ \sum_{a \in \mathbb{P}, a > y} \frac{1}{a} = \left\lfloor \frac{\mathbb{P}(n)}{n} \right\rfloor + \int_y^\infty \frac{\mathbb{P}(n)}{n^2} dn. \]

From [8], we have the upper bound

\[ |\mathbb{P}(n)| \leq n \exp\left( -\frac{1}{25} \sqrt{\log n \log \log n} \right) \]

for \( n \) larger than some \( n_0(\epsilon) \). Since

\[ \sum_{a \in \mathbb{P}, a > y} \frac{1}{a} \leq \int_y^\infty \frac{\mathbb{P}(t)}{t^2} dt \]

\[ \leq \int_y^\infty \frac{1}{te^{25/\sqrt{\log t \log \log t}}} dt \]

\[ = O\left( \left( \frac{\log y}{\log \log y} \right)^{1/2} \exp\left( -\frac{1}{25} (\log y \log \log y)^{1/2} \right) \right) \]

for sufficiently large \( y \).

Since the error vanishes as \( y \to \infty \), we have the following corollary.

**Corollary 2.** The density of abundant numbers can be expressed as the infinite sum

\[ \text{d}\mathcal{A} = \sum_{a_i \in \mathbb{P}} \frac{\varphi(c_i)}{c_i} \frac{1}{a_i}, \]

where the \( a_i \) are pnd’s and \( c_i = L_k/a_i \), as defined in (5), and where \( L_k \) is as defined in (4).
6 A generalization of pnd’s

The method used by Deléglise to determine \( d \alpha \) generalizes to \( \alpha \)-nondeficient numbers for any \( \alpha \). We may ask whether the pnd method may also be generalized to \( \alpha \)-pnd’s. However, to prove Theorem 4 we have used the result in \([8]\) which makes special use of the value \( \alpha = 2 \). In fact, Erdős in \([9]\) proves the existence of values of \( \alpha \) for which the sum of reciprocals of pnd’s does not converge. Such \( \alpha \) belong in the set of Liouville numbers.

A number \( \delta \) is called a \textit{Liouville number} if for each \( k \) there exists a rational number \( \frac{a_k}{b_k} \), with \( a_k, b_k \in \mathbb{Z} \), such that

\[
0 < \left| \delta - \frac{a_k}{b_k} \right| < \frac{1}{b_k^k}.
\]

This condition implies that \( \delta \) is irrational, and moreover that it is transcendental.

Let \( N_{\alpha}(n) \) denote the number of primitive \( \alpha \)-abundant numbers in \([1, n]\). Using the methods of Erdős \([8]\) it can be shown that if \( \alpha \) is not a Liouville number,

\[
N_{\alpha}(n) < \frac{n}{c_{\alpha}(\log n \log \log n)^{1/2}}
\]

for some positive constant \( c_{\alpha} \). This implies that for such \( \alpha \), the validity of the infinite series expression extends to \( \alpha \)-pnd’s as well.

Now suppose \( \alpha \) is Liouville. In this case by \([9]\) we are only guaranteed

\[
N_{\alpha}(n) = o\left(\frac{x}{\log x}\right),
\]

so the reciprocal sum of \( \alpha \)-pnd’s may not converge and the earlier argument fails. Nevertheless, we can show that the series expression for the density still applies. We will refer to the following lemma of Erdős \([7]\).

**Lemma 2.** The number of integers \( n \leq x \) that do not satisfy all of the following three conditions:

1. if \( p^e \mid n \) and \( e > 1 \), then \( p^e < (\log x)^{10} \),
2. the number of different prime factors of \( n \) is less than \( 10 \log \log x \),
3. the greatest prime factor of \( n \) is greater than \( x^{1/(20 \log \log x)} \),

is \( o(x/\log x)^2 \).
For convenience, we let
\[ F(x) = \prod_{p \leq x} \left( 1 - \frac{1}{p} \right), \]
and observe that by Mertens’ Theorem, we have
\[ F(x) \sim \frac{e^{-\gamma}}{\log x} \]
as \( x \to \infty \), where \( \gamma \) is the Euler-Mascheroni constant.

**Theorem 5.** Let \( \alpha \geq 1 \). Then the density of \( \alpha \)-abundant numbers is
\[ d \mathcal{A}_\alpha = \sum_{a_i \in \mathcal{P}_\alpha} \frac{\varphi(c_i)}{c_i} \frac{1}{a_i}, \]
where \( c_i \) is defined as above.

**Proof.** We partition the set of \( \alpha \)-pnd’s \( a_i \leq x \) into two classes. In the first class we have those not satisfying all three conditions listed in Lemma 2. Since the number of these is \( o(x/(\log x)^2) \), the reciprocal sum of these \( \alpha \)-pnd’s converges.

For the second class consisting of those \( \alpha \)-pnd’s that do satisfy the conditions listed in Lemma 2, we argue as follows. First we note that
\[ \frac{\varphi(L_k/a_i)}{L_k/a_i} \cdot \frac{1}{a_i} \leq \frac{\varphi(L_k)}{L_k} \cdot \frac{1}{\varphi(a_i)}. \]
Next we estimate \( \varphi(L_k)/L_k \) and \( \varphi(a_i) \). By condition 3, we have that \( a_i \), and thus also \( L_k \), contains primes greater than \( x^{1/(20 \log \log x)} \). Then by the definitions of \( L_k \) and \( F(x) \),
\[ \frac{\varphi(L_k)}{L_k} \leq F(x^{1/(20 \log \log x)}) = O \left( \frac{\log \log x}{\log x} \right), \]
By condition 2 we can bound \( \varphi(n)/n \) by
\[ \frac{\varphi(n)}{n} \geq F(p_{\omega(n)}) \geq F(10 \log \log n \log \log \log n) \sim \frac{e^{-\gamma}}{\log \log \log n}, \]
where here \( p_i \) denotes the \( i \)th prime.

Thus, for large \( x \), our \( a_i \) satisfy
\[ \frac{1}{\varphi(a_i)} \leq \frac{e^\gamma \log \log x}{a_i}. \]
Putting our estimates together, we find that
\[
\frac{\varphi(L_k/a_i)}{L_k/a_i} \cdot \frac{1}{a_i} \leq \frac{f(x)}{a_i}
\]
where
\[
f(x) = O\left(\frac{\log \log x \log \log \log x}{\log x}\right).
\]
Thus, the sum over \(a_i\) that are \(\alpha\)-pnd’s satisfying our conditions is
\[
\sum_{a_i \leq x} \frac{\varphi(L_k/a_i)}{L_k/a_i} \cdot \frac{1}{a_i} = O\left(\sum_{a_i \leq x} \frac{\log \log a_i \log \log \log a_i}{a_i \log a_i}\right).
\]
Since the number of \(\alpha\)-pnd’s up to \(x\) is bounded by
\[
o\left(\frac{x}{\log x}\right),
\]
the sum converges by partial summation. Thus, we have shown convergence of the sum over \(\alpha\)-pnd’s in the second class. We conclude that the density expression holds. \(\square\)

References


