Evaluation of the Mass Moment of Inertia Function

Before we actually “lift the hood” to examine this function’s “engine,” one more inspection of its “tires” and “exterior finish” will prove useful. After first collecting what is already known of this function

\[
\mathbb{I}_O(u) = \sum_{p \in B} m_p \mathbf{r}_{p/O} \times (\mathbf{u} \times \mathbf{r}_{p/O}) + \sum_{p \in B} m_p \mathbf{r}_{p/O}^2 \mathbf{u} - (\mathbf{r}_{p/O} \cdot \mathbf{u}) \mathbf{r}_{p/O}
\]  

(Y.1)

\[
\mathbb{I}_O(a_1, \mathbf{u}_1 + a_2 \mathbf{u}_2) = a_1 \mathbb{I}_O(\mathbf{u}_1) + a_2 \mathbb{I}_O(\mathbf{u}_2)
\]  

(Y.2)

it is easily established that

\[
\mathbb{I}_O(\mathbf{u}_1) = \sum_{p \in B} m_p \mathbf{r}_{p/O}^2 \mathbf{u}_1 - (\mathbf{r}_{p/O} \cdot \mathbf{u}_1) \mathbf{r}_{p/O}
\]  

(Y.3)

and

\[
\mathbf{u}_1 \cdot \mathbb{I}_O(\mathbf{u}_2) = \mathbf{u}_2 \cdot \mathbb{I}_O(\mathbf{u}_1)
\]  

(Y.4)

for any vector pair \((\mathbf{u}_1, \mathbf{u}_2)\). From this, the general symmetry property of the inertia function, expressed in the form

\[
\mathbf{u}_1 \cdot \mathbb{I}_O(\mathbf{u}_2) = \mathbf{u}_2 \cdot \mathbb{I}_O(\mathbf{u}_1)
\]  

(Y.5)

for any vector pair \((\mathbf{u}_1, \mathbf{u}_2)\), is apparent. In addition, it then follows that

\[
\mathbf{a} \cdot \mathbb{I}_O(\mathbf{a}) = \sum_{p \in B} m_p \mathbf{r}_{p/O}^2 (\mathbf{a} \cdot \mathbf{a}) - (\mathbf{r}_{p/O} \cdot \mathbf{a}) (\mathbf{r}_{p/O} \cdot \mathbf{a})
\]

\[
\mathbf{a} \cdot \mathbb{I}_O(\mathbf{a}) = \sum_{p \in B} m_p [\mathbf{r}_{p/O}^2 (\mathbf{a} \cdot \mathbf{a}) - (\mathbf{r}_{p/O} \cdot \mathbf{a}) (\mathbf{r}_{p/O} \cdot \mathbf{a})]
\]  

(Y.6)

for the choice of \(\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{a}\) for any unit vector \(\mathbf{a}\). With reference to the diagram below,
(Y.6) can be reinterpreted and recast in the more compact form

\[ I_O = \mathbf{a} \cdot \left[ \sum_{P} m_P d_P^2 \right], \]  

expressed in terms of the perpendicular distance \( d_P \) measured from the mass particle \( P \) to the \( \mathbf{a} \)-axis passing through the moment point \( O \). Many readers will recognize this as the familiar mass moment of inertia of the body with respect to the \( \mathbf{a} \)-axis. This correct observation serves to justify the naming of this function as it can evidently be used, through (Y.7), to determine the rotational mass (mass moment of inertia) of the rigid body \( B \) with respect to any axis (parallel to \( \mathbf{a} \)) passing through the moment point \( O \). This is one important application of this function.

In view of this result, it then follows from (Y.3)_2 and (Y.7) that

\[ \mathbf{a} \cdot \left[ \sum_{P} m_P d_P^2 \right] = M \left[ r_{G/O} \cdot \mathbf{a} \right] \]

\[ \mathbf{a} \cdot \left[ \sum_{P} m_P d_P^2 \right] - \mathbf{a} \cdot \left[ \sum_{P} m_P d_P^2 \right] = M \left[ r_{G/O} \cdot \mathbf{a} \right] \]

\[ I_O - I_G = M \left[ r_{G/O} \cdot \mathbf{a} \right]^2. \] 

By the same argument presented in the graphic supporting (Y.7) above, this is seen to confirm the (perhaps familiar) Parallel Axis Theorem

\[ I_O = I_G + M d^2 \] 

relating the mass moments of inertia with respect to parallel axes passing through the bodies own mass center \( G \) and an “offset” point \( O \), in terms of the bodies mass \( M \) and the perpendicular separation distance \( d \) between the parallel axes. This justifies the earlier remark following (X.17).

Apart from a number of such important general observations, evaluation of this function most often requires the introduction of a set of basis vectors with which to resolve all vectors into component form. To pursue this, suppose that

\[ \mathbf{w} = \mathbf{I}_O (\mathbf{u}) \] 

where the argument and image vectors \( \mathbf{u} \) & \( \mathbf{w} \) have the unique component representations
\[ \begin{align*}
\bar{u} &= u_1 \hat{e}_1 + u_2 \hat{e}_2 + u_3 \hat{e}_3 ; \\
\bar{w} &= w_1 \hat{e}_1 + w_2 \hat{e}_2 + w_3 \hat{e}_3 ;
\end{align*} \]
\[w_k = \bar{w} \cdot \hat{e}_k \quad \text{k=1,2,3} \quad \text{(Y.11)}\]

relative to a given set of orthonormal basis vectors \( \{ \hat{e}_k \}_{k=1}^3 \). In view of the linearity property (Y.2), (Y.10, 11) lead to the equivalent component relationship

\[ \bar{w} = \bar{I}_O(\bar{u}) \]
\[w_k = u_1 \cdot \bar{I}_O(\hat{e}_1) + u_2 \cdot \bar{I}_O(\hat{e}_2) + u_3 \cdot \bar{I}_O(\hat{e}_3) \]
\[w_k = \bar{I}_O(e_k) \cdot u_1 + \bar{I}_O(e_k) \cdot u_2 + \bar{I}_O(e_k) \cdot u_3 \quad \text{k=1,2,3} \quad \text{(Y.12)}\]

Thus, after selection of an orthonormal basis \( \{ \hat{e}_k \}_{k=1}^3 \), the argument and image vectors are represented by their respective (component) column vectors, the inertia function is represented by a 3x3 matrix of scalar array elements known as its representation matrix, and the action of the function is realized by matrix multiplication. Consequently, evaluation of the inertia function depends entirely on our ability to determine the nine (9) scalar coefficients which make up its representation matrix

\[ \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad \text{with} \quad I_{ij} = \hat{e}_i \cdot \bar{I}_O(\hat{e}_j) ; \quad i, j = 1, 2, 3 \quad \text{(Y.13)}\]

relative to the specified set of basis vectors.

In view of the general symmetry property (Y.5), the problem of determining these coefficients is simplified by the observation that only six (6) of the nine (9) are independent with
Thus, the ordering of the indices used to identify the various matrix coefficients is immaterial, rendering the desired representation matrix *symmetric*. In light of (Y.7) we may now focus our attention on the *diagonal* elements of this representation matrix, and observe that

\[ I_{kk} = \hat{e}_k \cdot \left[ I_{\text{O}}(\hat{e}_k) \right] = I_{\text{O}k} ; \quad k=1,2,3. \]  

(Y.15)

Consequently, the diagonal elements of this representation matrix are just the bodies own *mass moments of inertia* with respect to the axes (through O) which are parallel to the respective basis directions.

Precise formulae for determining these matrix coefficients are easily developed by introducing a Cartesian coordinate system having its origin at the moment point O, and its axes parallel to the selected basic directions.

With reference to (Y.4) & (Y.16), the matrix coefficient expressions (Y.13) take the expanded form:

\[ I_{ij} = \hat{e}_i \cdot \left[ I_{\text{O}}(\hat{e}_j) \right] = \sum_{p \in B} m_p \left[ (x_p^2 + y_p^2 + z_p^2) \hat{e}_i \cdot \hat{e}_j \right] - \left( \sum_{p \in B} m_p \right) \left[ x_p \hat{e}_1 + y_p \hat{e}_2 + z_p \hat{e}_3 \right] \cdot \hat{e}_i \]  

\[ \cdot \left[ \sum_{p \in B} m_p \left[ x_p \hat{e}_1 + y_p \hat{e}_2 + z_p \hat{e}_3 \right] \hat{e}_j \right] \]  

\[ ; \quad i \& j = 1,2,3. \]  

(Y.17)

Specifically, for \( i=j=1 \), this reduces to

\[ I_{11} = \sum_{p \in B} m_p \left[ (x_p^2 + y_p^2 + z_p^2) \hat{e}_1 \cdot \hat{e}_1 \right] - \left( \sum_{p \in B} m_p \right) \left[ x_p \hat{e}_1 + y_p \hat{e}_2 + z_p \hat{e}_3 \right] \cdot \hat{e}_1 \]  

\[ \cdot \left( \sum_{p \in B} m_p \left[ x_p \hat{e}_1 + y_p \hat{e}_2 + z_p \hat{e}_3 \right] \hat{e}_1 \right) \]  

\[ = \sum_{p \in B} m_p \left[ x_p^2 + y_p^2 + z_p^2 \right] \left( 1 - \left[ x_p \right] \left[ x_p \right] \right) \]  

\[ = \sum_{p \in B} m_p \left( y_p^2 + z_p^2 \right), \]  

(Y.18)

and similarly, for \( i=2 \& j=3 \):
\[
I_{23} = \sum_{p \in B} m_p \left[ \left( x_p^2 + y_p^2 + z_p^2 \right) (\hat{e}_2 \cdot \hat{e}_3) - \left( x_p \hat{e}_1 + y_p \hat{e}_2 + z_p \hat{e}_3 \right) \cdot \left( x_p \hat{e}_1 + y_p \hat{e}_2 + z_p \hat{e}_3 \right) \right] \\
= \sum_{p \in B} m_p \left[ \left( x_p^2 + y_p^2 + z_p^2 \right) (\theta) - \left[ (y_p) \right] \right] \\
I_{23} = I_{32} = -\left[ \sum_{p \in B} m_p (y_p z_p) \right].
\]

(Y.19)

In identical fashion, the complete set of *summation* formulae for the representation matrix coefficients

\[
\begin{align*}
I_{11} &= \sum_{p \in B} m_p \left( y_p^2 + z_p^2 \right) \\
I_{22} &= \sum_{p \in B} m_p \left( z_p^2 + x_p^2 \right) \\
I_{33} &= \sum_{p \in B} m_p \left( x_p^2 + y_p^2 \right) \\
I_{31} &= I_{13} = -\left[ \sum_{p \in B} m_p (z_p x_p) \right] \\
I_{12} &= I_{21} = -\left[ \sum_{p \in B} m_p (x_p y_p) \right]
\end{align*}
\]

(Y.20)

are confirmed. As has already been observed in (Y.15), the diagonal entries are just the *mass moments of inertia* of the body about the respective coordinate axes

\[
I_{11} = I_x \quad I_{22} = I_y \quad I_{33} = I_z
\]

(Y.21)

while the off-diagonal entries are associated with what are *traditionally* known as the bodies *products of inertia* with respect to the three coordinate planes:

\[
\begin{align*}
I_{23} = I_{32} = - (I_{yx}) \\ I_{21} = I_{13} = - (I_{xz}) \\ I_{12} = I_{31} = - (I_{xy}) \\
\end{align*}
\]

(Y.22)

\[
\begin{align*}
I_{yx} &= \sum_{p \in B} m_p (y_p z_p) = I_{xy} \\
I_{xz} &= \sum_{p \in B} m_p (z_p x_p) = I_{xz} \\
I_{xy} &= \sum_{p \in B} m_p (x_p y_p) = I_{yx}
\end{align*}
\]

\[
\left\{ \begin{array}{c}
\text{products of inertia}
\end{array} \right. \]

Thus, with the selection of an orthonormal basis and an associated Cartesian coordinate system originating at the moment point O, the *workings* of the *mass moment of inertia function* as it operates on a given *argument* vector are revealed as being equivalent to the action of a matrix multiplying a column vector. The appropriate *representation matrix* for the inertia function is formed from the *moments* and *products* of inertia of the body with respect to this Cartesian system as expressed in equations (Y.20, 21, 22).