ME 316  
Particle kinematics / Polar Coordinates

Example#1: 

At time $t=0$, the ball player throws the ball with an initial speed of 100 ft/sec at an angle of $30^\circ$ with the horizontal. Determine the quantities $r$, $\dot{r}$, $\ddot{r}$, $\theta$, $\dot{\theta}$, and $\ddot{\theta}$, all relative to the x-y axes shown, at time $t=0.5$ sec.

Solution:

Neglecting aerodynamic resistance, the ball can be assumed to be in a free flight trajectory for $t>0$ until it strikes the ground (or is caught). As you will recall from ME 215, this type of motion is characterized by constant acceleration, in this case  

$$\ddot{a} = \ddot{a}_0 = -g \hat{e}_y; \quad g = 32.2 \text{ ft/s}^2.$$  

Integration of the fundamental relations  

$$\frac{d\ddot{v}}{dt} = \ddot{a}$$  

$$\frac{d\ddot{r}}{dt} = \ddot{v}$$  

quickly leads to the general solution  

$$\ddot{v} = \ddot{v}(t) = \ddot{a}_0 t + \ddot{v}_0$$  

$$\ddot{r} = \ddot{r}(t) = \frac{1}{2} \ddot{a}_0 t^2 + \ddot{v}_0 t + \ddot{r}_0$$  

for this type of motion. For this example, the initial ($t=0$) position and velocity for the ball are given by  

$$\ddot{r}_0 = 6 \hat{e}_y$$  

$$\ddot{v}_0 = 100 \left( \frac{-3.22}{2} \hat{e}_x + \frac{1}{2} \hat{e}_y \right)$$  

which lead to the time solution  

$$\ddot{r}(t) = \frac{1}{2} (-32.2 \hat{e}_x) t^2 + [100 \left( \frac{-3.22}{2} \hat{e}_x + \frac{1}{2} \hat{e}_y \right)] t + (6 \hat{e}_y)$$  

$$= [50 \sqrt{3} t] \hat{e}_x + [6 + 50 t - 16.1 t^2] \hat{e}_y$$  

$$\ddot{v}(t) = (-32.2 \hat{e}_x) t + 100 \left( \frac{-3.22}{2} \hat{e}_x + \frac{1}{2} \hat{e}_y \right)$$  

$$= [50 \sqrt{3}] \hat{e}_x + [50 - 32.2 t] \hat{e}_y$$  

and specifically, at $t=0.5$  

$$\ddot{r} = 43.30 \hat{e}_x + 26.98 \hat{e}_y$$  

$$\ddot{v} = 86.60 \hat{e}_x + 33.90 \hat{e}_y$$  

Now, from the above position vector it is immediately evident that
and therefore
\[ r = \sqrt{x^2 + y^2} = 51.02 \text{ ft} \]
\[ \theta = \arctan \left( \frac{y}{x} \right) = 31.93^\circ \]
To determine the 1st and 2nd rates of change of these polar coordinates at this moment we now make use of the fact that the particle’s velocity and acceleration vectors admit multiple representations (please refer to page two of the General Course Outline)
\[
\dot{v} = 86.60 \hat{e}_x + 33.90 \hat{e}_y = \ddot{r} \hat{e}_r + r \ddot{\theta} \hat{e}_\theta
\]
\[
\ddot{a} = -32.2 \hat{e}_y = (\dddot{r} - r \ddot{\theta}^2) \hat{e}_r + (r \dddot{\theta} + 2 \ddot{r} \dot{\theta}) \hat{e}_\theta
\]
A particularly simple method for extracting the desired information from these vector expressions is to perform well-considered projection operations. By “dotting” both sides of the velocity equation with the radial unit vector we obtain
\[
86.60 (\hat{e}_x \cdot \hat{e}_r) + 33.90 (\hat{e}_y \cdot \hat{e}_r) = \ddot{r} (\hat{e}_r \cdot \hat{e}_r) + r \ddot{\theta} (\hat{e}_\theta \cdot \hat{e}_r)
\]
\[
86.60 \left[ \cos(\theta) \right] + 33.90 \left[ \sin(\theta) \right] = \ddot{r} (1) + r \ddot{\theta} (0)
\]
\[
\dot{r} = 86.60 \cos(\theta) + 33.90 \sin(\theta)
\]
\[
\ddot{r} = 86.60 \cos(31.93^\circ) + 33.90 \sin(31.93^\circ)
\]
\[
\ddot{r} = 91.43 \text{ ft/s}
\]
Observe that use has been made of the dot product relations summarized in the matrix table on page 2 of the General Course Outline. Proceeding in this manner, all other desired terms can be obtained:
\[
86.60 (\hat{e}_x \cdot \hat{e}_\theta) + 33.90 (\hat{e}_y \cdot \hat{e}_\theta) = r \ddot{\theta}
\]
\[
\ddot{r} = -86.60 \sin(\theta) + 33.90 \cos(\theta)
\]
\[
\ddot{\theta} = -86.60 \sin(31.93^\circ) + 33.90 \cos(31.93^\circ)
\]
\[
\ddot{\theta} = 51.02
\]
\[
\ddot{\theta} = -0.334 \text{ r/s}
\]
\[
-32.2 (\hat{e}_y \cdot \hat{e}_r) = \dddot{r} - r \dddot{\theta}
\]
\[
\dddot{r} = r \dddot{\theta} - 32.2 \sin(\theta)
\]
\[
\dddot{r} = 51.02(-0.334)^2 - 32.2 \sin(31.93^\circ)
\]
\[
\dddot{r} = -11.35 \text{ ft/s}^2
\]
\[
-32.2 (\hat{e}_y \cdot \hat{e}_\theta) = r \dddot{\theta} + 2 r \dddot{\theta}
\]
\[
\dddot{r} = -2 r \dddot{\theta} - 32.2 \cos(\theta)
\]
\[
\dddot{\theta} = \frac{-2(91.43)(-0.334) - 32.2 \cos(31.93^\circ)}{51.02}
\]
\[
\dddot{\theta} = 0.661 \text{ r/s}^2
\]
Observe that this technique requires that you dot (project) the equation with the unit vector associated with the particular term that you wish to isolate.