Import Concepts Pertaining to the Difficult Topic of Relative Motion

**Point or Particle** - An identifiable entity which is, at each moment in time, associated with a specific location in space having no discernible extent (size) or orientation.

**Reference Frame** - Any place (rigid platform) from which physical observations (scientific measurements) can be made.

**Notation**: Regular uppercase capital letters (or subscripted capitals) \( A, B, C, \ldots, P_1, P_2, P_3, \ldots \) shall be used to denote (label) **points** or **particles**. Uppercase script capitals (or subscripted script capitals) \( \mathcal{F}, \mathcal{G}, \mathcal{H}, \ldots, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \ldots \) shall be used to denote **reference frames**.

**Comment**: A point is **not** a reference frame and a reference frame is **not** a point. A reference frame \( \mathcal{F} \) is essentially a platform to which observers and/or scientific instruments can be affixed for the purpose of making scientific measurements (observations). As such, a reference frame \( \mathcal{F} \) is comprised of an infinite number of points known as **fixed \( \mathcal{F} \)-frame locations**. A point \( P \) is said to be a fixed \( \mathcal{F} \)-frame location, or more simply a point of reference frame \( \mathcal{F} \) \((P \in \mathcal{F})\), if \( P \) is perceived as being spatially fixed by an (any) \( \mathcal{F} \)-frame observer.

**Notation**:

\[
\frac{d}{dt} = \text{ (time rate of change as perceived (measured) by a (any) ground frame observer)}
\]

\[
\frac{\mathcal{d}}{dt} = \text{ (time rate of change as perceived (measured) by a (any) frame \( \mathcal{F} \) observer)}
\]

**FAQ#1**: Why is it necessary to introduce this type of notation? Isn’t a derivative just a derivative no matter who computes it? Are you suggesting that some observer/scientist/student would compute the derivative of \( t^3 \) correctly as \( 3t^2 \), while another observer in a different frame might compute something else?

**Response**: If our analysis involved only the differentiation of scalar quantities then, as you suggest, this sort of notational distinction would be unnecessary. The derivative of \( t^3 \) would indeed be correctly computed as \( 3t^2 \) by all observers, regardless of their frame of reference. Students cannot use this as an excuse for not doing math homework while flying in a plane.

Differentiation of vector quantities, however, involves additional considerations. Specifically, we have shown that the derivative of an observable vector quantity depends on its rate of change of direction (rotation rate) in addition to its rate of change of scalar magnitude.

As depicted in the accompanying cartoon illustration, different observers in different reference frames will not necessarily agree as to the specification of fixed (time invariant) spacial directions. One observers “fixed” directions may be perceived as rotating (changing) by another. For this
reason, the derivative of an observable geometric vector object is generally dependent on the reference frame from which it is being observed. This fact makes it necessary to introduce the above sort of notation which is intended to identify the specific reference frame from which the time-evolving vector is being observed. As noted in class, however, these absolute and relative time derivatives will be identical for the very special circumstance wherein the reference frame \( F \) moves without any discernible rotation. This special type of reference frame motion is called pure translational motion.

Definitions & Nomenclature:

Particle velocity and acceleration:

\[ \mathbf{v}_p = \{ \text{absolute velocity of particle } P \text{ as perceived (measured) by a (any) ground frame observer} \} \]
\[ \mathbf{v}_p = \frac{d}{dt} \mathbf{r}_p; \quad \mathbf{r}_p = \mathbf{r}_{pO} \text{ for some (any) ground-fixed point (origin) } O. \]
\[ \mathbf{a}_p = \{ \text{absolute acceleration of particle } P \text{ as perceived (measured) by a (any) ground frame observer} \} \]
\[ \mathbf{a}_p = \frac{d}{dt} \mathbf{v}_p \]
\[ \mathbf{v}^F_p = \mathbf{v}^F_{pF} = \{ \text{apparent velocity of particle } P \text{ as perceived (measured) by a (any) frame } F \text{ observer} \} \]
\[ \mathbf{v}^F_p = \frac{d}{dt} \mathbf{r}^F_p; \quad \mathbf{r}^F_p = \mathbf{r}^F_{pO} \text{ for some (any) frame-fixed point (origin) } O' \in F. \]
\[ \mathbf{a}^F_p = \mathbf{a}^F_{pF} = \{ \text{apparent acceleration of particle } P \text{ as perceived (measured) by a (any) frame } F \text{ observer} \} \]
\[ \mathbf{a}^F_p = \frac{d}{dt} \mathbf{v}^F_p \]

Relative particle velocity and acceleration:

\[ \mathbf{v}_{B/A} = \frac{d}{dt} \mathbf{r}_{B/A} = \{ \text{absolute relative velocity of two particles as measured from the ground frame} \} \]
\[ \mathbf{v}_{B/A} = \frac{d}{dt} \mathbf{r}_{B/A} = \frac{d}{dt} (\mathbf{r}_B - \mathbf{r}_A) = \frac{d}{dt} \mathbf{r}_B - \frac{d}{dt} \mathbf{r}_A = \mathbf{v}_B - \mathbf{v}_A \]
\[ \mathbf{a}_{B/A} = \frac{d}{dt} \mathbf{v}_{B/A} = \{ \text{absolute relative acceleration of two particles as measured from the ground frame} \} \]
\[ \mathbf{a}_{B/A} = \frac{d}{dt} \mathbf{v}_{B/A} = \frac{d}{dt} (\mathbf{v}_B - \mathbf{v}_A) = \frac{d}{dt} \mathbf{v}_B - \frac{d}{dt} \mathbf{v}_A = \mathbf{a}_B - \mathbf{a}_A \]
\[ \mathbf{v}^F_{B/A} = \frac{d}{dt} \mathbf{r}^F_{B/A} = \{ \text{apparent relative velocity of two particles as measured from frame } F \} \]
\[ \mathbf{v}^F_{B/A} = \frac{d}{dt} \mathbf{r}^F_{B/A} = \frac{d}{dt} (\mathbf{r}^F_B - \mathbf{r}^F_A) = \frac{d}{dt} \mathbf{r}^F_B - \frac{d}{dt} \mathbf{r}^F_A = \mathbf{v}^F_B - \mathbf{v}^F_A \]
\[ \mathbf{a}^F_{B/A} = \frac{d}{dt} \mathbf{v}^F_{B/A} = \{ \text{apparent relative acceleration of two particles as measured from frame } F \} \]
\[ \mathbf{a}^F_{B/A} = \frac{d}{dt} \mathbf{v}^F_{B/A} = \frac{d}{dt} (\mathbf{v}^F_B - \mathbf{v}^F_A) = \frac{d}{dt} \mathbf{v}^F_B - \frac{d}{dt} \mathbf{v}^F_A = \mathbf{a}^F_B - \mathbf{a}^F_A \]
FAQ#2: If a point A is fixed within a moving reference frame $F \ (A \in F)$, then isn't it true that
\[ \mathbf{v}_P^F = \mathbf{v}_{P/A} = \mathbf{v}_p - \mathbf{v}_A ; \quad A \in F \]
and
\[ \mathbf{a}_P^F = \mathbf{a}_{P/A} = \mathbf{a}_p - \mathbf{a}_A ; \quad A \in F \]
In other words, isn’t it true that the vector quantities $\mathbf{v}_{P/A}$ & $\mathbf{a}_{P/A}$ describe (represent) the (apparent) velocity and acceleration of a particle P as it would appear to (be measured by) a stationary observer located at point A (A∈F) within reference frame $F$?

Response: Generally NOT! This unfortunate fact follows from the above response to FAQ#1. To see this note that the point $A \in F$ is an admissible (and convenient) choice for an $F$-frame origin. With this selection, $O'=A$, it would then follow from the above definitions that the particles apparent velocity is computed as
\[ \mathbf{v}_P^F = \mathbf{v}_{P/F} = \frac{d}{dt} \mathbf{r}_P^F = \frac{d}{dt} \mathbf{r}_{P/A} \]
while the absolute relative velocity of P with respect to A is
\[ \mathbf{v}_{P/A} = \frac{d}{dt} \mathbf{r}_{P/A} = \mathbf{v}_p - \mathbf{v}_A . \]

With the identification of the mutually observable vector $\mathbf{A}$ as the particle displacement
\[ \mathbf{A} = \mathbf{r}_{P/A} \]
and the previous revelation that, generally speaking
\[ \frac{d}{dt} \mathbf{A} \neq \frac{d}{dt} \mathbf{X} , \]
we are forced to conclude that the apparent velocity of P (as viewed from frame $F$) is generally not the same as the absolute relative velocity of P with respect to $A \in F$ as viewed from the ground. The same conclusion applies to acceleration.

As discussed in class, the very special case wherein the reference frame $F$ moves without any discernible rotation (pure translational motion) is an important exception. For this very special case, the simple relative motion relations posited in your question are, in fact, valid.