Asymptotic Behaviour of the q-Poisson Distributions Heine and Euler by Pointwise Convergence

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1 Extended Abstract

Kemp [7, 8, 9], introduced the q-Poisson distributions, Heine and Euler, with probability functions given respectively by

$$f_X^H(x) = e_q(-\lambda)\frac{q^{\binom{x}{2}}\lambda^x}{[x]_q!}, x = 0, 1, 2, \ldots,\ 0 < q < 1,\ 0 < \lambda < \infty \tag{1}$$

and

$$f_X^E(x) = E_q(-\lambda)\frac{\lambda^x}{[x]_q!}, x = 0, 1, 2, \ldots,\ 0 < q < 1,\ 0 < \lambda(1-q) < 1, \tag{2}$$

where

$$e_q(z) := \sum_{n=0}^{\infty}\frac{(1-q)^n z^n}{(q;q)_n} = \sum_{n=0}^{\infty}\frac{z^n}{[n]_q!} = \frac{1}{((1-q)z;q)_\infty},\ \ |z| < 1 \tag{3}$$

and

$$E_q(z) := \sum_{n=0}^{\infty}\frac{(1-q)^n q^{\binom{z}{2}} z^n}{(q;q)_n} = \sum_{n=0}^{\infty}\frac{q^{\binom{z}{2}} z^n}{[n]_q!} = (-(1-q)z;q)_\infty,\ \ |z| < 1. \tag{4}$$

Both these q-Poisson distributions are unimodal and logconcave with Euler being infinitely divisible but Heine not. Also, Heine is underdispersed but Euler overdispersed.
Charalambides [3], reproduced Heine as direct approximation, as $n \to \infty$, of the q-Binomial distribution and the negative q-Binomial one, with probability functions given respectively by

$$f_X^B(x) = \binom{n}{x}_q q^{\binom{x}{2}}\theta^x \prod_{j=1}^{n}(1+\theta q^{j-1})^{-1},\ x = 0, 1, \ldots, n, \tag{5}$$

and

$$f_X^{NB}(x) = \binom{n+x-1}{x}_q q^{\binom{x}{2}}\theta^x \prod_{j=1}^{n+x}(1+\theta q^{j-1})^{-1},\ x = 0, 1, \ldots, \tag{6}$$

where $\theta > 0,\ 0 < q < 1$.
Moreover, Charalambides [3], reproduced Euler as direct approximation of the q-Binomial and the

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negative q-Binomial distributions of the second kind one, as $n \to \infty$, with probability functions given respectively by

$$f_X^{BS}(x) = \binom{n}{x}_q \theta^x \prod_{j=1}^{n-x} (1 - \theta q^{j-1}),\ x = 0, 1, \ldots, n, \tag{7}$$

and

$$f_X^{NBS}(x) = \binom{n+x-1}{x}_q \theta^x \prod_{j=1}^{n} (1 - \theta q^{j-1}),\ x = 0, 1, \ldots, \tag{8}$$

where $0 < \theta < 1$ and $0 < q < 1$ or $1 < q < \infty$ with $\theta q^{n-1} < 1$.
Kyriakoussis and Vamvakari [11, 12], for q constant, established a q-Stirling formula and proved limit theorems for the q-binomial distribution (5) and negative q-Binomial distribution (6), by using pointwise convergence in a q-analogue sense of the DeMoivre-Laplace classical limit theorem. Specifically in [11], the pointwise convergence of the q-binomial distribution to a deformed Stieltjes-Wigert continuous distribution was proved. In detail, transferred from the random variable X of the q-binomial distribution (5) to the equal-distributed deformed random variable $Y = [X]_{1/q}$ and for $n \to \infty$, the q-binomial distribution was approximated by a deformed standardized continuous Stieltjes-Wigert distribution as follows

$$\begin{aligned} f_X^B(x) \quad \cong \quad & \frac{q^{1/8}(\log q^{-1})^{1/2}}{(2\pi)^{1/2}} \left(q^{-3/2}(1-q)^{1/2} \frac{[x]_{1/q} - \mu_q}{\sigma_q} + q^{-1} \right)^{1/2} \\ & \cdot \exp\left(\frac{1}{2\log q} \log^2 \left(q^{-3/2}(1-q)^{1/2} \frac{[x]_{1/q} - \mu_q}{\sigma_q} + q^{-1} \right) \right), \\ & x \geq 0, \end{aligned} \tag{9}$$

where $\theta = \theta_n,\ n = 0, 1, 2, \ldots$ such that $\theta_n = q^{-\alpha n}$ with $0 < a < 1$ constant and μ_q and σ_q^2 the mean value and variance of the random variable Y, respectively. Also in [12], a similar asymptotic result has been provided for the negative q-binomial distribution.
The aim of this work is to study the pointwise convergence of both Heine and Euler distributions as $\lambda \to \infty$. Specifically, the pointwise convergence of the Heine distribution to a deformed Stieltjes-Wigert continuous distribution and of the Euler distribution to a deformed Gauss are proved. Moreover, pointwise convergence of the q-binomial of the second kind and the negative q-binomial of the second kind, to a deformed Gauss are analogously deduced. Also, the associated q-orthogonal polynomials in respect of their weight functions to the above q-distributions and the related Stieltjes-Wigert polynomials moment problem are presented (see Andrews[1, 2], Christiansen [4, 5], Ismail[6]).

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