Fixed points of pattern-avoiding permutations

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Permutations that avoid given patterns are among the most classical objects in combinatorics and have strong connections to many fields of mathematics, computer science and biology. The fixed points of pattern-avoiding permutations have drawn special attention in the literature, see for example [1, 2, 3, 4, 6]. We show that the fixed points of pattern-avoiding permutations are related to Brownian excursion. This result is suprising because, although Brownian excusion is related to the bulk behavior of the graph of a pattern-avoiding permutation [5], the property of being a fixed point is a very local property.

In particular, we find an exact discription for a scaling limit of the number of fixed points for both 123- and 231-avoiding permutations, strengthening the recent results of Elizalde [3] and Miner and Pak [6].

Our main result is the following theorem, which as far as we know is the first to give a connection between the asymptotic distribution of fixed points of pattern-avoiding permutations and Brownian excursion.

Theorem 1. *Let* $(\mathbf{e}_t, 0 \leq t \leq 1)$ *be standard Brownian exursion and let* σ_n *and* ρ_n *be respectively a uniformly random* **231**-*avoiding permutation of* $[n]$ *and a uniformly random* **123**-*avoiding permutation of* $[n]$. *Then*

(a)
$$\lim_{n\to\infty} \frac{1}{n^{1/4}} \sum_{i=1}^{n} \delta_{i/n} \mathbf{1}_{\{\sigma_n(i)=i\}} =_d \frac{1}{2^{7/4}\pi^{1/2}} \mathbf{e}_t^{-3/2} dt,$$

where the convergence is with respect to weak convergence of finite measures on $[0, 1]$.

(b) *Let* A *and* B *be independent Bernoulli*$(1/4)$ *random variables, also jointly independent of* $(\mathbf{e}_t, 0 \leq t \leq 1)$. *Then*

$$\lim_{n\to\infty} \sum_{i=1}^{n} \delta_{\frac{i-\frac{n}{2}}{\sqrt{2n}}} \mathbf{1}_{\{\rho_n(i)=i\}} =_d A\delta_{-\mathbf{e}(1/2)/2} + B\delta_{\mathbf{e}(1/2)/2},$$

where the convergence is with respect to weak convergence of finite measures on $\mathbb{R}$.

This is joint work with Christopher Hoffman and Douglas Rizzolo.

References

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