Walks in the Quarter-Plane with Multiple Steps

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We consider nearest-neighbor lattice walks in the positive quadrant $(\mathbb{Z}_{\geq 0})^2$ where step sets may contain several distinguishable steps pointing in the same direction. Let $f_{i,j,n}$ denote the number of paths starting at the origin and consisting of n steps taken from the step set ending at the point (i, j). We concern ourselves with the nature of the corresponding generating function $f(x, y, t) = \sum_{n=0}^{\infty} \sum_{i,j} f_{i,j,n} x^i y^j t^n$. In particular, we are interested in the cases where $f(x, y, t)$ is algebraic or D-finite.

If we let $a_{u,v}$ denote the multiplicity of the direction $(u, v) \in \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$, then we have the following simple recurrence relation for $f_{i,j,n}$:

$$f_{i,j,n+1} = \sum_{u,v} a_{u,v} f_{i-u,j-v,n} \qquad (n \in \mathbb{N}, i, j \in \mathbb{N})$$

We provide a partial classification models that have algebraic or D-finite generating functions. The classification is based on the so-called group of the model discussed by Fayolle, Iasnogorodski, and Malyshev in [2] and further investigated by Bousquet-Mélou in [1]. The group of the model must be dihedral, and is either of order $2n$ for some $n \in \mathbb{N}$, or infinite. We classify the models that give rise to finite groups of order $4, 6$, and 8. This classification is presented in terms of families whose simple defining equations depend only on the multiplicities $a_{i,j}$: for example, all models of that have a group of order 4 are characterized by the equations $a_{1,0}a_{-1,1} = a_{-1,0}a_{1,1}$, $a_{1,-1}a_{-1,1} = a_{-1,-1}a_{1,1}$, and $a_{1,-1}a_{-1,0} = a_{-1,-1}a_{1,0}$.

All of the models in our classification have generating functions that are either D-finite or algebraic, which we prove using orbit sum and half-orbit sum methods based on the techniques of [1]. Finally, we provide some partial results for the $G = D_{10}$ case.

References

[1] Mireille Bousquet-Mélou and Marni Mishna. Walks with small steps in the quarter plane. *Contemporary Mathematics*, 520:1–40, 2010.

[2] Guy Fayolle, Roudolf Iasnogorodski, and Vadim Malyshev. *Random walks in the quarter-plane*, volume 40 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, 1999. Algebraic methods, boundary value problems and applications.

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