Lozenge tilings of a hexagon with three holes

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A *plane partition* is a rectangular array of non-negative entries so that all rows and columns are weakly decreasing. Plane partitions are usually identified with their 3-D interpretation—stacks of unit cubes fitting in a given box. MacMahon's classical theorem [1] says that the generating function of the volume of the stacks π fitting in an $a \times b \times c$ box is given by

$$\sum_{\pi} q^{\text{volume of }\pi} = \frac{\mathrm{H}_q(a)\,\mathrm{H}_q(b)\,\mathrm{H}_q(c)\,\mathrm{H}_q(a+b+c)}{\mathrm{H}_q(a+b)\,\mathrm{H}_q(b+c)\,\mathrm{H}_q(c+a)}, \tag{0.1}$$

where the *q-hyperfactorial function* $\mathrm{H}_q(n)$ is defined by $\mathrm{H}(n) := [0]_q! \cdot [1]_q! \dots [n-1]_q!$ and where $[n]!_q = \prod_{i=1}^{n} \frac{1-q^n}{1-q}$. On the other hand, the above stacks can be viewed as the lozenge tilings of a semi-regular hexagon of side-lengths a, b, c, a, b, c (in cyclic order) on the triangular lattice. Here, a *lozenge tiling* is a covering of the region by lozenges (union of two adjacent unit equilateral triangles) so that there are no gaps or overlaps.

In 1999, James Propp [2] published a list of 32 open problems in the field of enumeration of tilings. Problem 3 on the list asks for the number of lozenge of tilings of a hexagon of side-lengths $2n, 2n+3, 2n, 2n+3, 2n, 2n+3$ on the triangular lattice, where three central unit triangles have been removed from its long sides. Theresia Eisenkölbl solved this problem.

One can view the unit triangles removed in the Propp's problem as triangular holes of size 1. We now consider a more general situation when our hexagon has three triangular holes of *arbitrary* sizes on non-consecutive sides. Moreover, these triangular holes can be extended to bowtie-shaped holes, which consist of one up-pointing and one down-pointing triangular holes sharing a vertex.

Similar to the case of hexagon, the tiling of our region can be viewed as stacks of unit cubes fitting in a 'special' box, which consists of several adjacent smaller boxes. Interestingly, the generating function of the volume of those stacks is also given by a simple product formula in terms of q-hyperfactorials.

References

[1] P. A. MacMahon, *Combinatory Analysis*, Vol. 1 and 2, Cambridge Univ. Press, 1916, reprinted by Chelsea, New York, 1960.

[2] J. Propp, *Enumeration of matchings: Problems and progress*, New Perspectives in Geometric Combinatorics, Cambridge Univ. Press, 1999, 255–291.