Z Transform and Its Application to the Analysis of LTI Systems

• Z-transform is an alternative representation of a discrete signal.
• Z-Transform is important in the analysis and characterization of LTI systems
• Z-Transform play the same role in the analysis of discrete time signal and LTI systems as Laplace transform does in the analysis of continuous time signal and LTI systems.
• Z-transform provides us with a mean of characterizing an LTI system and its response to various signals by its pole-zero locations.
Z Transform and Its Application to the Analysis of LTI Systems

The direct Z-Transform

- The z-transform of a discrete time signal \( x(n) \) is defined as

\[
X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}
\]

The direct z-transform

where \( z \) is a complex variable.

\[ z = re^{j\theta} \]

The inverse procedure is called inverse z-transform. So relationship can give as

\[ x(n) \leftrightarrow X(z) \]

We also denote the z-transform as

\[ X(z) = Z[x(n)] \]

Z Transform and Its Application to the Analysis of LTI Systems

Since the z-transform is an infinite power series, it exists only for the series convergence. The region of convergence (ROC) of \( X(z) \) is the set of all values of \( z \) for which \( X(z) \) attains a finite value.

Example

\( x(n) = \{1, 2, 5, 7, 0, 1\} \)

Sequence

Z-transform

\[ X(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5} \]

ROC: \( z \neq 0 \)

\[ z^k (k > 0) \]

becomes unbounded for \( z = \infty \)

\[ z^{-k} (k > 0) \]

becomes unbounded for \( z = 0 \)

In many cases, we can express the sum of the finite or infinite series for z-transform in a closed-form expression.
Example: 
\[ x(n) = \left( \frac{1}{2} \right)^n u(n) \] 
Find the z-transform

\[ x(n) = \left\{ 1, \left( \frac{1}{2} \right), \left( \frac{1}{2} \right)^2, \left( \frac{1}{2} \right)^3, \ldots \right\} \]

\[ X(z) = 1 + \frac{1}{2} z^{-1} + \left( \frac{1}{2} \right)^2 z^{-2} + \left( \frac{1}{2} \right)^3 z^{-3} + \ldots \]

\[ X(z) = \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n z^{-n} = \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n \]

Z-transform

If \[ \left| \frac{1}{2} z^{-1} \right| < 1 \] or \[ |z| > \frac{1}{2} \]

\[ X(z) = \frac{1}{1 - \frac{1}{2} z^{-1}} \]

ROC: \[ |z| > \frac{1}{2} \]

Reminder,

\[ 1 + A + A^2 + A^3 + \ldots = \frac{1}{1-A} \text{ if } |A| < 1 \]

The complex variable \( z \) can be written in polar form as

\[ z = re^{j\theta} \]

The \( X(z) \) can be expressed as

\[ X(z) \bigg|_{z=e^{j\theta}} = \sum_{n=-\infty}^{\infty} x(n)r^{-n}e^{-j\theta n} \]

\[ |X(z)| = \sum_{n=-\infty}^{\infty} |x(n)r^{-n}| \leq \sum_{n=-\infty}^{\infty} |x(n)r^{-n}| < \infty \]

\[ \leq \sum_{n=-\infty}^{\infty} |x(n)r^{-n}| + \sum_{n=-\infty}^{\infty} \frac{\sum_{n=-\infty}^{\infty} |x(n)r^{-n}|}{r^n} = \sum_{n=-\infty}^{\infty} |x(n)r^{-n}| + \sum_{n=-\infty}^{\infty} \frac{\sum_{n=-\infty}^{\infty} |x(n)r^{-n}|}{r^n} \]
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In the first sum, there must be exist values of r small enough so that \( x(-n)r^n \) is absolutely summable.

In the second sum, there must exist values of r large enough so that \( x(n)/r^n \) is absolutely summable.

Since the convergence of \( X(z) \) requires that both sums, the ROC of \( X(z) \) is specified the common region where the both sum are finite.

If there is no common region, then \( X(z) \) does not exist.

Example.1: if \( x(n) = \alpha^n u(n) \)

\[
X(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n
\]

if \( |\alpha z^{-1}| < 1 \) the power series is

\[
X(z) = \frac{1}{1 - \alpha z^{-1}} \quad \text{ROC: } |z| > |\alpha|
\]
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Example 1: Find z-transform of $x(n)$

$$x(n) = -\alpha^n u(-n-1) = \begin{cases} 0 & n \geq 0 \\ -\alpha^n & n \leq -1 \end{cases}$$

$$X(z) = \sum_{n=-\infty}^{1} (-\alpha^n) z^{-n} = -\sum_{m=1}^{\infty} (\alpha^{-1} z)^m$$

if $|\alpha z^{-1}| < 1$ the power series is

$$X(z) = -\frac{\alpha^{-1} z}{1-\alpha^{-1} z} = \frac{1}{1-\alpha z^{-1}}$$

ROC: $|z| < |\alpha|$

Z-transform is

$$Z\{\alpha^n u(n)\} = Z\{-\alpha^n u(-n-1)\} = \frac{1}{1-\alpha z^{-1}}$$

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Example 1: Find z-transform of $x(n)$

$$x(n) = \alpha^n u(n) + b^n u(-n-1)$$

$$X(z) = \sum_{n=0}^{\infty} (\alpha^n) z^{-n} + \sum_{n=-\infty}^{-1} b^n z^{-n} = \sum_{n=0}^{\infty} (\alpha z)^n + \sum_{m=1}^{\infty} (b^{-1} z)^m$$

From the first power series if $|\alpha z^{-1}| < 1$ or $|z| > |\alpha|$ and from the second power series if $|b^{-1} z| < 1$ or $|z| < |b|$, then

We have two cases:

- $|b| < |\alpha|$ There is no common region, $X(z)$ does not exist.
- $|b| > |\alpha|$ There is common region, which is $|\alpha| < |z| < |b|$, Then we obtain

$$X(z) = \frac{1}{1-\alpha z^{-1}} - \frac{1}{1-b^{-1} z} = \frac{b-\alpha}{\alpha + b - z - ab z^{-1}}$$

ROC: $|\alpha| < |z| < |b|$
# Z Transform and Its Application to the Analysis of LTI Systems

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<tr>
<td>$a^n u(n)$</td>
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## The direct $Z$-Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$z = re^{i\theta}$

The inverse procedure is called inverse $z$-transform. So relationship can give as

$$x(n) \leftrightarrow X(z)$$

### Example:

$x(n) = (0.5)^n u(n)$

$$X(z) = \sum_{n=0}^{\infty} (0.5)^n z^{-n} = \sum_{n=0}^{\infty} (0.5z^{-1})^n$$

$$X(z) = \frac{1}{1-0.5z^{-1}}$$

ROC: $|z| > 0.5$
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Example 2: Find z-transform of $x(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} (-0.5^n) z^{-n} = -\sum_{n=1}^{\infty} (0.5^{-1})^n$$

if $|0.5z^{-1}| < 1$ the power series is

$$X(z) = \frac{-0.5^{-1}z}{1-0.5^{-1}z} = \frac{1}{1-0.5z^{-1}}$$

ROC: $|z| < |0.5|

Z-transform is

$$Z\{\alpha^n u(n)\} = Z\{-\alpha^n u(-n-1)\} = \frac{1}{1-\alpha z^{-1}}$$

Example 3: Find z-transform of $x(n)$

$$x(n) = \alpha^n u(n) + b^n u(-n-1)$$

$$X(z) = \sum_{n=0}^{\infty} (\alpha z^{-n}) + \sum_{n=-\infty}^{-1} b^n z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-n}) + \sum_{n=-1}^{\infty} (b^{-1}z)^n$$

From the first power series if $|\alpha z^{-1}| < 1$ or $|z| > |\alpha|$ and from the second power series if $|b^{-1}z| < 1$ or $|z| < |b|$, then

We have two cases:

- $|b| < |\alpha|$ There is no common region, $X(z)$ does not exist.
- $|b| > |\alpha|$ There is common region, which is $|z| < |b|$, then we obtain

$$X(z) = \frac{1}{1-\alpha z^{-1}} - \frac{1}{1-b^{-1}z} = \frac{b - \alpha}{\alpha + b - z - abz^{-1}} \quad \text{ROC} \quad |\alpha| < |z| < |b|$$
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The Inverse z-Transform:

$x(n)$ can be obtained from $X(z)$ using the Cauchy integral theorem. Multiplying both sides by $z^{n-1}$ and integrating both sides over a closed contour within the ROC of $X(z)$ which encloses the origin.

$$\oint_{c} X(z) z^{n-1} \, dz = \sum_{k=-\infty}^{\infty} x(k) \oint_{c} z^{n-1-k} \, dz$$

$$= \sum_{k=-\infty}^{\infty} x(k) \oint_{c} \frac{1}{z^{k+n-1}} \, dz$$

The Cauchy integral theorem, which states that

$$\oint_{c} \frac{1}{z^{k+n-1}} \, dz = \begin{cases} 1 & k = n \\ 0 & k \neq n \end{cases}$$

Applying this to the previous equation, we have the inversion formula

$$x(n) = \frac{1}{2\pi j} \oint_{c} X(z) z^{n-1} \, dz$$

Properties of z-Transform

**Linearity:**

If $x_1(n) \leftrightarrow X_1(z)$ and $x_2(n) \leftrightarrow X_2(z)$, then

$$x(n) = a_1 x_1(n) + a_2 x_2(n) \leftrightarrow a_1 X_1(z) + a_2 X_2(z)$$

**Example:**

$x(n) = [3(2^n) - 4(3^n)]u(n)$

$x_1(n) = 2^n u(n)$

$x_2(n) = 3^n u(n)$

$x(n) = 3x_1(n) - 4x_2(n)$

Its z-transform:

$$X(z) = 3X_1(z) - 4X_2(z)$$

$$\alpha^n u(n) \leftrightarrow \frac{1}{1 - \alpha z^{-1}} \quad \text{ROC: } |\alpha| > |\alpha|$$

The intersection of the ROC of $X_1(z)$ and $X_2(z)$ is $|z| > 3$.3

$$X(z) = \frac{3}{1 - 2z^{-1}} - \frac{4}{1 - 3z^{-1}} \quad \text{ROC: } |z| > 3$$
Z Transform and Its Application to the Analysis of LTI Systems

Properties of z-Transform

**Time Shifting**

If \( x(n) \leftrightarrow X(z) \)

Then \( x(n-k) \leftrightarrow z^{-k}X(z) \)

**ROC:** same as \( X(z) \) accept for \( |z|=0 \) if \( k>0 \) and \( |z|=\infty \) if \( k<0 \)

**Example:**
\[ x_i(n) = [1,2,5,7,0,1] \]

Its z-transform:
\[ X_1(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-4} \]
**ROC:** \( |z| \neq 0 \)

\[ x_2(n) = x_i(n+2) \]
\[ X_2(z) = z^2 + 2z^{-1} + 5 + 7z^{-1} + z^{-2} \]
**ROC:** \( |z| \neq \infty \), \( |z| \neq 0 \)

\[ x_3(n) = x_i(n-2) \]
\[ X_3(z) = z^{-1} + 2z^{-3} + 5z^{-4} + 7z^{-5} + z^{-2} \]
**ROC:** \( |z| \neq 0 \)

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**Scaling in the z-domain**

If \( x(n) \leftrightarrow X(z) \)

Then \( a^n x(n) \leftrightarrow X(a^{-1}z) \)

for any constant \( a \), real or complex

**Example:**
\[ x(n) = a^n (\cos \omega_0 n)u(n) \]

\[ (\cos \omega_0 n)u(n) \leftrightarrow \frac{1-z^{-1} \cos \omega_0}{1-2z^{-1} \cos \omega_0 + z^{-2}} \]
**ROC:** \( |z| > 1 \)

Its z-transform:
\[ X(z) = \frac{1-az^{-1} \cos \omega_0}{1-2az^{-1} \cos \omega_0 + a^2 z^{-2}} \]
**ROC:** \( |z| > |a| \)
Z Transform and Its Application to the Analysis of LTI Systems

Properties of z-Transform

**Time reversal**

If \( x(n) \leftrightarrow X(z) \)  
\[ \text{ROC: } |z| > \frac{r_1}{r_2} \]

Then \( x(-n) \leftrightarrow X(z^{-1}) \)  
\[ \text{ROC: } \frac{1}{r_1} < |z| < \frac{1}{r_2} \]

Example: \( x(n) = u(-n) \)

\[ u(n) \leftrightarrow \frac{1}{1-z^{-1}} \]  
\[ \text{ROC: } |z| > 1 \]

\[
z\text{-transform:} \quad u(-n) \leftrightarrow \frac{1}{1-z} \quad \text{ROC: } |z| < 1
\]

**Differentiation in the z-domain**

If \( x(n) \leftrightarrow X(z) \)  
\[ \text{ROC: } |z| < \frac{r_1}{r_2} \]

Then \( nx(n) \leftrightarrow -z \frac{dX(z)}{dz} \)  
\[ \text{ROC: } \frac{1}{r_1} < |z| < \frac{1}{r_2} \]

Example: \( x(n) = na^n u(n) \)

\[ a^n u(n) \leftrightarrow \frac{1}{1-az^{-1}} \]  
\[ \text{ROC: } |z| > |a| \]

\[
\text{z-transform:} \quad na^n u(n) \leftrightarrow X(z) = -\frac{d}{dz} \left( \frac{1}{1-az^{-1}} \right) = \frac{az^{-1}}{(1-az^{-1})^2} \quad \text{ROC: } |z| > |a| \]
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Properties of z-Transform

Convolution of two sequences:

If

\[ x_1(n) \leftrightarrow X_1(z) \quad \text{and} \quad x_2(n) \leftrightarrow X_2(z) \]

Then

\[ x(n) = x_1(n) * x_2(n) \leftrightarrow X(z) = X_1(z)X_2(z) \]

ROC: is the intersection of \( X_1(z) \) and \( X_2(z) \)

Example:

\[ x_1(n) = \{1,-2,1\} \]
\[ x_2(n) = \begin{cases} 1 & 0 \leq n \leq 5 \\ 0 & \text{elsewhere} \end{cases} \]

\[ X_1(z) = 1 - 2z^{-1} + z^{-2} \]
\[ X_2(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5} \]

z-transform:

\[ X(z) = X_1(z)X_2(z) = 1 - z^{-1} - z^{-2} + z^{-4} \]

ROC:

\[ |z| \neq 0 \]

\[ x(n) = \{1,-1,0,0,0,0,-1,1\} \]

Correlation of two sequences:

\[ r_{x_{12}}(l) = \sum_{n=-\infty}^{\infty} x_1(n)x_2(n-l) \leftrightarrow R_{x_{12}}(z) = X_1(z)X_2(z^{-1}) \]

Example:

The autocorrelation of signal

\[ x(n) = a^nu(n) \quad -1 < a < 1 \]

\[ X(z) = \frac{1}{1-az^{-1}} \quad \text{ROC:} \quad |z| > 1 \]

\[ X(z^{-1}) = \frac{1}{1-a} \quad \text{ROC:} \quad |z| > |a| \]

\[ R_{x_{12}}(z) = X_1(z)X_2(z^{-1}) \]

\[ R_{x_{12}}(z) = \frac{1}{1-az^{-1}} \frac{1}{1-a} = \frac{1}{1-a(z^{-1}) + a^2} \]
Z Transform and Its Application to the Analysis of LTI Systems

Properties of z-Transform

The initial Value Theorem:

If \( x(n) \) is causal (\( x(n)=0 \) for \( n<0 \))

\[
x(0) = \lim_{z \to \infty} X(z)
\]

Example: