This document starts with a summary of useful facts from vector calculus, and then uses them to derive Maxwell’s equations. First, definitions of vector operators.

1. **Gradient Operator**: The gradient operator is something that acts on a function \( f \) and produces a vector whose components are equal to derivatives of the function. It is defined by:

\[
\nabla f = \hat{x} \frac{\partial f}{\partial x} + \hat{y} \frac{\partial f}{\partial y} + \hat{z} \frac{\partial f}{\partial z}
\]  

(1)

2. **Divergence**: We can apply the gradient operator to a vector field to get a scalar function, by taking the dot product of the gradient operator and the vector function.

\[
\nabla \cdot \vec{V}(x, y, z) = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}
\]  

(2)

3. **Curl**: We can take the cross product of the gradient operator with a vector field, and get another vector that involves derivatives of the vector field.

\[
\nabla \times \vec{V}(x, y, z) = \hat{x} \left( \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) + \hat{y} \left( \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right) + \hat{z} \left( \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right)
\]  

(3)

4. We can also define a second derivative of a vector function, called the Laplacian. The Laplacian can be applied to any function of \( x, y, \) and \( z \), whether the function is a vector or a scalar. The Laplacian is defined as:

\[
\nabla^2 f(x, y, z) = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}
\]  

(4)

Notes that the Laplacian of a scalar function is equal to the divergence of the gradient.

Second, some useful facts about vector fields:

1. If a vector field \( \vec{V}(x, y, z) \) has zero curl, then it can be written as the gradient of a scalar function \( f(x, y, z) \). *i.e.* if \( \nabla \times \vec{V}(x, y, z) = 0 \) then we can find some scalar function \( f \) such that \( \vec{V}(x, y, z) = \nabla f(x, y, z) \).

   This fact doesn’t necessarily tell us *how* to find \( f(x, y, z) \) or what \( f \) is, just that \( f \) exists and if we work hard enough we can find it.

2. The opposite is also true: If a vector field \( \vec{V}(x, y, z) \) happens to be equal to the gradient of some scalar function \( f \), then its curl is zero, *i.e.*

\[
\nabla \times (\nabla f) = 0
\]  

(5)
3. Another interesting fact is that the divergence of a curl is zero, i.e. if \( \vec{G}(x, y, z) = \nabla \times \vec{V}(x, y, z) \) then \( \nabla \cdot \vec{G} = 0 \), or:

\[
\nabla \cdot \left( \nabla \times \vec{V}(x, y, z) \right) = 0
\]

(6)

4. The curl of the curl of a vector gives something related to the Laplacian:

\[
\nabla \times \left( \nabla \times \vec{V} \right) = \nabla \left( \nabla \cdot \vec{V} \right) - \nabla^2 \vec{V}
\]

(7)

We’ll use this to derive the wave equation from Maxwell’s equations.

Finally, two theorems that are crucial for deriving Maxwell’s equations:

1. **Divergence Theorem**: If we draw some closed surface around a region, and calculate the flux of a vector field coming out of the surface \( S \), that’s the same as calculating the volume integral of the divergence of the vector field inside the volume \( V \) enclosed by the surface.

\[
\oint_S \vec{E} \cdot d\vec{A} = \int_V \nabla \cdot \vec{E} dV
\]

(8)

where \( d\vec{A} \) is a vector normal to the surface \( S \) and pointing outward away from the interior (i.e. away from the volume \( V \)), and its magnitude is \( dA \), the area of the tiny surface element.

2. **Stokes’s Theorem**: If we draw a (non-closed) surface \( S \) in space, and calculate the line integral of a vector field around its edge \( C \) (\( C \) is for “curve” and “cookie”, either is good enough for me), that’s the same as calculating the flux of the curl of the vector field through the surface.

\[
\oint_C \vec{B} \cdot d\vec{s} = \oint_S \left( \nabla \times \vec{B} \right) \cdot d\vec{A}
\]

(9)

where \( d\vec{s} \) is a vector tangent to a small segment of the curve \( C \) on the edge of \( S \), with magnitude equal to the length \( ds \) of the segment.

In integral form, we can use Gauss’s theorem to write two of Maxwell’s equations as:

\[
\oint_S \vec{E} \cdot d\vec{A} = \int_V \left( \nabla \cdot \vec{E} \right) dV = \frac{1}{\varepsilon_0} \int_V \rho(x, y, z) dxdydz
\]

\[
\oint_S \vec{B} \cdot d\vec{A} = \int_V \left( \nabla \cdot \vec{B} \right) dV = 0
\]

(10) (11)
If these equations hold for any volume $V$ in space, with any shape we can think of (and as far as we know they do indeed hold!) then we can equate the integrands and get:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

(12)

$$\nabla \cdot \vec{B} = 0$$

(13)

We can use Stokes’s Theorem to write the other two equations as:

$$\oint_C \vec{E} \cdot d\vec{s} = \int_S (\nabla \times \vec{E}) \cdot d\vec{A} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A}$$

(14)

$$\oint_C \vec{B} \cdot d\vec{s} = \int_S (\nabla \times \vec{B}) \cdot d\vec{A} = \int_S \left( \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \cdot d\vec{A}$$

(15)

And if these equations also hold for any surface $S$ in space, with any shape we can think of (and as far as we know they do indeed hold!) then we can again equate the integrands and get:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

(16)

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

(17)

Now, we have the Maxwell equations in all their glory. Let’s suppose we work in free space, where $\rho$ and $\vec{J}$ are both zero. $\rho = 0$ means that $\nabla \cdot \vec{E} = 0$. If we take the curl of Eq. (16) we get:

$$\nabla \times \left( \nabla \times \vec{E} \right) = \nabla \left( \nabla \cdot \vec{E} \right) - \nabla^2 \vec{E} = -\nabla^2 \vec{E} = -\frac{\partial}{\partial t} \left( \nabla \times \vec{B} \right)$$

(18)

where we dropped $\nabla \left( \nabla \cdot \vec{E} \right)$ because $\nabla \cdot \vec{E} = 0$ if $\rho = 0$. But now we have this expression involving the magnetic field to contend with. That, however, is just minus the time derivative of Eq. (17). And the first term in Eq. (17) is zero, because we’re in empty space (i.e. no currents, so $\vec{J} = 0$), while the second term in Eq. (17) is the second derivative of the electric field. So, we get the wave equation:

$$\nabla^2 \vec{E} = \epsilon_0 \mu_0 \frac{\partial^2}{\partial t^2} \vec{E}$$

(19)

Knight does this in section 35.5, but his derivation involves taking the integrals around a very small loop and showing that they involve derivatives. It’s kind of confusing, more like a trick than an invocation of a basic rule in mathematics. This derivation introduces facts that show up all over the place in physics and math.