\[ \sum_{j} w_j C_j \]

1 \(\|\sum_{j} w_j C_j \) can be solved by ratio rule.

Sort the jobs in ascending order of \( P_j/w_j \)

\[ P_1/w_1 \leq P_2/w_2 \leq \ldots \leq P_n/w_n \]

Proof: By interchange argument. (Homework exercise)

\( P_2 \| \sum_{j} w_j C_j \) is NP-hard.

Reduction from partition problem

Give \( A = \{a_1, a_2, \ldots, a_n\} \), a set of \( n \) integer. Construct an instance of \( P_2 \| \sum_{j} w_j C_j \) as follows.

Let there be \( n \) jobs, each job \( i \) has processing time \( a_i \) and weight \( a_i \)

Let \( B = 1/2 \sum a_j \)

Let the threshold for \( P_2 \| \sum_{j} w_j C_j \) be

\[ L = B^2 + \frac{1}{2} \sum_{i=1 \text{ to } n} a_i^2 \]

Is there a schedule of the \( n \) jobs on 2 machines such that \( \sum_{j} w_j C_j \leq L \)?
Suppose jobs $i_1, i_2, \ldots, i_k$ are assigned on machine 1 and jobs $j_1, j_2, \ldots, j_k$ are assigned on machine 2.

\begin{align*}
\begin{array}{|c|c|c|c|}
\hline
i_1 & i_2 & \ldots & i_k \\
\hline
ai_1 & ai_2 & & ai_k \\
\hline
\end{array}
\end{align*}

\begin{align*}
ai_1(ai_1) \\
+ ai_2(ai_1 + ai_2) \\
\vdots \\
+ ai_k(ai_1 + ai_2 + \ldots + ai_k)
\end{align*}

\[
\sum_{x=1}^{k} aix^2 + \sum_{x \neq y} aix ay
\]

Similarly, on machine 2

\[
\sum_{x=1}^{l} ajx^2 + \sum_{x \neq y} ajx ay
\]

Therefore,

\[
\sum w_j C_j = \sum_{x=1}^{k} aix^2 + \sum_{y=1}^{l} ajy^2 + \sum_{x \neq y} aix ay + \sum_{x \neq y} ajx ay
\]

\[
= \frac{1}{2} (a_1 + a_2 + \ldots + a_k)^2 + \frac{1}{2} (a_1 + a_2 + \ldots + a_l)^2 + \frac{1}{2} \sum_{i=1}^{n} ai^2
\]

Let $\alpha = \sum_{i=1}^{k} ai$

\[
\sum w_j C_j = \frac{1}{2} [ \alpha^2 + (2B - \alpha)^2 + \sum_{i=1}^{n} ai^2 ]
\]

\[
= \frac{1}{2} [ 4B^2 - 4B\alpha + 2\alpha^2 + \sum_{i=1}^{n} ai^2 ]
\]

\[
= L + \frac{1}{2} (2B^2 - 4B\alpha + 2\alpha^2)
\]

\[
= L + (B - \alpha)^2
\]

Thus, the schedule has $\sum w_j C_j \leq L$ iff $B = \alpha$ or there is a partition.