With this important parallel/perpendicular decomposition expressed in the explicit form
\[ \mathbf{B} = \mathbf{B}_p + \mathbf{B}_\perp = (\mathbf{B} \cdot \mathbf{a}) \mathbf{a} + \mathbf{a} \times (\mathbf{B} \times \mathbf{a}), \]
the following two identities:

\[ \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{C} \cdot \mathbf{A}) \mathbf{B} - (\mathbf{B} \cdot \mathbf{A}) \mathbf{C} \]
\[ (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = (\mathbf{B} \times \mathbf{C}) \cdot \mathbf{A} \]
can now be established.

The proof of the first identity involving the double-cross product proceeds as follows:
\[
\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{A} \times [(\mathbf{B}_p + \mathbf{B}_\perp) \times (\mathbf{C}_p + \mathbf{C}_\perp)]
\]
\[
= \mathbf{A} \times [(\mathbf{B}_p \times \mathbf{C}_p) + (\mathbf{B}_p \times \mathbf{C}_\perp) + (\mathbf{B}_\perp \times \mathbf{C}_p) + (\mathbf{B}_\perp \times \mathbf{C}_\perp)]
\]
\[
= \mathbf{A} \times (\mathbf{B}_p \times \mathbf{C}_p) + \mathbf{A} \times (\mathbf{B}_p \times \mathbf{C}_\perp) + \mathbf{A} \times (\mathbf{B}_\perp \times \mathbf{C}_p) + \mathbf{A} \times (\mathbf{B}_\perp \times \mathbf{C}_\perp)
\]
\[
= \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{A} \times (\mathbf{B} \times \mathbf{C})
\]
\[
= \mathbf{A} \times (\mathbf{B} \times \mathbf{C})
\]
\[
(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = (\mathbf{B} \times \mathbf{C}) \cdot \mathbf{A}
\]
As an exercise, confirm each step in this derivation by applying results derived earlier in this section. The most challenging step is step #3 in which the fourth term is judged to be identically zero.

The formal proof of the second identity involving the so-called triple scalar product follows fairly easily by simple algebraic manipulation after resolving the vector \( \mathbf{B} \) into its parallel and perpendicular components relative to \( \mathbf{C} \) and making use of the just derived CAB-BAC identity.
\[
(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = [(\mathbf{A} \times (\mathbf{B}_p + \mathbf{B}_\perp))] \cdot \mathbf{C}
\]
\[
= (\mathbf{A} \times \mathbf{B}_p) \cdot \mathbf{C} + (\mathbf{A} \times \mathbf{B}_\perp) \cdot \mathbf{C}
\]
\[
= (\mathbf{B} \cdot \mathbf{a}) \mathbf{a} \times (\mathbf{B} \times \mathbf{a}) + \mathbf{a} \times (\mathbf{B} \times \mathbf{a})
\]
\[
= (\mathbf{B} \cdot \mathbf{a}) \mathbf{a} \times (\mathbf{B} \times \mathbf{a})
\]
\[
(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = (\mathbf{B} \times \mathbf{C}) \cdot \mathbf{A}
\]
As an exercise, confirm each step in this derivation by applying results derived earlier in this section. Pay particular attention to the conclusions regarding vanishing terms in step #3 and step #7.