ULTIMATE STRENGTH OF AIRCRAFT STRUCTURES

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ABSTRACT

Strength analysis of aircraft structures focuses on static strength, fatigue, and damage tolerance of the materials. Countless hours are invested in quantifying the static strength, which is allowed to fully yield under ultimate loads so that the structure can be designed as light and efficient as possible. This analysis is critical to a lightweight initial design, but is also of great importance in the evaluation of engineering modifications and repairs. Analysis of the critical sections often resorts to plastic bending analysis, and use of E.F. Bruhn’s Iterative Slice Method, or Cozzone’s Simplified Method for Symmetric Sections, is often employed. Yet these methods fall short when the critical section includes thin flanges that buckle or cripple prior to ultimate failure, as is generally the case for frames, floor beams, stringers, and other structural members used on aircraft. This paper presents a solution to this shortfall, and introduces a hybrid procedure for calculating the ultimate strength of a cross section that accounts for material non-linearity, flange stability, and other effects.

INTRODUCTION

One of the cornerstones of stress analysis methods for aircraft and other lightweight structures is ultimate analysis.

The Federal Aviation Administration (FAA) in its Code of Federal Regulations (CFA) for aviation defines the maximum load that a structure could be subjected to as the limit load, and it defines the ultimate load as the limit load multiplied by an appropriate factor of safety.

Much of the static analysis performed for aircraft structural substantiation focuses on the strength of structures subjected to ultimate loads. While the CFA does not allow detrimental permanent deformation to occur under limit loads, it does allow permanent deformation to occur under ultimate loads as long as these loads can be withstood for 3 seconds or more.

Typical ultimate analysis for aircraft compares elastic stresses due to ultimate loads to the ultimate allowable in tension, compression, or shear using values from [1] or equivalent. This comparison of an elastic stress to an ultimate material allowable is valid for a member subjected to axial loading, but it only approximates the strength of a member subjected to bending since it ignores the local yielding effects of the material between the yield and ultimate strengths of the material. However, this approximation is conservative for the ductile materials used in lightweight structures, and is generally accepted as a valid, conservative comparison by industry experts and by the certifying authorities.

However, when elastic bending stresses due to ultimate loads are computed that approach or exceed the ultimate allowable of a material, a more accurate analysis is typically performed that takes the plasticity of the material into account. This type of analysis is commonly referred to as “ultimate” or “plastic” analysis of the structure.

Many Mechanics of Materials textbooks provide a basic foundation for the determination of an allowable for the “ultimate” or “plastic” moment of a simple cross section. The methods presented generally are valid only for pure bending, require that the cross section be symmetric and that the stress-strain curve of the material be identical in both tension and compression, and require the analyst to integrate the stresses across the cross section.

Cozzone [2] proposed an easier approach that approximates the stress distribution across the section as a trapezoidal one. His method is also valid only for pure bending, and also requires the cross section to be symmetrical and the stress-strain curve of the material be identical in both tension and compression. Yet due to its simplicity his method...
was presented by both Bruhn [3] and Peery [4], and is in widespread use today.

In addition to presenting Cozzone’s approach, Bruhn [3] went a step further and documented a tabular iterative slice method that enables the analyst to evaluate non-symmetric sections as well as materials with stress-strain curves that differ in tension and compression. While his method is shown for pure bending it also handles sections subjected to both bending and axial load.


These methods all focus on the stress-strain response of the member in the inelastic range, and utilize the observation that in plastic analysis, it is the strain that varies in a linear fashion across the section and not the stress as is expected in elastic analysis.

Unfortunately, all these methods fall short in that they do not account for the instability of flanges of the cross section in the ultimate analysis. While Cozzone [2] and Peery [4] both mention crippling of tubes in their discussion, neither provides a method to address the problem. Since many aircraft and other lightweight structures utilize extruded, forged, and bent-up sheet sections with thin flanges that buckle or cripple before ultimate load, this is a significant shortfall in the fundamental documentation relied on by strength analysts in the field.

This paper presents a simple correction to this shortfall, and proposes a simple modification to Bruhn’s iterative slice procedure that has been implemented successfully to various degrees at McDonnell Douglas, Boeing, US Technical, and elsewhere.

**NOMENCLATURE**

- $A_i$ = Area of element i
- $b$ = Element width dimension
- $F_{cc}$ = Crippling Allowable
- $F_{cr}$ = Buckling Allowable
- $F_{tu}$ = Ultimate Allowable Tension
- $h$ = Element height dimension
- $H$ = Cross Section Total Height
- $\delta$ = Deflection
- $\varepsilon$ = Strain
- $L_0$ = Undeformed length segment
- $P$ = Force
- $M$ = Moment
- $M_{ult}$ = Ultimate Allowable Moment
- $R$ = Radius of Curvature to neutral axis
- $\sigma$ = Stress
- $y$ = Distance above neutral axis
- $Y$ = Distance above lower surface
- $Y_{CG}$ = Distance of Centroid above lower surface
- $Y_{NA}$ = Distance of Neutral Axis above lower surface

**FUNDAMENTALS**

**Strain of Straight Beams Subjected to Pure Bending**

Recall the classical case of an initially straight beam subjected to a pure bending, as shown in Figure 1.

- Imagine that we had first marked off two sections AA and BB across the initially-straight beam such that an arbitrary undeformed length $L_0$ is defined as shown. We can see that as the bending moment deforms the beam into a circular arc around the center of curvature at O, the cross section BB rotates to a new position B‘B’ about the neutral axis of the beam where the strain is zero, and the radius of curvature $R$ defines the shape of the beam along the neutral axis where the strain is zero, as shown.

- If we imagine that the beam is sliced into a series of thin elemental strips from the top to the bottom surfaces of the beam, then we can see that each strip will deform an amount $\delta$ that varies from the top to the bottom of the beam. The strain at any distance $y$ from the neutral axis is given by

$$\varepsilon = \frac{\delta}{L_0}$$

A closer look at the deflection of the beam about the radius of curvature $R$, and of the deflection of the cross section $\delta$ as it rotates about the neutral surface provides the geometric curvature relations shown in Figure 2.
If we apply the principle of similar triangles to Figure 2a and 2b we find that the deflection of any strip $\delta$ relative to its distance from the neutral axis $y$ is proportional to the distance $L_0$ relative to the radius of curvature $R$ of the neutral surface.

This results in

$$\frac{\delta}{L_0} = \frac{y}{R}$$

This can be written

$$\frac{\delta}{L_0} = \frac{y}{R}$$

Combining (1) and (3) we find a relation of the strain as a function of the distance from the neutral surface $y$ and the radius of curvature $R$.

$$\varepsilon = \frac{y}{R}$$

Armed with this relation we will now take a detour to develop a couple of other key relationships.

**Moment as a Function of Stress**

Returning to consider our thin elemental strips of material defined from the top to the bottom of the beam along $B'B'$, we can identify the stress on each strip as $\sigma_i$, and identify its area as $A_i$, and we can see that the force on each strip of our beam can be written

$$P_i = \sigma_i A_i$$

The incremental moment that each of these slices induces about the neutral axis of the beam can be written

$$M_i = P_i y_i = \sigma_i A_i y_i$$

The total moment across the section can then be written

$$M_{\text{total}} = \Sigma M_i = \Sigma \sigma_i A_i y_i$$

This $M_{\text{total}}$ is the total moment in the cross section due to the stresses across the section.

**Moment as a Function of Elastic Strain**

Now if our stresses are in the elastic range, then we recall that the stress is proportional to the strain, and is defined by the linear relationship

$$\sigma_i = E_i \varepsilon_i$$

Inserting (4) we find

$$\sigma_i = E_i \frac{y_i}{R}$$

Inserting this into (7) we find

$$M_{\text{total}} = \Sigma \left( E_i \frac{y_i}{R} \right) A_i y_i$$

which can be written

$$M_{\text{total}} = \Sigma E_i A_i y_i^2$$

If the material is homogeneous, then $E$ is constant across the section, and both $E$ and $R$ can move outside the summation. Noting that $\Sigma A_i y_i^2$ is simply the second moment of the area, more commonly referred to as the moment of inertia of the section $I$, and rearranging slightly, we see that

$$\frac{M_{\text{total}}}{I} = \frac{E}{R}$$

If we rearrange (9) for a homogeneous material to the form

$$\sigma_i = \frac{E}{R} y_i$$

Then it is easy to see that combining (11) and (12) produces

$$\sigma_i = \frac{M_{\text{total}} y_i}{I}$$

This is our familiar equation relating the stress at any position to the product of the applied moment and fiber position over the moment of inertia, and is valid for a homogeneous material subjected to elastic bending.

A stress analyst is typically only interested in the maximum stress in the cross section, and this occurs at the extreme fibers where $y_i$ is maximum. In this case, we find the familiar equation

$$\sigma_{\text{max}} = \frac{M_{\text{total}} y_{\text{max}}}{I}$$

While it is customary to evaluate the maximum stress against the corresponding allowable in this fashion, we could have just as easily have inserted the allowable stress in lieu of $\sigma_{\text{max}}$, and written the allowable moment as follows.

$$M_{\text{all}} = \frac{\sigma_{\text{all}} I}{y_{\text{max}}}$$

Therefore, to evaluate the strength of homogeneous sections subjected to elastic bending, we can either compare (14) to the material stress allowable at the extreme fiber, or insert the allowable stress at the extreme fiber into (15) to compute the allowable moment, and compare the applied moment to this value.

Most structural analysis uses the former approach, which is valid for elastic analysis where $E$ is constant across the cross section. However, this approach is used almost universally for ultimate analysis as well by simply inserting the ultimate allowable $F_{tu}$ into (14). This is not strictly correct, since $E$ varies across the section in all regions where stresses rise above the elastic limit, but it is conservative since it assumes a linear increase in stress up to the allowable, effectively ignoring the additional load-carrying capability of the material produced by yielding. This concept will be clearer in the next section, where we will look at the moment that develops in a cross section as the strains move beyond the elastic limit.

**Moment as a Function of Plastic Strain**

When an initially-straight beam is loaded in pure bending such that the stresses produced exceed the elastic limit of the material, it deflects in the same manner as before, producing
strains that relate to position as defined by equations (1), (2), (3), and (4).

We can rearrange (4) and write
\[ \frac{1}{R} = \frac{\varepsilon}{y} \] (16)

If we consider again Figure 1, and define \( \varepsilon_{\text{max}} \) as the strain in the extreme fiber where \( y \) is maximum, then application of similar triangles to the strain in the section reveals the relation
\[ \frac{\varepsilon_i}{y_i} = \frac{\varepsilon_{\text{max}}}{y_{\text{max}}} \] (17)

where the subscript \( i \) has been added to \( \varepsilon \) and \( y \) to emphasize the idea that \( \varepsilon_i \) represents the average strain in the \( i \)th elemental strip of material which is located at \( y_i \). This equation is valid for both positive and negative values of \( y \), as long as \( y_{\text{max}} \) corresponds with the position of the maximum strain \( \varepsilon_{\text{max}} \).

This can be rewritten to define the strain as a function of position as follows.
\[ \varepsilon_i = \frac{\varepsilon_{\text{max}}}{y_{\text{max}}} y_i \] (18)

This relation is the key to "plastic", or "ultimate" analysis of members subjected to bending. It means that once the ultimate strain of a material is known, we can immediately determine the strain at any point \( y \) in the cross section. It means that the strains are linear whether the stresses are above or below the elastic limit of the material, and therefore it applies to both elastic and plastic analysis. When the stresses are below the elastic limit, then \( E \) is a constant across the section, and the stresses also vary in a linear fashion along the cross section. When the stresses rise above the elastic limit, then the stress distribution becomes nonlinear, while the strain distribution remains linear.

Armed with this understanding we are now ready to evaluate the allowable of this inelastic section. The way this is done is by enforcing a linear strain distribution on the cross section, starting with the maximum strain at the extreme fiber.

First, we idealize our section by dividing it into a user-selected number of elements, identifying each element from top to bottom, as shown in Figure 3.

Next we record the width \( b_i \), height \( h_i \), and centroid relative to the lower surface \( Y_i \) of each idealized strip into a table such as that shown in Figure 4.

The area for each strip \( A_i \) can then be calculated and tabulated, and the center of gravity \( Y_{CG} \) of the section can be determined \( (Y_{CG} = \Sigma A_i Y_i / \Sigma A_i) \).

Initially, the neutral axis \( Y_{NA} \) will be assumed to coincide with the center of gravity \( (Y_{CG}) \) of the section. This assumption will remain valid for symmetric sections of materials with stress-strain curves that are the same in tension and compression. However, if either condition is violated then the neutral axis will migrate from the center of gravity to the location where the strain is zero.

The distance \( y_i \) is simply \( Y_i - Y_{NA} \).

Next, the strain on each element \( \varepsilon_i \) can be determined from (18) by inserting the maximum strain \( \varepsilon_{\text{max}} \) and its distance from the neutral axis \( y_{\text{max}} \) into the equation. For a symmetrical section with a stress-strain curve that is identical in tension and compression, this is simple, and involves reading the maximum strain \( \varepsilon_{\text{max}} \) from the stress-strain curve (alternately the maximum percent elongation from [1] can be used), and inserting this and the maximum distance \( y \) to the furthest extreme fiber into (18) for each element. This means \( y_{\text{max}} \) will be equal to \( Y_{NA} \), and \( Y_{NA} = H - Y_{NA} \) for a symmetrical section.

This is also straightforward for a non-symmetrical section, as long as \( y_{\text{max}} \) is taken as the larger of either \( Y_{NA} \) or \( H - Y_{NA} \).

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![Fig. 3: Nomenclature.](image1)

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one producing the minimum ultimate moment as the correct solution.

Once the strain in each element is determined and tabulated, the stress in each is simply read from the stress-strain curve. This step can be performed by hand, or can be automated using the Ramburg-Osgood coefficients for the material, or by performing a table-lookup function on a tabulated set of points for the stress-strain curve. It appears performing an interpolation function on 6 to 10 points performs sufficiently accurate results.

Next, the force in each strip of material is computed as $P_i = \sigma_i A_i$, and the moment of this force is $M_i = P_i \gamma_i$.

When the section is symmetric and the stress-strain curve is the same in tension and compression, then this completes the process, and the ultimate moment allowable of the section can be obtained by summing the moment contribution of each element $M_{all} = \sum M_i$.

However, if the section is not symmetric, or if the stress-strain curve in compression and tension are different, then additional work is needed to determine the correct ultimate moment. This error is made apparent by summing the forces $P_i$ on each element in Figure 4. If the sum of the forces is not zero, then the neutral axis is actually in a different place than what was assumed, and it needs to be moved to the correct location. This is usually done by iteration. If the net force is positive, then the neutral axis must move upward. If the net force is negative, then it must move downward.

The correct ultimate moment is found by adjusting the assumed location of the neutral axis $Y_{NA}$ until the sum of the forces equals zero, or until the net force drops below some error parameter.

One easily determined error parameter involves taking the absolute ratio of the sum of the forces to the sum of the moments $|\sum P / \sum M|$. Hand analysis can easily get this ratio below 1%, and automated analysis using Excel, MATLAB or equivalent can reduce this error far below the negligible range.

This procedure efficiently finds the allowable ultimate moment $M_{all}$ for any cross-section, and accurately handles both symmetric and unsymmetric sections, whether the stress-strain curve is the same or different in tension and compression.

This procedure can also be used to evaluate a cross section subjected to combined axial load and bending. The only difference is that the neutral axis for this case is moved until the sum of forces equals the applied axial load, rather than zero as before. The ultimate moment is then the moment that can be applied in addition to some force $P_{axial}$, and as such, should perhaps be identified as such. For example, if the axial load is 1000 lbs, then the ultimate moment could be designated as $M_{alt} = F_{axial} \times 1000$, or something similar.

This ultimate moment determination procedure is valid for a wide variety of applications, and is valid for symmetric and unsymmetric sections, whether or not the stress-strain curve is the same in tension and compression. However, this procedure fails to accurately account for the same when the stresses in the elements exceed the stability allowables of the webs or flanges, as is often the case for aircraft and other lightweight structures. However, a slight adjustment to the method extends its capability to cover this condition as well, as discussed in the following section.

**Ultimate Moment That Includes Stability Effects**

Often, in aircraft and other lightweight structures, the thickness of the flanges and webs of the cross section are so thin that they experience stability failure prior to reaching the stresses otherwise supportable by the material. Buckling, crippling, and Euler-Johnson allowables frequently limit the strength for these thin flanges. Lateral stability can also dominate as the limiting failure mode, and needs to be evaluated.

Fortunately, these effects can be easily accounted for using the method presented above with a slight variation. All that is needed is to add an additional column, such as that shown in Figure 5, and to record any cutoff or limiting values on the stresses for those elements.

![Fig. 5: Ultimate Bending Calculation w/Limits](image)

For example, the cross section of Figure 3 is idealized with two elements for the upper flange, three for the lower flange, and an unspecified number of elements for the web. Since these flanges all appear relatively thin, the crippling allowable should be determined for each flange and for the web. The crippling allowable $F_{cc}$ of the upper flanges should be entered into the LIMITS column for element 1 and 2, the crippling allowable for the lower flanges should be entered for elements n-2 through n, and the crippling allowable for the web should be entered for elements 3 through n-3. Web buckling $F_{crw}$ should also be checked, and flange bucking to the nearest support $F_{crf}$, and if these produce lower values than the crippling allowables, then these should be entered in the LIMITS column in lieu of the crippling allowable for the affected elements.

Once this is done and the LIMITS column is populated, then the forces in each element $P_i$ are computed as the product of the area $A_i$ and the minimum of $\sigma_i$ and $F_{cc}$ or $F_{cr}$ for the element.

The moment of each element about the neutral axis is again computed as $M_i = P_i \gamma_i$. 
Once again, the neutral axis is adjusted upwards or downward until the sum of the elemental forces \( P_i \) match the applied axial load \( P_{ax} \) (or zero if there is none).

When this procedure is complete, the ultimate moment \( M_{ult} \) is simply the sum of the moments \( EM \), as before.

This modified procedure effectively and accurately determines the ultimate moment for any cross section, and easily addresses symmetrical and unsymmetrical sections, stress-strain curves that can be the same or different in tension and compression, and also stability limitations of the section.

This procedure directly addresses the gap in the existing documentation for plastic bending analysis, and outlines a correct procedure that can be effectively used to address any ultimate analysis need for aircraft or other lightweight structure.

The procedure can easily be implemented with hand analysis, performs even more efficiently in an EXCEL or other spreadsheet-type analysis, and can also be readily programmed to maximum the efficiency of use.

**EXAMPLE**

This method can be illustrated with a typical example. One common cross section evaluated by aerospace strength engineers is the frame of an aircraft fuselage. Suppose a typical frame such as that shown in Figure 6 is subjected to a bending moment that causes compression on the inner cap. This particular frame is lifted from the skin by shear clips to allow the stringers to be uninterrupted. Therefore the shear clip is not continuous, and is not included in our section, and is shown with phantom lines below.

![Fig. 6: Typical Frame](image)

The first step in our analysis procedure is to determine the section properties of the section. For section properties, we typically only need 4 elements to evaluate this section, one for each frame cap, one for the skin, and one for the frame web. It is convenient, however, to use the same idealization for plastic bending analysis as was used in determining section properties.

In plastic bending analysis, our simplified approach uses a discrete idealization of the frame elements, and our error diminishes as the number of elements increases. For (vertically) thin elements, a single element is generally sufficient, since the variation of the strain across the element is small. However, for a vertically tall element, such as the frame web in our section, treating it as a single element introduces significant error into our calculation. Therefore, the web is idealized as 4 elements, both in our section properties calculation, and in our plastic bending analysis. This results in the section properties table shown in Figure 7, which can be easily generated by hand, or using EXCEL.

![Fig. 7: Section Properties](image)

This table determines the area and moment of inertia of the section, and also clearly shows the dimensions and vertical position of each idealized element for use in our subsequent plastic bending analysis.

We now evaluate our material properties. Suppose this frame is made of a 7075-T73 Aluminum Die Forging, and [1] provides properties for this material as shown in Figure 8.

![Fig. 8: Frame Material Properties](image)

This table shows the modulus of elasticity and yield stress for tension and compression. We can easily use our method to evaluate the skin element with the correct stress-strain curve, but suppose we decide to ignore these differences, concluding that the oversight will produce negligible error (we will evaluate this assumption later).
While we now have key material properties for our section, we also need the stress-strain curve to perform our plastic bending assessment. Full-range stress-strain curves are available in [1], which can be read by hand. However, Ramberg-Osgood coefficients can also be used to determine the stress-strain relationship using solely the information from Figure 8. Using either approach, sufficient accuracy can usually be obtained by identifying 6 or more points from the stress-strain curve for use in assessing the stress-strain relationship. For our material, the stress-strain curve is nearly identical in tension and compression, so we define the following points for our analysis, as shown in Figure 9.

<table>
<thead>
<tr>
<th>Tension Stress-Strain Points:</th>
<th>Compression Stress-Strain Points:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Point</td>
<td>Strain</td>
</tr>
<tr>
<td>P₁₇</td>
<td>0.0000</td>
</tr>
<tr>
<td>P₂</td>
<td>0.0012</td>
</tr>
<tr>
<td>P₃</td>
<td>0.0024</td>
</tr>
<tr>
<td>P₄</td>
<td>0.0036</td>
</tr>
<tr>
<td>P₅</td>
<td>0.0065</td>
</tr>
<tr>
<td>P₆</td>
<td>0.0700</td>
</tr>
<tr>
<td>P₇</td>
<td>0.4200</td>
</tr>
</tbody>
</table>

**Fig. 9: Section Properties**

It is noted that in addition to the 6 points used to define the tension and compression stress-strain relationship, one additional point can be used when an automated lookup function is used in EXCEL. This keeps the automated functions from becoming confused when the strain computed exceeds the maximum elongation identified for the material. Care must be used when auto-coding an adjustment such as this one, since we should not generally allow the actual strain in the material to exceed the elongation. However, when the exceedance occurs on an element that has buckled or crippled, this is acceptable since after the element becomes unstable it continues to hold its load, but deforms without significant straining of the material. This is a bit of a simplification, but seems to reflect the ultimate behavior of thin ductile sections fairly well.

We are now ready to construct our first plastic bending subtable, as shown in Figure 10.

Before constructing this table, we must identify any additional limits on the element stresses beyond the stress-strain relationship. Usually, this involves imposing limits on the crippling, buckling, or lateral stability allowables of the element or section.

Suppose the inner cap of our section is supported such that lateral stability is not an issue, and the crippling allowable of the inner flange is determined to be 39,745 psi. Although our section is loaded in bending, since this flange is vertically thin, the stress level will not vary significantly across the element, so we place this value in the LIMITS column of our spreadsheet.

When subjected to bending, however, the frame web experiences a steep stress gradient. Therefore, stability of the web elements can be represented by computing the buckling allowable of a flat plate subjected to bending. The allowable computed for this web exceeds the compressive strength of the material, so no additional limit is imposed.

However, it is noted that when loaded in compression beyond the yield, the modulus of the material will begin to drop, resulting in lower stability allowables than would be computed using the elastic modulus. While these allowables can all be accurately determined and input into our limits column, a faster and more conservative method is to simply limit each element to its yield strength in compression. While this is not required, it does provide a conservative result at a considerable savings of analysis time. This approach was used for this example, and is shown in Figure 10 as well.

The next step is to determine the neutral axis of the section. Initially, this can be assumed to coincide with the centroid of the section.

Using this neutral axis, the distance from each element to the neutral axis can be computed as $y_i = Y_{NA} - y_i$. For bending producing compression on the upper flange, this will result in a positive value of $y_i$ for any element below the neutral axis, and a negative $y_i$ for any element above the neutral axis.

We then set the strain at the extreme fiber experiencing tension (the lowermost fiber) to the ultimate strain, and compute the strain in each element using Equation (18), which imposes a linear distribution of strain from this extreme fiber. This populates the 4th column of our table in Figure 10.

We are now ready to determine the stress in each element. We can do this manually by consulting a full-range stress-strain curve from [1] for each element, or we can do it automatically with an EXCEL or MATLAB or similar program acting on our 6 points of the tension and compression stress-strain curve. This produces column 5 of the table in Figure 10.

The force in each element is simply $F_i = \sigma_i A_i$, and the moment of each elemental force about the neutral axis is simply $M_i = y_i F_i$, as shown in columns 6 and 7 of Figure 10.

Before accepting the moment from the lower right corner of Figure 10 as our ultimate allowable, we must check the resultant force on the section (the summation at the bottom of column 6) against our applied force. The frame in our example is in pure bending, so the forces should sum to zero. Since the
summation performed after assuming that the neutral axis is at the centroid produces a net tension, this means that the assumed neutral axis is incorrect for this loading. This can be corrected by simply moving the neutral axis until the net tension matches our applied normal force (which is zero). If we continue to move the assumed neutral axis downward until the net tension is zero, we eventually determine that the neutral axis for this section and loading occurs at 1.522" above the lowermost fiber, and our table evolves into that shown in Figure 11.

Since the sum of our forces now equals our applied force (zero), our ultimate moment allowable is now revealed as the summation of column 7, or \( M_{ult} = 87,114 \text{ in-lb} \).

Having produced a value for the ultimate moment, it is important to reflect on the values obtained to ensure they are meaningful.

Column 1 shows the idealized numbering of each element, with 4 elements representing the web. Okay so far.

Column 2 records limits for each element. The crippling allowable was used for element 1, and the yield allowable was used as a compressive limit for all other elements. Fine.

Column 3 shows the distance from each element to the neutral axis, with negative values above the neutral axis and positive values below it. Okay.

Column 4 shows the strain in each element. Element 7 is limited by the ultimate strain (same as max elongation) of the frame material, since we used the 7075-T73 properties to approximate the skin properties. This is conservative since the skin is able to strain a good deal more than this value. The strain distribution is linear, as intended. However, the strains in elements 1 through 3 exceed the ultimate strain of our material. This should cause us to stop and consider.

This strain exceedance is numerically caused by setting a strain limit on the lower fiber, assuming a linear strain distribution, and rotating this linear strain line such that the neutral axis eventually falls on the one spot where the net tension is zero. This is correct and intended. However, the result in this case produces a strain exceeding the maximum elongation (ultimate strain) of the material at the other extreme fiber of the section. While our numerical approach is sound, this means we need to think about this.

For all three elements exceeding the ultimate strain, the exceedance is for compression. For element 1, before reaching the maximum value shown, the element cripples. This means that it holds its load, but begins to squirm increasingly out of the way without the typical increase in force. This means that deflections can become large without the usual corresponding increase in stress. This is acceptable and representative of stability behavior of the flange. Therefore this is okay for this element.

The two web elements are limited by the yield allowable. However, this yield allowable is imposed as a value representative of the stability of the web acting as a flat sheet in bending. While the stability allowable for this web appears huge when the elastic modulus is used to compute it, it quickly diminishes to something near the yield stress when the tangent modulus is used in its determination. Therefore, this limit is also representative of stability of the element, and therefore it is acceptable to conclude that the additional strain beyond the ultimate is representative of stability movement of the elements rather than compressive straining beyond the ultimate. Therefore, these values look acceptable as well.

When we consider the stresses in column 5, we see that all are within our material allowables and our stability limits. Therefore this column is fine.

The forces on each element look reasonable, as do the moments.

Therefore, our subtable looks correct and we can accept the ultimate moment from the bottom of column 7 as an accurate reflection of the ultimate bending allowable due to the tension limit.

We can now construct a similar table by imposing the ultimate compressive strain on the uppermost fiber of our section, and imposing a linear strain distribution from this value, and adjusting the slope of the linear strain curve until the neutral axis is located. This results in the subtable of Figure 12.

This reveals an ultimate moment of \( M_{ult} = 83,341 \text{ in-lb} \) due to the compressive limit. Scrutinizing the accuracy of the table as before introduces no concerns, so our ultimate moment for bending in this direction is taken as the minimum of the two values determined, or \( M_{ult} = 83,341 \text{ in-lb} \).
Repeating this exercise using the correct stress-strain curve for the skin element (not shown), we would have found ultimate moments of 87,377 in-lb for the tension limit, and of 83,455 in-lb for the compressive limit, revealing a 0.3% and 0.14% error in our previous calculation, respectively. This indicates that our judgment was sound in neglecting the differences between the two materials in our previous approach.

Therefore, the proposed approach accurately determined the ultimate moment for our section.

If instead of using the proposed approach, we had simply computed the allowable moment by inserting \( F_{tu} \) as the tension allowable of the skin into Equation (15), we would have overstated our allowable by an unconservative 30%, which is inappropriate and incorrect.

If we had instead inserted \( F_{cc} \) as the compression allowable of the inner flange into Equation (15), we would have understated our allowable by a grossly-conservative 44%, which is unnecessary and inefficient for the design.

If we had incorrectly applied Cozzone’s Method [2] to this non-symmetric section, we would have come up with an incorrect result that is 22% unconservative.

Therefore, it is concluded that the proposed method is accurate, appropriate, and the best alternative for assessing the ultimate static strength of aircraft structures.

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CONCLUSION

This paper develops a procedure to accurately and efficiently determine the ultimate allowable for a cross-section subjected to bending and axial load. The procedure effectively handles both symmetric and non-symmetric sections; it accurately evaluates stresses into the inelastic range for any material without requiring the stress-strain curve to be the same in tension and compression, and it correctly addresses stability effects.

This approach is of direct use for any ultimate-bending analysis need, and can be used for aircraft or other lightweight structural analysis where ultimate analysis is frequently performed. This approach directly addresses a shortfall in the documentation, which does not provide a clear, single procedure that accurately handles all of these effects.

REFERENCES


