Metric Dimension and a Puzzle on a Chessboard

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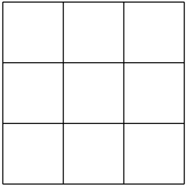
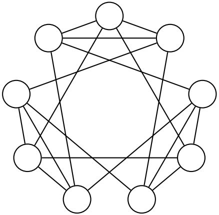
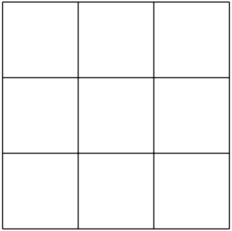
Chapter 1: The Puzzle

in which a puzzle on a chessboard is seen to be equivalent to a problem on a graph



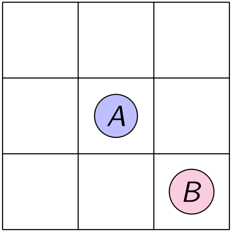


Puzzle: Location on a Chessboard



3×3 *board*

Puzzle: Location on a Chessboard



Place landmarks in some of the cells

Puzzle: Location on a Chessboard

	A	B
A	A	A, B
B	A, B	B

Label each remaining cell with list of landmarks seen from that cell, i.e., appearing in the same row or column

Puzzle: Location on a Chessboard

	A	B
A	A	A, B
B	A, B	B

Is the board resolved?

Puzzle: Location on a Chessboard

	A	B
A	A	A, B
B	A, B	B

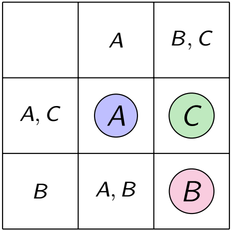
Is the board resolved? No! Some cells see the same set of landmarks.

Puzzle: Location on a Chessboard

	A	B
A	A	A, B
B	A, B	B

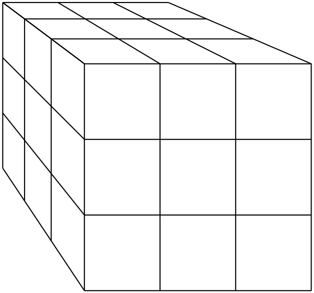
How many landmarks are needed to resolve the board?

Puzzle: Location on a Chessboard



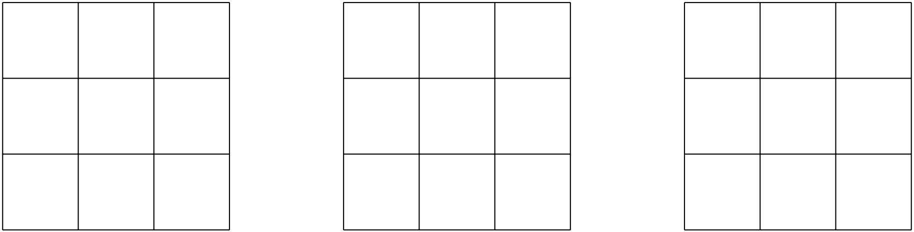
We can resolve the 3×3 *board with* 3 *landmarks, and no fewer.*

Puzzle: Location on a 3D-Chessboard



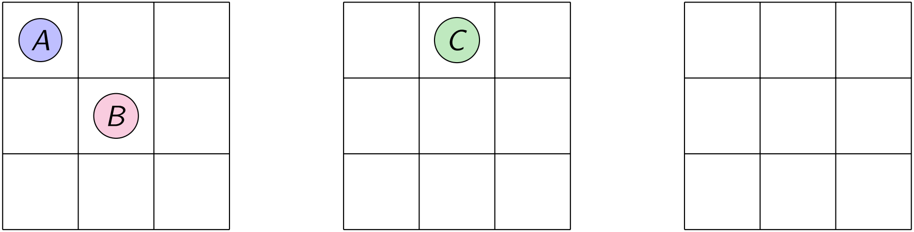
$3 \times 3 \times 3$ *board*

Puzzle: Location on a 3D-Chessboard



$3 \times 3 \times 3$ *board*

Puzzle: Location on a 3D-Chessboard



Place landmarks in some of the cells

Puzzle: Location on a 3D-Chessboard

A	A	A
A	A	A
A	A	A

A	A	A
A		
A		

A	A	A
A		
A		

Label each remaining cell with list of landmarks seen from that cell, i.e., appearing in the same row, column, or layer

Puzzle: Location on a 3D-Chessboard

A	A, B, C	A, B, C
A, B	B	A, B
A, B	A, B, C	A, B

A, C	C	A, C
A, B, C	B, C	B, C
A, C	B, C	C

A, C	A, B, C	A, C
A, B	B, C	B
A	B, C	

Is the board resolved?

Puzzle: Location on a 3D-Chessboard

A	*A*, *B*, *C*	*A*, *B*, *C*
A, *B*	*B*	*A*, *B*
A, *B*	*A*, *B*, *C*	*A*, *B*

A, *C*	*C*	*A*, *C*
A, *B*, *C*	*B*, *C*	*B*, *C*
A, *C*	*B*, *C*	*C*

A, *C*	*A*, *B*, *C*	*A*, *C*
A, *B*	*B*, *C*	*B*
A	*B*, *C*	

Is the board resolved? No! Some cells see the same set of landmarks.

Puzzle: Location on a 3D-Chessboard

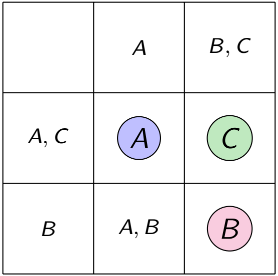
A	A, B, C	A, B, C
A, B	B	A, B
A, B	A, B, C	A, B

A, C	C	A, C
A, B, C	B, C	B, C
A, C	B, C	C

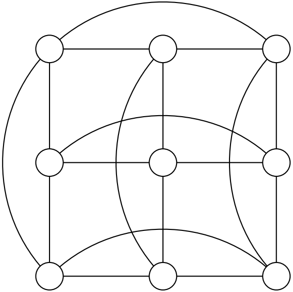
A, C	A, B, C	A, C
A, B	B, C	B
A	B, C	

How many landmarks are needed to resolve the board?

2D Puzzle and Metric Dimension of a Graph



3 × 3 puzzle



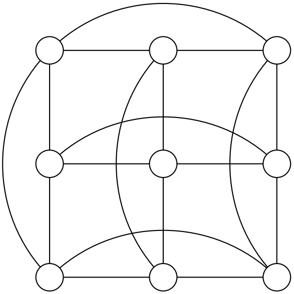
$K_3 \square K_3$

How many landmarks are needed to resolve the graph?

2D Puzzle and Metric Dimension of a Graph

222	122	211
121	A	C
212	112	B

3×3 puzzle



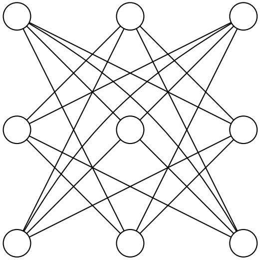
$K_3 \square K_3$

How many landmarks are needed to resolve the graph?

2D Puzzle and Metric Dimension of a Graph

111	211	122
212	*A*	*C*
121	221	*B*

3×3 puzzle



$K_3 \times K_3$

How many landmarks are needed to resolve the graph?

Metric Dimension of $K_m \square K_n$

Theorem (Cáceres, et.al., 2007)

The metric dimension of $K_m \square K_n$ is

$$\dim(K_m \square K_n) = \begin{cases} \lfloor \frac{2}{3}(m+n-1) \rfloor & \text{if } m \leq n \leq 2m-1 \\ n-1 & \text{if } n \geq 2m-1 \end{cases}$$

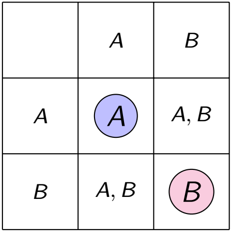
Note: For $m, n \geq 3$, the Cartesian product $K_m \square K_n$ and the direct product $K_m \times K_n$ have the same minimum resolving sets and metric dimension.

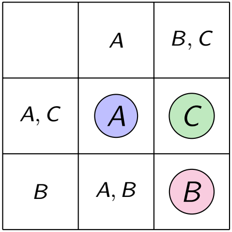
Metric Dimension of $K_m \square K_n$

Theorem (Cáceres, et al., 2007)

For $m, n \geq 2$, a placement of landmarks resolves the $m \times n$ board iff

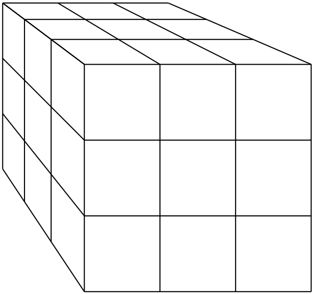
- *there is at most one empty row and at most one empty column*
- *there is at most one lonely landmark*
- *if there is an empty row and an empty column, then there is no lonely landmark*



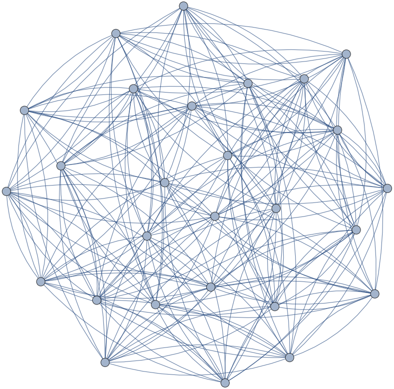


3D Puzzle and Metric Dimension of a Graph

What is the metric dimension of a direct product of three complete graphs?



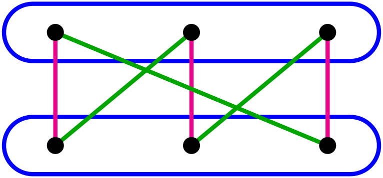
$3 \times 3 \times 3$ *board*



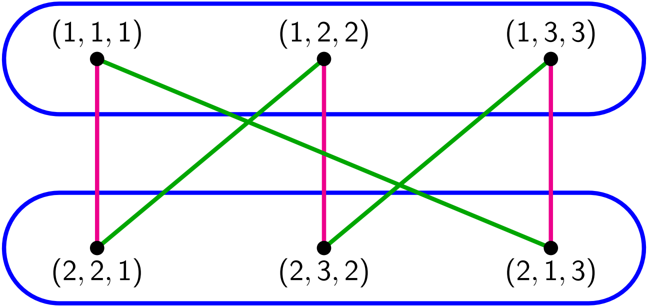
$K_3 \times K_3 \times K_3$

Chapter 2: The Landmark Hypergraph

in which we graph relationships between landmarks and use edge coloring to characterize resolving sets



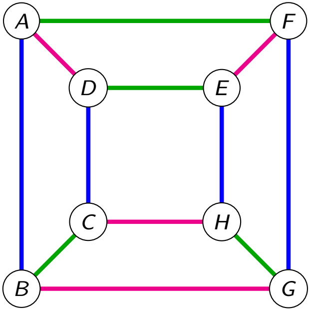
Defining the Landmark Hypergraph

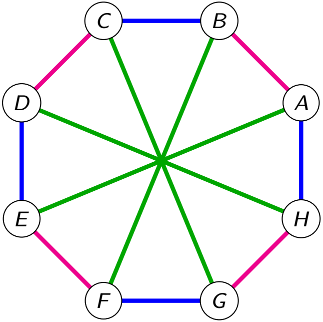


Landmark hypergraph of $W \subseteq V(K_3^3)$

Hyperedges group landmarks with a common coordinate:
blue (1st coordinate), green (2nd coordinate), pink (3rd coordinate)

Make a Guess!



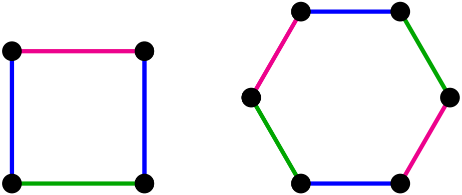


Which could be the landmark graph of a resolving set for K_4^3*?*

Forbidden Subgraph Theorem

Theorem (F-G.,Uhl, 2022)

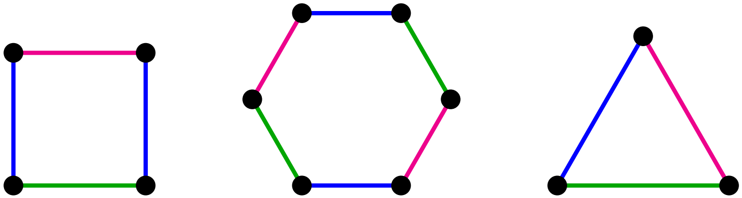
Suppose the landmark graph of $W \subseteq V(K_n^3)$ is a simple graph with n edges of each color. Then W resolves K_n^3 if and only if the landmark graph of W avoids forbidden 4-cycles and forbidden 6-cycles.



Forbidden Subgraph Theorem

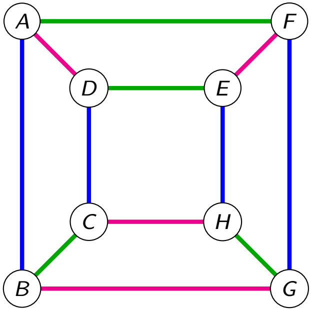
Theorem (F-G.,Uhl, 2022)

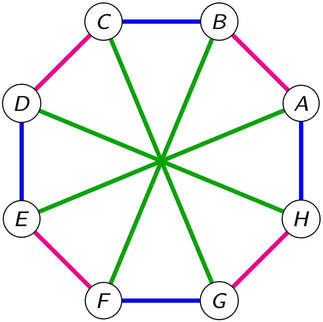
Suppose the landmark graph of $W \subseteq V(K_n^3)$ *is a simple graph with* n *edges of each color. Then* W *resolves* K_n^3 *if and only if the landmark graph of* W *avoids forbidden 4-cycles and forbidden 6-cycles.*



Bonus! If the landmark graph also avoids rainbow triangles, then $W \cup \{(n+1, n+1, n+1)\}$ resolves K_{n+1}^3.

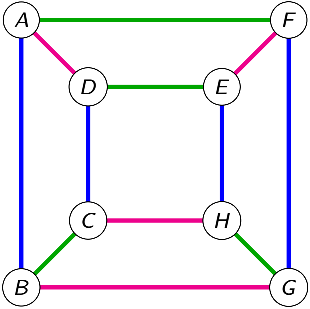
Forbidden Subgraphs

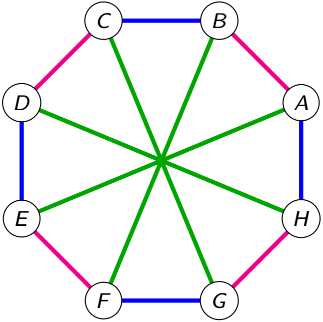




Which could be the landmark graph of a resolving set for K_4^3*?*

Forbidden Subgraphs



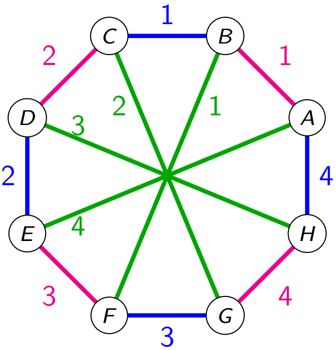


Which could be the landmark graph of a resolving set for K_4^3?
Not the cube. The Möbius graph!

Chapter 3: Metric Dimension of K_n^3

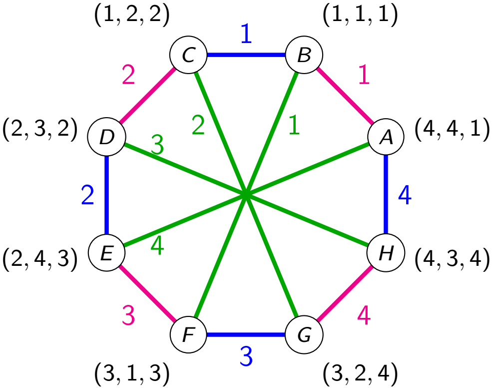
in which we construct minimum resolving sets of K_n^3
and show the metric dimension is $2n-1$ (usually)

Solving the $4 \times 4 \times 4$ Puzzle



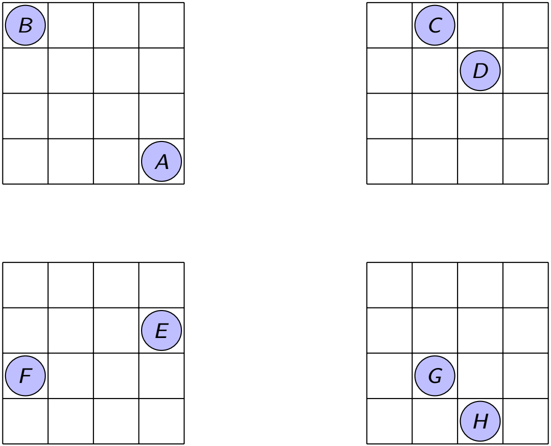
Given graph that avoids forbidden 4-cycles and forbidden 6-cycles...
Number edges of each color 1–4 and determine coordinates of landmarks.

Solving the $4 \times 4 \times 4$ Puzzle

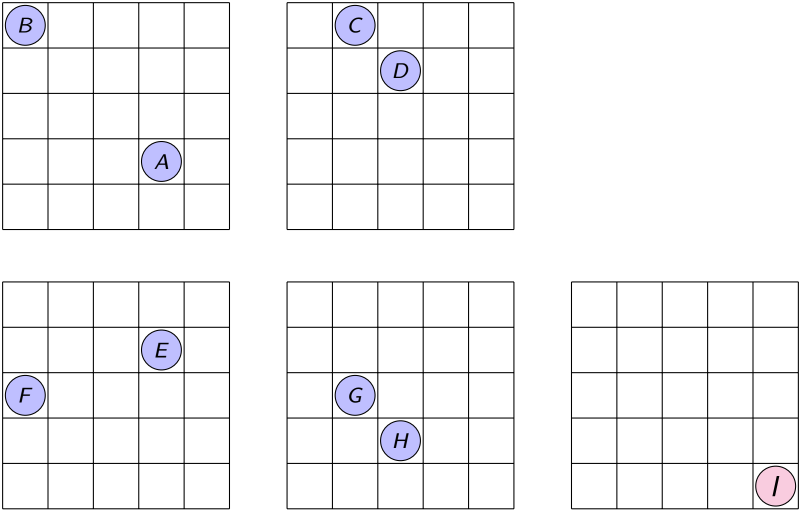


Given graph that avoids forbidden 4-cycles and forbidden 6-cycles. . .
Number edges of each color 1–4 and determine coordinates of landmarks.

Puzzle Solution ($4 \times 4 \times 4$ board)



Puzzle Solution ($5 \times 5 \times 5$ board)

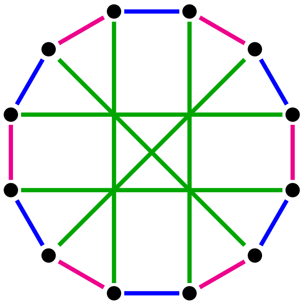


Metric Dimension of $K_n \times K_n \times K_n$

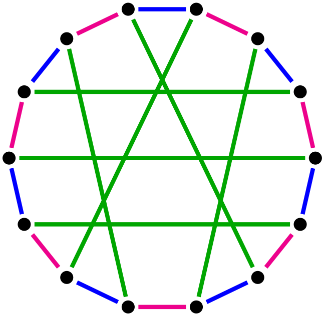
Theorem (F-G.,Uhl, 2022)

The metric dimension of the direct product $K_n^3 = K_n \times K_n \times K_n$ *is*

$$\dim(K_n^3) = \begin{cases} 2n & \text{if } n \in \{3,4\} \\ 2n-1 & \text{if } n \geq 5 \end{cases}$$

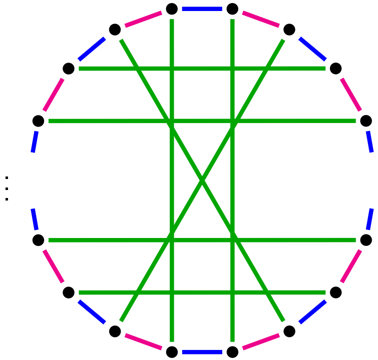


$n = 6$ (bonus $n = 7$)

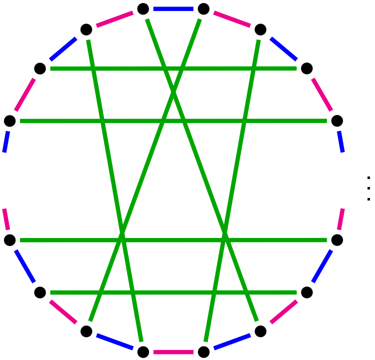


$n = 7$ (bonus $n = 8$)

Landmark Graphs of Resolving Sets



$n = 2k$ (bonus $n = 2k + 1$)



$n = 2k + 1$ (bonus $n = 2k + 2$)

Infinite family of graphs avoiding forbidden 4-cycles, forbidden 6-cycles, and rainbow triangles

Further Exploration

Current work on $n_1 \times n_2 \times n_3$ puzzle

- generalization of Forbidden Subgraph Theorem
- constructions of resolving sets

Questions

- What other equivalent combinatorial problems are there?
- Which graphs achieve lower bound $2\max(n_1, n_2, n_3) - 1$?
- What if we vary constraints on landmark hypergraph?
- What about direct product of more than three complete graphs?

Thanks!