WEIGHTED GRAPHS IN THE SENSE OF JOHN AND A GLOBAL POINCARÉ INEQUALITY

FERNANDO LÓPEZ-GARCÍA AND JOHN RODRIGUEZ

ABSTRACT. In this paper, we establish a condition on weighted graphs with finite measure that guarantees the validity of a global Poincaré inequality. This condition can be viewed as a discrete analogue of the criterion introduced by J. Boman in 1982 for Whitney cubes, which in turn characterizes the condition originally proposed by F. John in his seminal 1961 work.

1. INTRODUCTION

The class of John domains in $\mathbb{R}^n$ is the largest class of domains, under very general assumptions, for which the improved Poincaré inequality and the Sobolev-Poincaré inequality hold, together with many related results. For details, we refer the readers to [1, 4, 7, 12] and the references therein. This class of domains was introduced by Fritz John in 1961 (see [8]) via the twisted cone condition, which asserts that any point x in the domain can be connected to a fixed base point x_0 by a path such that, at any point along the path, the distance to the boundary is comparable to the distance traveled from x. In 1982, J. Boman provided in [3] an equivalent formulation in terms of chains of Whitney cubes. Later, the first author of this manuscript introduced in [11] another equivalent definition of John domains, similar to that in [3], where chains are replaced by a tree structure. In this paper, we adapt the notion from [11] to weighted graphs. This formulation will be developed in Section 3; however, for convenience, we include the definition here. Given a connected graph $G = (V, E)$, equipped with a positive vertex function $\mu : V \to \mathbb{R}$, we are interested in graphs for which there exists a rooted spanning tree (V, E_T) of G (i.e. a connected and acyclic subgraph that contains every vertex in V with a distinguished element a as its root) and a constant $c \geq 1$ such that

$$\mu(S_t) \leq c\mu(t) \quad \text{for all } t \in V, \tag{1.1}$$

where $S_t := \{s \in V : s \succeq t\}$ is the *shadow* of t. In this definition, we say that $s \succeq t$ if the unique path in the spanning tree from s to the root a contains t.

Following the Euclidean case, we show that this condition implies a global version of the Poincaré inequality. This implication is obtained through a *local-to-global* argument studied in [10], which is based on decomposition of functions. In addition, we provide an estimate for the constant appearing in the Poincaré inequality in terms of the constant in (1.1). These estimates are of particular interest for bounding the first nonzero eigenvalue of the Laplacian. This relationship is well known in both continuous and discrete settings; see [5, 6] and references therein for details.

Date: November 26, 2025.

2020 *Mathematics Subject Classification.* Primary: 26D15; Secondary: 05C75, 05C22, 05C63, 05C05.

Key words and phrases. Analysis on weighted graphs, John domains, Poincaré inequality, local-to-global, trees.

The notion of derivatives we use here is standard and defined as follows. Given a function $f: V \to \mathbb{R}$, the *length of the gradient* of f is the map $|\nabla f|: V \to \mathbb{R}_{\geq 0}$ given by

$$|\nabla f|(t) = \sum_{s:s\sim t} |f(t) - f(s)| \quad \text{for all } t \in V,$$

where $s \sim t$ indicates that the vertices s and t are adjacent in the graph $G = (V, E)$.

As a slight abuse of notation, we write (V, μ) for the weighted graph (V, E, μ).

Our main result is the following:

Theorem. *Let (V, μ) be a weighted graph that satisfies the condition in Definition 3.1, whose spanning tree has degree uniformly bounded. Also, let $p \in [1, \infty)$. Then, there exists a constant $C_P > 0$ such that the inequality*

$$\|f\|_{\ell^p(V,\mu)} \leq C_P \||\nabla f|\|_{\ell^p(V,\mu)}$$

holds for any $f \in \ell^p(V, \mu)$ that sums zero with respect to μ on V.

The paper is organized as follows. In Section 2, we study the continuity of a Hardy-type operator, which plays a central role in our approach. Section 3 is devoted to a function decomposition that allows us to pass the validity of the Poincaré inequality on edges to the entire weighted graph. Finally, in Section 4, we establish the global Poincaré inequality.

2. Some basic definitions and an averaging operator on Graphs with Rooted Spanning Trees

Let us recall some basic and well-known definitions on graphs and introduce an averaging operator that we use later.

A graph G is a pair (V, E), where V is a set of *vertices*, i.e., an arbitrary set whose elements are called vertices, and E is a set of *edges* where the elements of E are unordered pairs (x, y) of vertices $x, y \in V$. We write $x \sim y$ if $(x, y) \in E$ and say that x is *adjacent* to y. The edge (x, y) is also denoted by xy, and x, y are called the *endpoints* of this edge. In this work, we assume that the vertex set V is countable.

A *chain* in G is a finite sequence of vertices $t_0, t_1, \ldots, t_n$ such that t_i is adjacent to t_{i-1} for any $1 \leq i \leq n$.The graph $G = (V, E)$ is *connected* if for any $s, t \in V$ there exists a chain connecting s to t.

A graph G is called *locally finite* if every vertex x has finitely many adjacent vertices. The *degree* of a vertex x, denoted $\deg(x)$, is the number of edges incident to x. We say that G has *bounded degree* if there exists an integer M such that $\deg(x) \leq M$ for all $x \in V$. A graph G is called *finite* if the number of its vertices is finite.

A graph is a *tree* if it is connected and has no cycles. A *rooted tree* is a tree with a distinguished vertex a, called the *root*. If G is a rooted tree, it is possible to define a partial order $\succeq$ on V by declaring that $s \succeq t$ if the unique path from s to the root a contains t. We write $s \succ t$ if $s \succeq t$ and $s \neq t$.

A vertex k is the *parent* of a vertex t if $t \succ k$ and k is adjacent to t. For simplicity, we write $k = t_p$. Analogously, t is a *child* of k if $k = t_p$.

Finally, a *spanning tree* of a connected graph $G = (V, E)$ is a minimal connected subgraph of G that includes all vertices and has no cycles. We use the following notation:

$$G_T := (V, E_T), \tag{2.1}$$

where $E_T \subseteq E$ is the edge set of the spanning tree. In this case, we say $x \sim_T y$ if the pair $(x, y) \in E_T$. Naturally, if $x \sim_T y$ then $x \sim y$.

A weighted graph is a graph $G = (V, E)$ provided with a positive function $\mu : V \to \mathbb{R}$. By abuse of notation, we denote weighted graphs by $G = (V, \mu)$. We say that (V, μ) has *finite measure* if

$$\mu(V) = \sum_{s \in V} \mu(s) < \infty.$$

Furthermore, let $\Omega \subseteq V$. Then a function $f : V \to \mathbb{R}$ *sums zero with respect to* μ *on* Ω if

$$\sum_{s \in \Omega} f(s)\mu(s) = 0.$$

The space $\ell^q(V, \mu)$, or simply $\ell^q(V)$, for $1 \leq q < \infty$, consists of all functions $f : V \to \mathbb{R}$ such that

$$\|f\|_q := \left(\sum_{s \in V} |f(s)|^q \mu(s) \right)^{1/q} < \infty.$$

For $q = \infty$, the space $\ell^\infty(V)$ is defined as the set of all bounded functions $f : V \to \mathbb{C}$, with norm

$$\|f\|_\infty := \sup_{s \in V} |f(s)|.$$

Notice that in weighted graphs with finite measure $\ell^q(V) \subseteq \ell^1(V)$, for any $1 \leq q \leq \infty$.

Finally, let us define the following averaging operator; also known in this manuscript as a Hardy-type operator.

Definition 2.1. *Let* (V, E, μ) *be a weighted, connected graph with finite measure and a distinguished spanning tree* (V, E_T) *that induces a partial order* $\succeq$ *on the vertices. We define the following* Hardy-type operator T *on* $\ell^1(V, \mu)$

$$Tf(t) := \frac{1}{\mu(S_t)} \sum_{s : s \succeq t} |f(s)|\mu(s),$$

for all $t \in V$*, where* $S_t := \{s \in V : s \succeq t\}$ *is called the* shadow *of the vertex* t.

Remark 2.2. *Notice that this averaging operator* T *depends on the chosen spanning tree.*

The following theorems state that the operator T defined above is strong (∞, ∞) and weak $(1, 1)$ continuous. We then prove the strong (q, q) continuity of T, for $1 < q < \infty$, using the Marcinkiewicz interpolation technique from [14].

Theorem 2.3. *The Hardy-type operator* T *in Definition 2.1 is strong* (∞, ∞) *continuous, with constant* $C = 1$*. Namely,*

$$\|Tf\|_{\ell^\infty(V)} \leq \|f\|_{\ell^\infty(V)},$$

for any $f \in \ell^\infty(V)$.

Proof. This result follows from the fact that T is an averaging operator. Indeed, given a function $f \in \ell^\infty(V)$, we acquire the following

$$\begin{aligned}
\|Tf\|_{\ell^\infty(V)} &= \sup_{t \in V} |Tf(t)| \\
&= \sup_{t \in V} \frac{1}{\mu(S_t)} \sum_{s \succeq t} |f(s)|\mu(s) \\
&\leq \|f\|_{\ell^\infty(V)} \sup_{t \in V} \frac{1}{\mu(S_t)} \sum_{s \succeq t} \mu(s)
\end{aligned}$$

$$
\begin{aligned}
&= \|f\|_{\ell^\infty(V)} \sup_{t\in V} \frac{\mu(S_t)}{\mu(S_t)} \\
&= \|f\|_{\ell^\infty(V)}.
\end{aligned}
$$

Therefore, T is bounded with $\|T\|_{\ell^\infty(V)\to\ell^\infty(V)} \leq 1$, and thus strong (∞,∞) continuous. □

We proceed by proving the following lemma, which is used in the argument to conclude the weak $(1,1)$ continuity of T.

Lemma 2.4. *Let $G = (V, E)$ be a graph with a rooted spanning tree (V, E_T) that induces on V the partial order $\succeq$. Let $t_1, t_2 \in V$. If $S_{t_1} \cap S_{t_2} \neq \emptyset$, then either $S_{t_1} \subseteq S_{t_2}$ or $S_{t_2} \subseteq S_{t_1}$.*

Proof. Since $S_{t_1} \cap S_{t_2} \neq \emptyset$, there exists $t \in S_{t_1} \cap S_{t_2}$. It follows that $t \succeq t_1$ and $t \succeq t_2$. Now, let us assume, to the contrary, that $t_1 \not\succeq t_2$ and $t_2 \not\succeq t_1$. Then the path from t through t_1 to the root a is distinct from the path from t through t_2 to the root a. The existence of these two distinct paths from t to the root a implies the existence of a cycle, which contradicts the fact that the partial order is defined by a spanning tree. Therefore, either $t_1 \succeq t_2$ or $t_2 \succeq t_1$ which gives that $t_1 \in S_{t_2}$ or $t_2 \in S_{t_1}$. It follows that $S_{t_1} \subseteq S_{t_2}$ or $S_{t_2} \subseteq S_{t_1}$. □

Now, we prove the weak $(1,1)$ continuity of T.

Theorem 2.5. *The Hardy-type operator T in Definition 2.1 is weak $(1,1)$ continuous with constant $C = 1$. Namely, for all $f \in \ell^1(V,\mu)$ and $\lambda > 0$,*

$$\mu(V_{Tf}(\lambda)) < \frac{1}{\lambda}\|f\|_{\ell^1(V,\mu)},$$

where the subset of vertices $V_{Tf}(\lambda) := \{t \in V : |Tf(t)| > \lambda\}$.

Proof. Let us define the subset of *minimal vertices* $M_{Tf}(\lambda) \subseteq V_{Tf}(\lambda)$ as

$$M_{Tf}(\lambda) := \{t \in V_{Tf}(\lambda) : s \notin V_{Tf}(\lambda) \text{ for all } a \preceq s \prec t\}.$$

It follows from its definition that

$$V_{Tf}(\lambda) \subseteq \bigcup_{t\in M_{Tf}(\lambda)} S_t.$$

Consequently,

$$\mu(V_{Tf}(\lambda)) \leq \mu\left(\bigcup_{t\in M_{Tf}(\lambda)} S_t\right) = \sum_{t\in M_{Tf}(\lambda)} \mu(S_t).$$

The last identity is justified by Lemma 2.4 and by the definition of the set of minimal vertices $M_{Tf}(\lambda)$. More specifically, no two shadows of vertices in $M_{Tf}(\lambda)$ intersect.

Now, since $M_{Tf}(\lambda) \subseteq V_{Tf}(\lambda)$, we find that

$$
\begin{aligned}
\mu(V_{Tf}(\lambda)) &\leq \sum_{t\in M_{Tf}(\lambda)} \mu(S_t) \\
&< \sum_{t\in M_{Tf}(\lambda)} \mu(S_t)\frac{Tf(t)}{\lambda} \\
&= \frac{1}{\lambda}\sum_{t\in M_{Tf}(\lambda)} \sum_{s\succeq t} |f(s)|\mu(s) \\
&\leq \frac{1}{\lambda}\|f\|_{\ell^1(V,\mu)}.
\end{aligned}
$$

We conclude that T is weak $(1,1)$ continuous. $\square$

Now, let us prove the strong (q,q) continuity of T. This is commonly shown through the Marcinkiewicz interpolation technique. We have not found its discrete version in the literature, so we adapt the proof from [14] to our case.

Theorem 2.6. *The Hardy-type operator T in Definition 2.1 is strong (q,q) continuous for $1<q<\infty$. Moreover, for all $f \in \ell^q(V,\mu)$,*

$$\|Tf\|_{\ell^q(V,\mu)} \le \left(\frac{2^q q}{q-1}\right)^{1/q} \|f\|_{\ell^q(V,\mu)}.$$

Proof. Let $f \in \ell^q(V,\mu)$ and $\lambda \in (0,\infty)$. Then an application of Hölder's inequality gives

$$\begin{aligned}
\|f\|_{\ell^1(V,\mu)} &= \sum_{t\in V} |f(t)|\mu(t)^{1/q+1/p} \\
&\le \left(\sum_{t\in V} |f(t)|^q \mu(t)\right)^{1/q} \left(\sum_{t\in V} \mu(t)\right)^{1/p} \\
&= \|f\|_{\ell^q(V,\mu)} \mu(V)^{1/p},
\end{aligned} \tag{2.2}$$

where $1/q + 1/p = 1$. Thus, using that (V,μ) has finite measure, we can conclude that $\|f\|_{\ell^1(V,\mu)} < \infty$. Now, we decompose the function f into two parts:

$$f_1(t) = f(t)\chi_{V_f(\lambda/2)}(t) \quad \text{and} \quad f_2(t) = f(t)\chi_{V\setminus V_f(\lambda/2)}(t),$$

defined for all $t \in V$, where $V_f(\lambda) = \{t \in V : |f(t)| > \lambda\}$. Hence, (2.2) and the fact that f_1 is a restriction of f imply that $\|f_1\|_{\ell^1(V,\mu)} < \infty$. Then, from the weak continuity of the operator T in $\ell^1(V,\mu)$, stated in Theorem 2.5, it follows that

$$\mu(V_{Tf_1}(\lambda/2)) < \frac{2}{\lambda}\|f_1\|_{\ell^1(V,\mu)}. \tag{2.3}$$

Also notice that $\|f_2\|_{\ell^\infty(V)} \le \lambda/2 < \infty$ by the definition of f_2. So through an application of the strong continuity of the operator T in $\ell^\infty(V)$, proved in Theorem 2.3, we have

$$Tf_2(t) \le \sup_{t\in V} Tf_2(t) = \|Tf_2\|_{\ell^\infty(V)} \le \|f_2\|_{\ell^\infty(V)} \le \frac{\lambda}{2}.$$

for all $t \in V$. Hence, no vertex t satisfies that $Tf_2(t) > \lambda/2$. Hence, $\mu(V_{Tf_2}(\lambda/2)) = 0$. It is also important to note that $f = f_1 + f_2$ and T is a sublinear operator which follows from the triangle inequality. Thus,

$$Tf(t) \le Tf_1(t) + Tf_2(t),$$

for each $t \in V$. The inequality above now implies $V_{Tf}(\lambda) \subseteq V_{Tf_1}(\lambda/2) \cup V_{Tf_2}(\lambda/2)$. This fact coupled with (2.3) and that $V_{Tf_2}(\lambda/2)$ is the empty set imply that

$$\begin{aligned}
\mu(V_{Tf}(\lambda)) &\le \mu(V_{Tf_1}(\lambda/2) \cup V_{Tf_2}(\lambda/2)) \\
&\le \mu(V_{Tf_1}(\lambda/2)) + \mu(V_{Tf_2}(\lambda/2)) \\
&\le \frac{2}{\lambda}\|f_1\|_{\ell^1(V,\mu)}.
\end{aligned} \tag{2.4}$$

Observe that for all $\lambda \in (0,\infty)$, $x^q = \int_0^\infty q\lambda^{q-1}\chi_{\{x>\lambda\}}(x,\lambda)\,\mathrm{d}\lambda$ where $\chi_{\{x>\lambda\}}(x,\lambda)$ is the characteristic function over the set $\{(x,\lambda) \in \mathbb{R}^2_{>0} : x > \lambda\}$. Indeed, we have

$$\int_0^\infty q\lambda^{q-1}\chi_{\{x>\lambda\}}(x,\lambda)\,\mathrm{d}\lambda = \int_0^x q\lambda^{q-1}\,\mathrm{d}\lambda = x^q.$$

Notice in the following lines, we have a countable sum and an integral of positive functions so we can interchange them. Thus,

$$\begin{aligned}
\|Tf\|^q_{\ell^q(V,\mu)} &= \sum_{t\in V} |Tf(t)|^q \mu(t) \\
&= \sum_{t\in V} \left(\int_0^\infty q\lambda^{q-1}\chi_{\{Tf(t)>\lambda\}}(t,\lambda)\,\mathrm{d}\lambda \right) \mu(t) \\
&= \int_0^\infty q\lambda^{q-1} \left(\sum_{t\in V} \chi_{\{Tf(t)>\lambda\}}(t,\lambda)\mu(t) \right) \mathrm{d}\lambda \\
&= \int_0^\infty q\lambda^{q-1} \left(\sum_{t:Tf(t)>\lambda} \mu(t) \right) \mathrm{d}\lambda \\
&= \int_0^\infty q\lambda^{q-1}\mu(V_{Tf}(\lambda))\,\mathrm{d}\lambda.
\end{aligned}$$

Recall that $\|f_1\|_{\ell^1(V,\mu)} < \infty$. Hence, the following integral and sum, which arise from (2.4), may be interchanged, allowing us to achieve our final estimate:

$$\begin{aligned}
\|Tf\|^q_{\ell^q(V,\mu)} = \int_0^\infty q\lambda^{q-1}\mu(V_{Tf}(\lambda))\,\mathrm{d}\lambda &\le \int_0^\infty q\lambda^{q-1} \left(\frac{2}{\lambda}\|f_1\|_{\ell^1(V,\mu)} \right) \mathrm{d}\lambda \\
&= 2\int_0^\infty q\lambda^{q-2} \left(\sum_{t\in V} |f(t)|\chi_{V_f(\lambda/2)}(t,\lambda)\mu(t) \right) \mathrm{d}\lambda \\
&= 2\sum_{t\in V} |f(t)| \left(\int_0^\infty q\lambda^{q-2}\chi_{\{|f(t)|>\lambda/2\}}(t,\lambda)\,\mathrm{d}\lambda \right) \mu(t) \\
&= 2\sum_{t\in V} |f(t)| \left(\int_0^{2|f(t)|} q\lambda^{q-2}\,\mathrm{d}\lambda \right) \mu(t)
\end{aligned}$$

where we now obtain

$$\begin{aligned}
\|Tf\|^q_{\ell^q(V,\mu)} &\le \frac{2q(2^{q-1})}{(q-1)} \sum_{t\in V} |f(t)||f(t)|^{q-1}\mu(t) \\
&= \frac{2^q q}{q-1} \sum_{t\in V} |f(t)|^q \mu(t) \\
&= 2^q \frac{q}{q-1} \|f\|^q_{\ell^q(V,\mu)}.
\end{aligned}$$

Therefore, we achieve the bound on the operator T given by

$$\|Tf\|_{\ell^q(V,\mu)} \le 2\left(\frac{q}{q-1}\right)^{1/q} \|f\|_{\ell^q(V,\mu)}.$$

Hence, the Hardy-type operator T is strong (q, q) continuous for $q \in (1, \infty)$. $\square$

The continuity of this operator, which is fundamental in this local-to-global methodology, was also studied in [13] for different weights.

3. A Decomposition of Functions Technique

In this work, we use a local-to-global argument to prove the validity of a certain Poincaré-type inequality on weighted graphs, relying on the fact that they are valid on segments or edges. This technique requires writing functions with vanishing mean value on the entire graph as sums of functions supported on segments that also have vanishing mean value. The *continuity* of this decomposition depends on the following *geometric* condition on the graph:

Definition 3.1. *Let $G = (V, \mu)$ be a weighted, connected, and locally finite graph with a summable weight function $\mu : V \to \mathbb{R}_{>0}$. In this work, we are interested in graphs G for which there exist a rooted spanning tree and a positive constant c such that*

$$\mu(S_t) \leq c\mu(t) \quad \textit{for all } t \in V, \tag{3.1}$$

where $S_t := \{s \in V : s \succeq t\}$ is the shadow *of t.*

This geometric definition on graphs is inspired by the equivalent definition of John domains proved in [11] on Euclidean domains.

It is also important to note that weighted graphs (V, μ) satisfying condition (3.1) have finite measure, since

$$\mu(V) = \mu(S_a) \leq c\mu(a) < \infty,$$

where a denotes the root of the spanning tree.

Also, notice that any finite weighted graph (V, μ) satisfies (3.1) with a constant

$$c = \frac{\mu(V)}{\min_{t \in V} \mu(t)}.$$

Next, we present two weighted graphs: the first one satisfies (3.1), whereas the second does not.

Examples 3.2. *Let (V, μ) be a weighted rooted k-ary tree (i.e. a tree where each vertex has k children), where the weight is defined by $\mu(t) = \alpha^{\mathrm{dist}(t,a)}$, for every $t \in V$, for some $0 < \alpha < 1/k$. Here, $dist(t, a)$ denotes the number of edges in the chain that connects t to the root a. Then (V, μ) is John. Indeed, for any $t \in V$, we have the following estimate:*

$$\frac{\mu(S_t)}{\mu(t)} = \frac{\sum_{s \in S_t} \mu(s)}{\mu(t)} = \frac{\sum_{s \succeq t} \alpha^{\mathrm{dist}(s,a)}}{\alpha^{\mathrm{dist}(t,a)}} = \sum_{s \succeq t} \alpha^{\mathrm{dist}(s,a) - \mathrm{dist}(t,a)} = \sum_{s \succeq t} \alpha^{\mathrm{dist}(s,t)}$$

where the last line holds since $s \succeq t$ and therefore contains t on its unique path to the root a. Now, (V, μ) is a k-ary tree, so each vertex has k children. Namely, we can continue the estimation in the following way:

$$\frac{\mu(S_t)}{\mu(t)} = \sum_{s \succeq t} \alpha^{\mathrm{dist}(s,t)} \leq \sum_{n \geq 0} k^n \alpha^n = \sum_{n \geq 0} (k\alpha)^n = \frac{1}{1 - k\alpha}.$$

So k-ary trees, with the weight introduced above, verify condition (3.1) *with a constant estimated by* $1/(1 - k\alpha)$.

Examples 3.3. *Given a bounded domain $\Omega \subset \mathbb{R}^n$ and a Whitney decomposition $\{Q_t\}_{t\in V}$, we define the graph $G = (V, E)$, where the vertices correspond to the Whitney cubes, and two vertices t and s are adjacent in G if and only if Q_t and Q_s intersect along an $(n-1)$-dimensional face of one of them. We refer the reader to [15] for details on the existence and properties of Whitney cubes. Furthermore, we define the weight $\mu : V \to \mathbb{R}$ as the Lebesgue measure of each Whitney cube (i.e. $\mu(t) := |Q_t|$).*

Now, it was shown in [11] that if Ω is a John domain in the sense of Fritz John there exists a Whitney decomposition and a spanning tree (V, E_T) of the graph of Whitney cubes such that

$$Q_s \subseteq KQ_t$$

for any $s, t \in V$, with $s \succeq t$, where KQ_t is a K-dilation of Q_t. Hence, using that Whitney cubes have disjoint interiors, we conclude that

$$\begin{aligned}\mu(S_t) &= \sum_{s\succeq t} \mu(s) = \sum_{s\succeq t} |Q_s| \\ &= \left|\bigcup_{s\succeq t} Q_s\right| \leq |KQ_t| = K^n|Q_t| = K^n\mu(t).\end{aligned}$$

Therefore, this graph satisfies the condition in Definition 3.1 with constant $c = K^n$.

As we mentioned, every weighted graph that satisfies the condition in Definition 3.1 has finite measure. However, not every weighted graph of finite measure satisfies condition (3.1). Let us show an example of this fact.

Examples 3.4. *Let (V, μ) be the weighted path graph on $V = \mathbb{N}_{\geq 2}$, where adjacency is given by consecutive integers. The weight is defined by $\mu(n) = n^{-1}\ln(n)^{-\gamma}$ for all $n \in \mathbb{N}_{\geq 2}$, where $\gamma > 1$.*

Now, let us consider the function $f(x) = x^{-1}\ln(x)^{-\gamma}$. Observe that $f(x)$ is continuous, positive, and decreasing in $[2, \infty)$. Since this holds, the integral test implies that

$$\mu(V) = \mu(\mathbb{N}_{\geq 2}) = \sum_{n\geq 2} \mu(n) = \sum_{n\geq 2} \frac{1}{n(\ln(n))^\gamma} < \infty.$$

So (V, μ) has a finite measure. Now, let $n \in \mathbb{N}_{\geq 2}$ and observe that

$$\frac{\mu(S_n)}{\mu(n)} = \frac{\sum_{k\geq n} \frac{1}{k(\ln(k))^\gamma}}{\frac{1}{n(\ln(n))^\gamma}} = n(\ln(n))^\gamma \sum_{k\geq n} \frac{1}{k(\ln(k))^\gamma}.$$

The integral of the function $f(x)$ previously mentioned provides a lower estimate for the sum above in the following way:

$$\begin{aligned}\frac{\mu(S_n)}{\mu(n)} &= n(\ln(n))^\gamma \sum_{k\geq n} \frac{1}{k(\ln(k))^\gamma} \\ &\geq n(\ln(n))^\gamma \int_n^\infty \frac{1}{x(\ln(x))^\gamma}\, \mathrm{d}x \\ &= n(\ln(n))^\gamma \frac{1}{(\gamma-1)(\ln(n))^{\gamma-1}} \\ &= \frac{n\ln(n)}{\gamma - 1}.\end{aligned}$$

Hence, $\mu(S_n)/\mu(n)$ is not uniformly bounded and (V, μ) does not satisfy (3.1).

Now, let us define the decomposition of functions that we use in this local-to-global argument to prove the $\ell^p(V,\mu)$-Poincaré inequality for $1 \le p < \infty$.

Definition 3.5. *Let (V,μ) be a weighted graph with finite measure with respect to μ, and let it be equipped with a rooted spanning tree that defines a partial order $\succeq$. Given $f \in \ell^1(V,\mu)$, we introduce the collection of functions $\{f_t\}_{t\in V^*}$, where $V^* = V \setminus \{a\}$. For each $t \in V^*$, the function $f_t : V \to \mathbb{R}$ is defined by*

$$f_t(s) := \begin{cases} 0 & s \neq t \text{ and } s \neq t_p, \\ f(t) + \dfrac{1}{\mu(t)} \displaystyle\sum_{k \succ t} f(k)\mu(k) & s = t, \\ -\dfrac{1}{\mu(t_p)} \displaystyle\sum_{k \succeq t} f(k)\mu(k) & s = t_p, \end{cases} \tag{3.2}$$

for any $s \in V$.

Observe that this collection of functions is well defined since $f \in \ell^1(V,\mu)$. The following results state some other properties that the functions in (3.2) verify.

Theorem 3.6. *Let (V,μ) be a weighted graph satisfying the assumptions stated in Definition 3.5. Now, let $f \in \ell^1(V,\mu)$, a function with sum zero with respect to μ on V. Then the functions f_t in the collection $\{f_t\}_{t\in V^*}$ presented in (3.2) describe a decomposition of f supported on the segments $\{t,t_p\}$ such that $f = \sum_{t\in V^*} f_t$ and f_t sums zero with respect to μ on V for every $t \in V^*$.*

Proof. Notice that f_t is indeed supported on the segment $\{t,t_p\}$ for each $t \in V^*$. Now, let s be a vertex other than the root. Then, by using (3.2), we attain the following

$$\begin{aligned} \sum_{t\in V^*} f_t(s) &= \sum_{t:s=t \text{ or } s=t_p} f_t(s) \\ &= f_s(s) + \sum_{t:s=t_p} f_t(s) \\ &= f(s) + \frac{1}{\mu(s)} \sum_{k\succ s} f(k)\mu(k) + \sum_{t:s=t_p} \left(-\frac{1}{\mu(s)} \sum_{k\succeq t} f(k)\mu(k) \right). \end{aligned}$$

Furthermore, the sum over all vertices t such that $s = t_p$ converges since the shadow of each vertex t is disjoint and $f \in \ell^1(V,\mu)$. We now derive the following

$$\begin{aligned} \sum_{t\in V^*} f_t(s) &= f(s) + \frac{1}{\mu(s)} \sum_{k\succ s} f(k)\mu(k) - \frac{1}{\mu(s)} \sum_{t:s=t_p} \sum_{k\succeq t} f(k)\mu(k) \\ &= f(s) + \frac{1}{\mu(s)} \sum_{k\succ s} f(k)\mu(k) - \frac{1}{\mu(s)} \sum_{k\succ s} f(k)\mu(k) \\ &= f(s). \end{aligned}$$

If s is the root, that is $s = a$, then $f_t(a) = 0$ for each t that is not a child of the root a. Additionally,

$$\sum_{t\in V^*} f_t(s) = \sum_{t:a=t_p} f_t(a)$$

$$
\begin{aligned}
&= \sum_{t:a=t_p} \left(-\frac{1}{\mu(a)} \sum_{k\succeq t} f(k)\mu(k) \right) \\
&= -\frac{1}{\mu(a)} \sum_{t:a=t_p} \sum_{k\succeq t} f(k)\mu(k) \\
&= -\frac{1}{\mu(a)} \sum_{k\succ a} f(k)\mu(k) \\
&= f(a),
\end{aligned}
$$

where the last line holds since f has sum zero with respect to μ on V. Whence, we conclude that f can be decomposed by f_t for $t \in V^*$. We now aim to show that each f_t sums zero with respect to μ on V. To that end, let $t \in V^*$. Then

$$
\begin{aligned}
\sum_{s\in V} f_t(s)\mu(s) &= \sum_{s:s=t \text{ or } s=t_p} f_t(s)\mu(s) \\
&= \left(f(t) + \frac{1}{\mu(t)} \sum_{k\succ t} f(k)\mu(k) \right) \mu(t) - \left(\frac{1}{\mu(t_p)} \sum_{k\succeq t} f(k)\mu(k) \right) \mu(t_p) \\
&= \sum_{k\succeq t} f(k)\mu(k) - \sum_{k\succeq t} f(k)\mu(k) = 0,
\end{aligned}
$$

as desired. □

Notice that the previous proof does not require the validity of condition (3.1); however, this condition is used in the proof of the following result. The next corollary follows from the continuity of the Hardy-type operator T.

Corollary 3.7. *Let (V,μ) be a weighted graph that satisfies* (3.1)*, whose spanning tree has degree bounded by M. Then, for every $g \in \ell^q(V,\mu)$, with $q \in (1,\infty)$, the collection of functions $\{g_t\}_{t\in V^*}$ defined in (3.2) satisfies the following estimate:*

$$
\sum_{t\in V^*} \|g_t\|^q_{\ell^q(V,\mu)} \le c^q M \frac{2^q q}{q-1} \|g\|^q_{\ell^q(V,\mu)}, \tag{3.3}
$$

where c is the constant that appears in Definition 3.1.

Proof. We begin our estimation argument as follows:

$$
\sum_{t\in V^*} \|g_t\|^q_{\ell^q(V,\mu)} = \sum_{t\in V^*} |g_t(t)|^q \mu(t) + |g_t(t_p)|^q \mu(t_p), \tag{3.4}
$$

where we wish to estimate both $g_t(t)$ and $g_t(t_p)$ by the Hardy-type operator in Definition 2.1. The strong (q,q) continuity of T implies the final estimation. Thus, we proceed as follows:

$$
|g_t(t)| = \left| g(t) + \frac{1}{\mu(t)} \sum_{k\succ t} g(k)\mu(k) \right| = \left| \frac{1}{\mu(t)} \sum_{k\succeq t} g(k)\mu(k) \right| \le \frac{\mu(S_t)}{\mu(t)} \frac{1}{\mu(S_t)} \sum_{k\succeq t} |g(k)|\mu(k).
$$

Then $|g_t(t)| \le cTg(t)$, where c is the constant in Definition 3.1. Similarly, we give a bound for $|g_t(t_p)|$:

$$
|g_t(t_p)| = \left| -\frac{1}{\mu(t_p)} \sum_{k\succeq t} g(k)\mu(k) \right| \le \frac{\mu(S_{t_p})}{\mu(t_p)} \frac{1}{\mu(S_{t_p})} \sum_{k\succeq t_p} |g(k)|\mu(k).
$$

Hence, $|g_t(t_p)| \leq cTg(t_p)$. Now, observe that Theorem 2.6 and the fact that $g \in \ell^q(V,\mu)$ implies that $Tg \in \ell^q(V,\mu)$. Hence, from (3.4) we can conclude that

$$\begin{aligned}
\sum_{t\in V^*} \|g_t\|^q_{\ell^q(V,\mu)} &\leq c^q \sum_{t\in V^*} |Tg(t)|^q \mu(t) + c^q \sum_{t\in V^*} |Tg(t_p)|^q \mu(t_p) \\
&= c^q \sum_{s\in V^*} |Tg(s)|^q \mu(s) + c^q \sum_{s\in V} |Tg(s)|^q \mu(s) \#\{t \in V^* \colon t_p = s\} \\
&= c^q M \sum_{s\in V} |Tg(s)|^q \mu(s) \\
&\leq c^q M \frac{2^q q}{q-1} \|g\|^q_{\ell^q(V,\mu)},
\end{aligned}$$

which concludes the proof. □

4. A global weighted Poincaré Inequality on Graphs with Rooted Spanning Trees

In this section, we prove the validity of a global $\ell^p(V,\mu)$-Poincaré inequality on weighted graphs (V,μ), that satisfy the condition in Definition 3.1, via a local-to-global argument. This technique requires us to prove that the $\ell^p(V,\mu)$-Poincaré inequality is satisfied locally. In particular, we prove that it is satisfied on any edge of a general weighted graph (V,μ). We prove this fact since the collection of decomposed functions $\{g_t\}_{t\in V^*}$ in Definition 3.5 is supported exactly on the segments $\{t, t_p\} \subseteq V$ for a spanning tree in the graph. Through a dual argument, we are able to utilize the decomposition and thus the validity of the local Poincaré inequality to prove the global Poincaré inequality.

Before we lead to the proofs of the local and global $\ell^p(V,\mu)$-Poincaré inequalities, we recall the notion of gradient that we use in this work.

Definition 4.1. *Given a locally finite graph $G = (V,E)$ and a function $f : V \to \mathbb{R}$, we define the* length of the gradient *of f as the function $|\nabla f| : V \to \mathbb{R}_{\geq 0}$ defined by*

$$|\nabla f|(t) = \sum_{s:s\sim t} |f(s) - f(t)| \quad \textit{for all } t \in V. \tag{4.1}$$

This is a standard notion on analysis on graphs and it plays the role that gradients play in analysis on metric spaces [9]. Now, we denote the average of a function $f : V \to \mathbb{R}$ on a subset of vertices $\Omega \subseteq V$ with finite measure by

$$f_\Omega = \frac{1}{\mu(\Omega)} \sum_{t\in\Omega} f(t)\mu(t).$$

We begin our discussion by proving that a $\ell^p(V,\mu)$-Poincaré inequality is satisfied locally on the segments $\{t, t_p\} \subseteq V$.

Lemma 4.2. *Let $G = (V,\mu)$ be a weighted graph with a distinguished spanning tree that defines on V a partial order $\preceq$. Now, given a function $f \in \ell^p(V,\mu)$, $p \in [1,\infty)$, that sums zero with respect to μ on every segment $\{t, t_p\} \subseteq V$ for certain $t \in V$, it follows that*

$$\|f\|_{\ell^p(\{t,t_p\},\mu)} \leq \| |\nabla(f|_{\{t,t_p\}})| \|_{\ell^p(\{t,t_p\},\mu)}.$$

Proof. Let $t \in V$ and suppose that the hypotheses are satisfied. Since f sums zero with respect to μ on $\{t, t_p\}$, we have $f(t)\mu(t) + f(t_p)\mu(t_p) = 0$. Now the weight function μ is positive, so without loss of generality, $f(t) > 0$ and $f(t_p) < 0$. This implies that $|f(t)| \leq |f(t) - f(t_p)|$ and $|f(t_p)| \leq |f(t) - f(t_p)|$. Therefore, we can attain the following

$$\begin{aligned}
\|f\|^p_{\ell^p(\{t,t_p\},\mu)} &= |f(t)|^p\mu(t) + |f(t_p)|^p\mu(t_p) \\
&\leq |f(t) - f(t_p)|^p\mu(t) + |f(t) - f(t_p)|^p\mu(t_p) \\
&= ||\nabla(f|_{\{t,t_p\}})|(t)|^p\mu(t) + ||\nabla(f|_{\{t,t_p\}})|(t_p)|^p\mu(t_p) \\
&= \||\nabla(f|_{\{t,t_p\}})|\|^p_{\ell^p(\{t,t_p\},\mu)}
\end{aligned}$$

as desired. □

Theorem 4.3. *Let (V, μ) be a weighted graph that satisfies the condition in Definition 3.1, whose spanning tree has degree bounded by M. Also, let $p \in [1, \infty)$. Then, there exists a constant $C_P > 0$ such that the inequality*

$$\|f\|_{\ell^p(V,\mu)} \leq C_P \||\nabla f|\|_{\ell^p(V,\mu)} \tag{4.2}$$

holds for any $f \in \ell^p(V, \mu)$ that sums zero with respect to μ on V. Furthermore, the constant in the Poincaré inequality is upper bounded by a multiple of the geometric constant in (3.1). *Indeed,*

$$\begin{aligned}
&C_P \leq c\, M\, 2p^{1-1/p} && \textit{if } p \in (1, \infty) \\
&C_P \leq c\, 2 && \textit{if } p = 1,
\end{aligned}$$

where c is the constant that appears in Definition 3.1.

Proof. First, let us consider the case $p > 1$. We estimate $\|f\|_{\ell^p(V,\mu)}$ by a dual argument. That is, we have

$$\|f\|_{\ell^p(V,\mu)} = \sup_{\|g\|_{\ell^q(V,\mu)} \leq 1} \sum_{s\in V} f(s)g(s)\mu(s),$$

where $1/p + 1/q = 1$.

Furthermore, the fact that f sums zero with respect to μ on V implies that

$$\sum_{s\in V} f(s)(g(s) - g_V)\mu(s) = \sum_{s\in V} f(s)g(s)\mu(s) - g_V \sum_{s\in V} f(s)\mu(s) = \sum_{s\in V} f(s)g(s)\mu(s).$$

Thus,

$$\|f\|_{\ell^p(V,\mu)} = \sup_{\|g\|_{\ell^q(V,\mu)} \leq 1} \sum_{s\in V} f(s)(g(s) - g_V)\mu(s),$$

where $g - g_V$ sums zero with respect to μ on V. Moreover, notice that

$$\begin{aligned}
\|g - g_V\|_{\ell^q(V,\mu)} &\leq \|g\|_{\ell^q(V,\mu)} + \|g_V\|_{\ell^q(V,\mu)} = \|g\|_{\ell^q(V,\mu)} + |g_V|\mu(V)^{1/q} \\
&\leq \|g\|_{\ell^q(V,\mu)} + \mu(V)^{1/q-1} \sum_{s\in V} |g(s)|\mu(s) \\
&\leq \|g\|_{\ell^q(V,\mu)} + \mu(V)^{1/q-1}\mu(V)^{1/p}\|g\|_{\ell^q(V,\mu)} \leq 2\|g\|_{\ell^q(V,\mu)} \leq 2.
\end{aligned}$$

Therefore, using Theorem 3.6, we can decompose $g - g_V$ according to (3.2). For simplicity, we write the decomposed functions as g_t for $t \in V^*$. Thus,

$$\|f\|_{\ell^p(V,\mu)} = \sup_{\|g\|_{\ell^q(V,\mu)} \leq 1} \sum_{s\in V} f(s) \left(\sum_{t\in V^*} g_t(s) \right) \mu(s) = \sup_{\|g\|_{\ell^q(V,\mu)} \leq 1} \sum_{s\in V} \sum_{t\in V^*} f(s)g_t(s)\mu(s). \tag{4.3}$$

Notice that the double sum stated above is absolutely convergent. In fact, we use the support of each function g_t and (3.3):

$$\begin{aligned}\sum_{s\in V}\sum_{t\in V^*}|f(s)||g_t(s)|\mu(s) &= \sum_{t\in V^*}\sum_{\substack{s:s=t \text{ or}\\ s=t_p}}|f(s)||g_t(s)|\mu(s)\\ &\le \left(\sum_{t\in V^*}\sum_{\substack{s:s=t \text{ or}\\ s=t_p}}|f(s)|^p\mu(s)\right)^{1/p}\left(\sum_{t\in V^*}\sum_{\substack{s:s=t \text{ or}\\ s=t_p}}|g_t(s)|^q\mu(s)\right)^{1/q}\\ &= M^{1/p}\|f\|_{\ell^p(V,\mu)}\left(\sum_{t\in V^*}\|g_t(s)\|^q_{\ell^q(V,\mu)}\right)^{1/q}.\end{aligned}$$

Observe in the middle line that each term $|f(s)|^p\mu(s)$, with $s\in V$ such that $s=t$ or $s=t_p$ for some $t\in V^*$, appears multiple times. Actually, the number of repetitions equals the number of children plus its parent if there is one. Therefore, the double sum in (4.3) converges absolutely, and we may interchange the sums. Now, note that $f_{\{t,t_p\}}$ is the average of f on the segment $\{t,t_p\}$ and so $f-f_{\{t,t_p\}}$ sums zero with respect to μ on the segment $\{t,t_p\}$. With these notes, we proceed from (4.3) as follows

$$\begin{aligned}\|f\|_{\ell^p(V,\mu)} &= \sup_{\|g\|_{\ell^q(V,\mu)}\le 1}\sum_{t\in V^*}\sum_{s\in V}f(s)g_t(s)\mu(s)\\ &= \sup_{\|g\|_{\ell^q(V,\mu)}\le 1}\sum_{t\in V^*}f(s)g_t(s)\mu(s)\\ &= \sup_{\|g\|_{\ell^q(V,\mu)}\le 1}\sum_{t\in V^*}\sum_{\substack{s:s=t \text{ or}\\ s=t_p}}(f(s)-f_{\{t,t_p\}})g_t(s)\mu(s)\\ &\le \sup_{\|g\|_{\ell^q(V,\mu)}\le 1}\sum_{t\in V^*}\|f-f_{\{t,t_p\}}\|_{\ell^p(\{t,t_p\},\mu)}\|g_t\|_{\ell^q(\{t,t_p\},\mu)}. \qquad (4.4)\end{aligned}$$

The third equality holds because g_t has zero sum with respect to μ on V by Theorem 3.6; hence, g_t also has zero sum with respect to μ on the segment $\{t,t_p\}$. The last line follows from Hölder's inequality. Now, using Lemma 4.2, we obtain

$$\begin{aligned}|\nabla((f-f_{\{t,t_p\}})\big|_{\{t,t_p\}})|(s) &= |(f(t)-f_{\{t,t_p\}})-(f(t_p)-f_{\{t,t_p\}})|\\ &= |f(t)-f(t_p)| = |\nabla(f|_{\{t,t_p\}})|(s),\end{aligned}$$

for $s=t$ or $s=t_p$. Henceforth, using the local Poincaré inequality in Lemma 4.2, Hölder's inequality and Corollary 3.7, we can conclude that

$$\begin{aligned}\|f\|_{\ell^p(V,\mu)} &\le \sup_{\|g\|_{\ell^q(V,\mu)}\le 1}\sum_{t\in V^*}\||\nabla(f|_{\{t,t_p\}})|\|_{\ell^p(\{t,t_p\},\mu)}\|g_t\|_{\ell^q(\{t,t_p\},\mu)}\\ &\le \sup_{\|g\|_{\ell^q(V,\mu)}\le 1}\left(\sum_{t\in V^*}\||\nabla(f|_{\{t,t_p\}})|\|^p_{\ell^p(\{t,t_p\},\mu)}\right)^{1/p}\left(\sum_{t\in V^*}\|g_t\|^q_{\ell^q(\{t,t_p\},\mu)}\right)^{1/q}\\ &\le cM^{1/q}\left(\frac{2^q q}{q-1}\right)^{1/q}\left(\sum_{t\in V^*}\||\nabla(f|_{\{t,t_p\}})|\|^p_{\ell^p(\{t,t_p\},\mu)}\right)^{1/p}\end{aligned}$$

$$
\begin{aligned}
&= cM^{1/q}\left(\frac{2^q q}{q-1}\right)^{1/q}\left(\sum_{t\in V^*}|f(t)-f(t_p)|^p\mu(t)+|f(t)-f(t_p)|^p\mu(t_p)\right)^{1/p}\\
&= cM^{1/q}\left(\frac{2^q q}{q-1}\right)^{1/q}\left(\sum_{k\in V}\sum_{s:s\sim_T k}|f(s)-f(k)|^p\mu(k)\right)^{1/p}\\
&\le cM^{1/q}\left(\frac{2^q q}{q-1}\right)^{1/q}\left(\sum_{k\in V}M\left(\sum_{s:s\sim_T k}|f(s)-f(k)|\right)^p\mu(k)\right)^{1/p}\\
&\le cM\left(\frac{2^q q}{q-1}\right)^{1/q}\left(\sum_{k\in V}\left(|\nabla f|(k)\right)^p\mu(k)\right)^{1/p}.
\end{aligned}
$$

Recall that $s\sim_T k$ means that s and k are adjacent in the spanning tree (V,E_T) that appears in Definition 2.1. The last estimate follows from the fact that $s\sim_T k$ implies $s\sim k$. Actually, we prove a slightly stronger result where the length of the gradient in the original graph (V,E) is replaced by that on the spanning tree (V,E_T), which is smaller since $E_T\subseteq E$. Hence, this completes the proof for $1<p<\infty$.

Finally, consider the case $p=1$. We begin as we did in the case $p>1$ and estimate $\|f\|_{\ell^1(V,\mu)}$ by a dual argument. That is, consider the following

$$
\|f\|_{\ell^1(V,\mu)}=\sup_{\|g\|_{\ell^\infty(V)}\le 1}\sum_{s\in V}f(s)g(s)\mu(s)=\sup_{\|g\|_{\ell^\infty(V)}\le 1}\sum_{s\in V}f(s)(g(s)-g_V)\mu(s),
$$

where, similarly to the first case, Hölder's inequality and the fact that f sums zero with respect to μ on V imply that the two sums above are equal. Observe that $g-g_V$ sums zero with respect to μ on V as in the previous case. Thus, according to Theorem 3.6, we can decompose $g-g_V$ following (3.2) and writing g_t for simplicity. Hence, we have

$$
\|f\|_{\ell^1(V,\mu)}=\sup_{\|g\|_{\ell^\infty(V)}\le 1}\sum_{s\in V}f(s)\left(\sum_{t\in V^*}g_t(s)\right)\mu(s)=\sup_{\|g\|_{\ell^\infty(V)}\le 1}\sum_{s\in V}\sum_{t\in V^*}f(s)g_t(s)\mu(s).
$$

Next, since the double sum converges absolutely (as in the case $p>1$), we may interchange the order of summation. That is,

$$
\begin{aligned}
\|f\|_{\ell^1(V,\mu)} &= \sup_{\|g\|_{\ell^\infty(V)}\le 1}\sum_{t\in V^*}\sum_{s\in V}f(s)g_t(s)\mu(s)\\
&= \sup_{\|g\|_{\ell^\infty(V)}\le 1}\sum_{t\in V^*}\sum_{\substack{s:s=t \text{ or}\\ s=t_p}}f(s)g_t(s)\mu(s)\\
&= \sup_{\|g\|_{\ell^\infty(V)}\le 1}\sum_{t\in V^*}\sum_{\substack{s:s=t \text{ or}\\ s=t_p}}(f(s)-f_{\{t,t_p\}})g_t(s)\mu(s)\\
&\le \sup_{\|g\|_{\ell^\infty(V)}\le 1}\sum_{t\in V^*}\|f-f_{\{t,t_p\}}\|_{\ell^1(\{t,t_p\},\mu)}\|g_t\|_{\ell^\infty(\{t,t_p\})},
\end{aligned}
$$

where the third line holds since g_t sums zero with respect to μ on V and thus on the segment $\{t,t_p\}$. We now invoke Lemma 4.2 to continue this string of inequalities by writing

$$
\|f\|_{\ell^1(V,\mu)}\le\sup_{\|g\|_{\ell^\infty(V)}\le 1}\sum_{t\in V^*}\||\nabla(f|_{\{t,t_p\}})|\|_{\ell^1(\{t,t_p\},\mu)}\|g_t\|_{\ell^\infty(V)}
$$

$$\leq \sup_{\|g\|_{\ell^\infty(V)}\leq 1} \left(\sum_{t\in V^*} \| |\nabla(f|_{\{t,t_p\}})| \|_{\ell^1(\{t,t_p\},\mu)} \right) \left(\sup_{t\in V^*} \|g_t\|_{\ell^\infty(V)} \right).$$

Before we proceed, we must estimate $\sup_{t\in V^*}\|g_t\|_{\ell^\infty(V)}$. We note that $|g_t(s)| \leq c|T(g - g_V)(s)|$ for all $s \in V$. Hence,

$$\|g_t\|_{\ell^\infty(V)} \leq c\|T(g-g_V)\|_{\ell^\infty(V)} \leq c\|g-g_V\|_{\ell^\infty(V)} \leq c(\|g\|_{\ell^\infty(V)} + \|g_V\|_{\ell^\infty(V)}),$$

where the second inequality follows from Theorem 2.3, the strong (∞,∞) continuity of the Hardy operator T. Furthermore,

$$\|g_t\|_{\ell^\infty(V)} \leq c\left(\|g\|_{\ell^\infty(V)} + \frac{1}{\mu(V)}\sum_{s\in V} g(s)\mu(s) \right) \leq c(2\|g\|_{\ell^\infty(V)}).$$

This holds for all $t \in V^*$, so we adjust our estimate of $\|f\|_{\ell^1(V,\mu)}$ as follows

$$\begin{aligned}
\|f\|_{\ell^1(V,\mu)} &\leq c \sup_{\|g\|_{\ell^\infty(V)}\leq 1} \left(\sum_{t\in V^*} \| |\nabla(f|_{\{t,t_p\}})| \|_{\ell^1(\{t,t_p\},\mu)} \right) \left(2\|g\|_{\ell^\infty(V)}\right) \\
&\leq 2c \sum_{t\in V^*} \Big(|\nabla(f|_{\{t,t_p\}})|(t)\mu(t) + |\nabla(f|_{\{t,t_p\}})|(t_p)\mu(t_p) \Big) \\
&= 2c \sum_{t\in V^*} \left(|f(t)-f(t_p)|\mu(t) + |f(t_p)-f(t)|\mu(t_p) \right) \\
&= 2c \sum_{k\in V} \sum_{s: s\sim_T k} |f(s)-f(k)|\mu(k) \\
&\leq 2c \sum_{k\in V} \sum_{s: s\sim k} |f(s)-f(k)|\mu(k) \\
&= 2c \sum_{t\in V} |\nabla f(t)|\mu(t) = 2c\| |\nabla f| \|_{\ell^1(V,\mu)}.
\end{aligned}$$

Hence, $C_P \leq 2c$ in the special case when $p = 1$, and the proof is complete. □

Remark 4.4. *Notice that the weighted k-ary trees described in Example 3.2, where the weight depends on $\alpha < 1/k$, satisfy the condition in Definition 3.1 with constant $\frac{1}{1-k\alpha}$. Therefore, we can assert that the global $\ell^p(V,\mu)$-Poincaré inequality stated in Theorem 4.3 is satisfied for $1 \leq p < \infty$ and the constant can be estimated by*

$$C_P \leq \begin{cases} \dfrac{2\,(k+1)\,p^{1-1/p}}{1-k\alpha} & \text{if } p > 1, \\[2ex] \dfrac{2}{1-k\alpha} & \text{if } p = 1. \end{cases}$$

Remark 4.5. *Theorem 4.3 is shown in [2, Lemma 2.2], where the vertices of the graph are the natural numbers and consecutive numbers are adjacent. This version of the discrete Poincaré inequality was used to establish the validity of a certain weighted version of Korn's inequality on bounded domains in $\mathbb{R}^n$ with a boundary singularity.*

Remark 4.6. *A natural problem that arises after this theorem is determining whether the validity of the Poincaré inequality stated in* (4.2) *implies* (3.1). *An analogous question was studied by S. Buckley and P. Koskela in [4], who proved that, under general geometric assumptions on domains in $\mathbb{R}^n$, the validity of the Sobolev–Poincaré inequality implies that the domain is a John domain.*

ACKNOWLEDGEMENTS

The first author gratefully acknowledges Prof. Ojea, the Departamento de Matemática of the Universidad de Buenos Aires, and the Instituto de Investigaciones Matemáticas Luis A. Santaló (IMAS, CONICET–UBA) for their warm hospitality during his 2025 sabbatical visit, when this manuscript was partially written.

REFERENCES

[1] Gabriel Acosta, Ricardo G. Durán, and María A. Muschietti. Solutions of the divergence operator on John domains. *Adv. Math.*, 206(2):373–401, 2006.

[2] Gabriel Acosta and Ignacio Ojea. Korn's inequalities for generalized external cusps. *Math. Methods Appl. Sci.*, 39(17):4935–4950, 2016.

[3] Jan Boman. L^p-estimates for very strongly elliptic systems. Technical Report Report No. 3, Department of Mathematics, University of Stockholm, 1982.

[4] S. Buckley and P. Koskela. Sobolev-Poincaré implies John. *Math. Res. Lett.*, 2(5):577–593, 1995.

[5] Fan R. K. Chung. *Spectral graph theory*, volume 92 of *CBMS Regional Conference Series in Mathematics*. Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1997.

[6] Alexander Grigor'yan. *Introduction to Analysis on Graphs*. University Lecture Series. American Mathematical Society, 2018.

[7] Ritva Hurri-Syrjänen. An improved Poincaré inequality. *Proc. Amer. Math. Soc.*, 120(1):213–222, 1994.

[8] Fritz John. Rotation and strain. *Comm. Pure Appl. Math.*, 14:391–413, 1961.

[9] M. Levi, F. Santagati, A. Tabacco, and M. Vallarino. Poincaré Inequalities on Graphs. *Anal. Math.*, 49(2):529–544, 2023.

[10] Fernando López-García. A Decomposition Technique for Integrable Functions with Applications to the Divergence Problem. *Journal of Mathematical Analysis and Applications*, 418:1–19, 08 2013.

[11] Fernando López-García. Weighted Korn inequality on John domains. *Studia Mathematica*, 241:1–23, 12 2016.

[12] Fernando López-García. Weighted generalized Korn inequalities on John domains. *Math. Methods Appl. Sci.*, 41(17):8003–8018, 2018.

[13] Fernando López-García and Ignacio Ojea. Weighted Discrete Hardy Inequalities on Trees and Applications. *Potential Analysis*, 59:1–29, 01 2022.

[14] Tuan Anh Nguyen. An L^p comparison, $p \in (1, \infty)$, on the Finite Differences of a Discrete Harmonic Function at the Boundary of a Discrete Box. *Potential Analysis*, 56:1–56, 12 2020.

[15] Elias M. Stein. *Singular integrals and differentiability properties of functions*, volume No. 30 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1970.

DEPARTMENT OF MATHEMATICS AND STATISTICS, CALIFORNIA STATE POLYTECHNIC UNIVERSITY POMONA, POMONA, CA 91768, US
Email address: `fal@cpp.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WA 98195, US
Email address: `jrodri7@uw.edu`