The state covariance matrix P

$$X = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \begin{matrix} \rightarrow \text{position} \\ \rightarrow \text{velocity} \end{matrix}$$

Initial displacement $X_0 = 50\ m$

Initial velocity $\dot{X}_0 = V_0 = 5\ m/s$

Acceleration $a_0 = 2\ m/s$

process variation - standard deviation

$\sigma_x = 0.5\ m \Rightarrow \sigma_x^2 = (0.5)^2 = 0.25$

$\sigma_{\dot{x}} = 0.2\ m/sec. \Rightarrow \sigma_{\dot{x}}^2 = (0.2)^2 = 0.04$

Covariance $\sigma_x \sigma_{\dot{x}} = (0.5)(0.2) = 0.1$

The stat covariance matrix P is given by:

$$P = \begin{bmatrix} \sigma_x^2 & \sigma_{x\dot{x}} \\ \sigma_{\dot{x}x} & \sigma_{\dot{x}}^2 \end{bmatrix}$$

$$P = \begin{bmatrix} 0.25 & 0.1 \\ 0.1 & 0.04 \end{bmatrix}$$

If the estimate error for the one variable x (position) is completely independent of other variable $\dot{x}$ (velocity) Then the covariance elements are zero.

$$P = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.04 \end{bmatrix}$$

Example

Tracking an aerial vehicle.

For simulating noise in the model of the system only. It is not used in Kalman Filter algorithm

Initial state

X_o

P_o

Previous state

X_{k-1}

P_{k-1}

New predicted state

(1) $X_{kp} = F X_{k-1} + B u_k + w_k$

(2) $P_{kp} = F P_{k-1} F^T + Q_k$

(3) process covariance matrix

Includes the noise (w_k) for the Kalman filter algorithm.

(4) $K = \dfrac{P_{kp} H^T}{H P_{kp} H^T + R}$

(5) $Y_k = C Y_{km} + Z_m$

measurement

(6) $X_k = X_{kp} + K [Y - H X_k]$

updating with new measurement and Calculating Kalman gain K and calculating the new state X_k

(7) $P_k = (I - KH) P_{kp}$

X_k

(8) current value becomes previous

The given values are :

Initial velocity $V_{ox} = 280$ m/s

Initial position $X_o = 4000$ m

Measured data for position and velocity using the sensors:

Observation

$X_0 = 4000\ m$ $\qquad V_{0x} = 280\ m/s$

$X_1 = 4260\ m$ $\qquad V_{1x} = 282\ m/s$

$X_2 = 4550\ m$ $\qquad V_{2x} = 285\ m/s$

$X_3 = 4860\ m$ $\qquad V_{3x} = 286\ m/s$

$X_4 = 5110\ m$ $\qquad V_{4x} = 290\ m/s$

Initial conditions:

$a_x = 2\ m/s^2$ $\qquad \Delta t = 1\ sec.$

$V_x = 280\ m/s$ $\qquad \Delta x = 25\ m$

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process errors in process covariace matrix

Process $\rightarrow$ predicted model (mathematical model)

$\Delta P_x = 20\ m$

$\Delta P_{V_x} = 5\ m/s$

observation errors (measurement errors)

$$\Delta x = 25 \ m$$

$$\Delta V_x = 6 \ m/s$$

① The predicted state

$$X_{kp} = F X_{k-1} + B u_k + w_k$$

Start with zero in this example

$$= \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ V_{x_0} \end{bmatrix} + \begin{bmatrix} \frac{1}{2}\Delta t^2 \\ \Delta t \end{bmatrix} [a_{x_0}] + 0$$

will add later

$$X_{kp} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4000 \\ 280 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} [2] + 0$$

$$= \begin{bmatrix} 4281 \\ 282 \end{bmatrix}$$

② The initial process covariance matrix

$$\Delta x = 20 \ m$$

$$\Delta V_x = 5 \ m/s$$

$$P_{k-1} = \begin{bmatrix} \Delta x^2 & \Delta x \Delta V \\ \Delta x \Delta V & \Delta V^2 \end{bmatrix} = \begin{bmatrix} 20^2 & 20 \times 5 \\ 20 \times 5 & 5^2 \end{bmatrix}$$

$$P_{k-1} = \begin{bmatrix} 400 & 100 \\ 100 & 25 \end{bmatrix}$$

If the position and velocity errors are independent of each other we can assume $\Delta x \, \Delta V = 0$, therefore:

$$P_{k-1} = \begin{bmatrix} 400 & 0 \\ 0 & 25 \end{bmatrix}$$

3 The predicted process covariance matrix

$$P_{kp} = F P_{k-1} F^T + Q_k$$

zero for now

we will add later in the next example

$$= \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 400 & 0 \\ 0 & 25 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \Delta t & 1 \end{bmatrix} + 0$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 400 & 0 \\ 0 & 25 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 400 & 25 \\ 0 & 25 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 425 & 25 \\ 25 & 25 \end{bmatrix} \longrightarrow \begin{bmatrix} 425 & 0 \\ 0 & 25 \end{bmatrix}$$

assumption: Independent error between position and velocity

④ calculate the Kalman Gain

$$K = \frac{P_{Kp} H^T}{H P_{Kp} H^T + R}$$

$$= \frac{\begin{bmatrix} 425 & 0 \\ 0 & 25 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 425 & 0 \\ 0 & 25 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 625 & 0 \\ 0 & 36 \end{bmatrix}}$$

$$= \frac{\begin{bmatrix} 425 & 0 \\ 0 & 25 \end{bmatrix}}{\begin{bmatrix} 1050 & 0 \\ 0 & 61 \end{bmatrix}} = \begin{bmatrix} 425/1050 & 0 \\ 0 & 25/61 \end{bmatrix}$$

$$K = \begin{bmatrix} 0.405 & 0 \\ 0 & 0.410 \end{bmatrix}$$

5 The new observation (measurment by the sensor)

$$Y_K = C\, Y_{\cancel{K}m} + Z_K$$

zero for this example (will add in the next example)

$$Y_K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4260 \\ 282 \end{bmatrix} + 0$$

6 caculating the current state

$$X_K = X_{Kp} + K\left(Y_K - H X_{Kp}\right)$$

$$X_K = \begin{bmatrix} 4281 \\ 282 \end{bmatrix} + \begin{bmatrix} 0.405 & 0 \\ 0 & 0.410 \end{bmatrix} \left(\begin{bmatrix} 4260 \\ 282 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4281 \\ 282 \end{bmatrix} \right)$$

we had:

$$\left\{ \begin{array}{l} K = \begin{bmatrix} 0.405 & 0 \\ 0 & 0.410 \end{bmatrix} \\[1em] X_{Kp} = \begin{bmatrix} 4281 \\ 282 \end{bmatrix} \\[1em] Y_K = \begin{bmatrix} 4260 \\ 282 \end{bmatrix} \end{array} \right.$$

$$X_K = \begin{bmatrix} 4281 \\ 282 \end{bmatrix} + \begin{bmatrix} 0.405 & 0 \\ 0 & 0.410 \end{bmatrix} \begin{bmatrix} -21 \\ 0 \end{bmatrix}$$

$$X_K = \begin{bmatrix} 4272.5 \\ 282 \end{bmatrix}$$

7. updating the process covariance matrix

$$P_K = (I - KH) P_{Kp}$$

$$P_K = \left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \underbrace{\begin{bmatrix} 0.405 & 0 \\ 0 & 0.410 \end{bmatrix}}_{K} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right] \underbrace{\begin{bmatrix} 425 & 0 \\ 0 & 25 \end{bmatrix}}_{P_{Kp}}$$

$$P_K \approx \begin{bmatrix} 253 & 0 \\ 0 & 14.8 \end{bmatrix}$$

8. The current state becomes the previous state

$$X_p = \begin{bmatrix} 4272.5 \\ 282 \end{bmatrix} \xRightarrow{\text{Becomes}} X_{K-1}$$

$$P_K = \begin{bmatrix} 253 & 0 \\ 0 & 148 \end{bmatrix} \qquad P_{K-1}$$

$$X_{Kp} = F X_{K-1} + B u_K + w_K$$

$$P_{Kp} = F P_{K-1} F^T + Q_R$$

1-2 Repeat calculations

$$X_{Kp} = F X_{K-1} + B u_K + w_K$$

$$= \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4272.5 \\ 282 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}\Delta t^2 \\ \Delta t \end{bmatrix} [2] + 0$$

$$= \begin{bmatrix} 4555.5 \\ 284 \end{bmatrix}$$

$$P_{Kp} = F P_{K-1} F^T + Q_R$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 253 & 0 \\ 0 & 14.8 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + 0$$

$$= \begin{bmatrix} 257.8 & 0 \\ 0 & 14.8 \end{bmatrix}$$

④ Repeat Kalman gain Calculation

$$K = \frac{P_{kp} H^T}{H P_{kp} H^T + R}$$

$$K = \frac{\begin{bmatrix} 267.8 & 0 \\ 0 & 14.8 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 267.8 & 0 \\ 0 & 14.8 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 625 & 0 \\ 0 & 36 \end{bmatrix}}$$

$$K = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.291 \end{bmatrix}$$

Repeating ⑤ - ⑥ :

$$Y_K = C Y_K + Z_K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4550 \\ 285 \end{bmatrix} + 0$$

$$Y_K = \begin{bmatrix} 4550 \\ 285 \end{bmatrix}$$

current state matrix:

$$X_K = X_{Kp} + K[Y_K - H X_{pK}]$$

$$X_K = \begin{bmatrix} 4555.5 \\ 284 \end{bmatrix} + \begin{bmatrix} 0.3 & 0 \\ 0 & 0.291 \end{bmatrix} \left(\begin{bmatrix} 4550 \\ 285 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4555.5 \\ 284 \end{bmatrix} \right)$$

$$X_K = \begin{bmatrix} 4553.8 \\ 284.3 \end{bmatrix}$$

Repeat 7 & 8

$$P_K = (I - KH) P_{Kp}$$

$$P_K = \begin{bmatrix} 187.5 & 0 \\ 0 & 10.5 \end{bmatrix}$$

Becomes P_{K-1} In the next step of calculation

$$X_K = \begin{bmatrix} 4553.8 \\ 284.3 \end{bmatrix} \xrightarrow{\text{Becomes}} X_{K-1}$$

$$X_{Kp} = \begin{bmatrix} 4839.1 \\ 2863 \end{bmatrix}$$

$$P_{Kp} = F P_{K-1} F^T + Q_K$$

$$= \begin{bmatrix} 187.5 & 0 \\ 0 & 10.5 \end{bmatrix}$$

$$K = \begin{bmatrix} 0.231 & 0 \\ 0 & 0.226 \end{bmatrix}$$

$$X_{Kp} = \begin{bmatrix} 4839.1 \\ 286.3 \end{bmatrix}$$

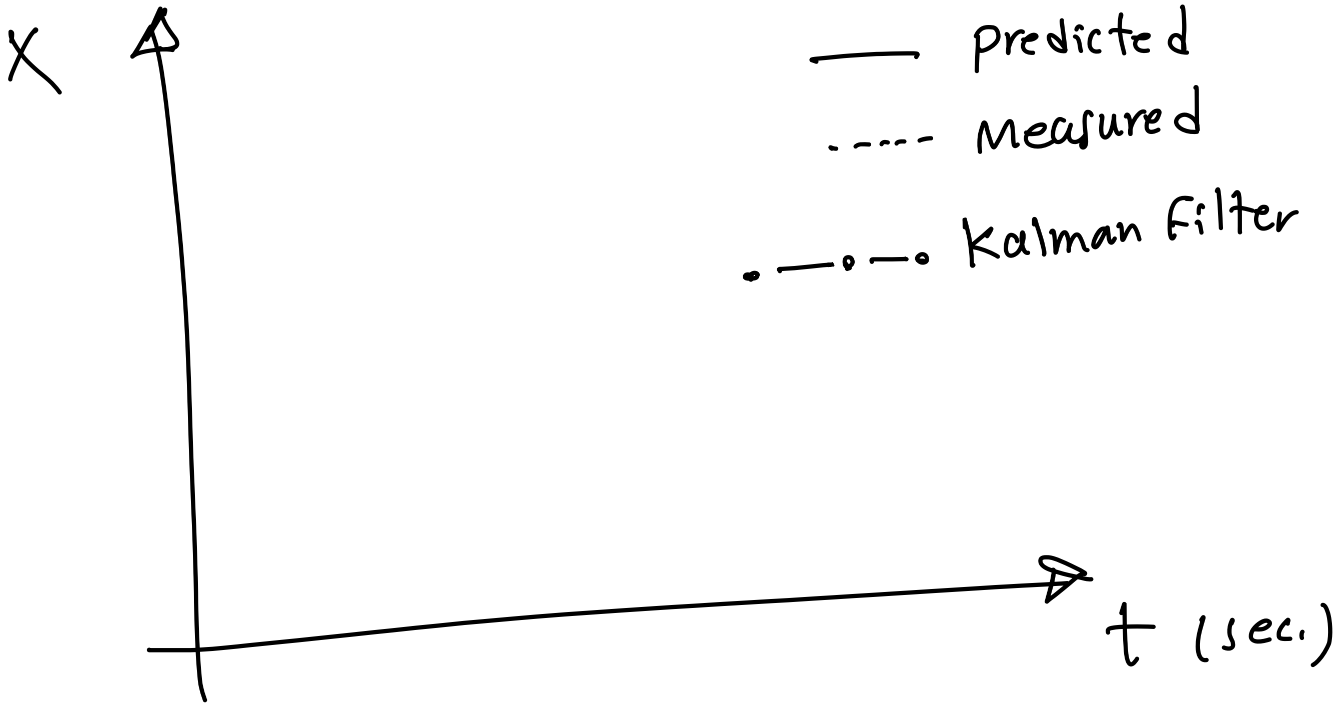
$$P_{Kp} = \begin{bmatrix} 187.5 & 0 \\ 0 & 10.5 \end{bmatrix}$$

$$K = \begin{bmatrix} 0.231 & 0 \\ 0 & 0.226 \end{bmatrix}$$

$$Y_K = \begin{bmatrix} 4860 \\ 286 \end{bmatrix}$$

$$X_K = \begin{bmatrix} 4843.9 \\ 286.2 \end{bmatrix}$$

$$P_K = \begin{bmatrix} 144.2 & 0 \\ 0 & 8.1 \end{bmatrix}$$



x
predicted
Measured
Kalman filter
t (sec.)