Linear Quadratic Gaussian (LQG) Controller

A Linear Quadratic Gaussian (LQG) Controller is an optimal control strategy for linear systems with Gaussian (random) noise and incomplete state information.

LQG combines

- Linear Quadratic Regulator (LQR) - optimal state feedback control

- Kalman Filter - optimal state estimation

Therefore,

$$LQG = LQR + \text{Kalman Filter}$$

Most real systems:

- have noise in measurements (e.g. sensor noise on a drone altimeter)
- have noise in dynamics (The mathematical model is not exact) - for example it may not model wind gusts, or modeling errors)
- cannot measure all states directly. (for example, an accelerometer does not measure position, or a GPS does not measure acceleration).
- require optimal performance with minimal error and input control effort (energy for the actuators).

LQG provides optimal control and optimal state estimation under Gaussian noise.

system Model

The LQG framework assumes a linear time-invariant (LTI) system described by:

state dynamics:

$$x_{k+1} = F x_k + B u_k + w_k$$

Transition matrix

Control Input

process noise

Measurements

$$y_k = C x_k + v_k$$

measurement noise

measurements

state vector

Noise Covariance matrices:

Q_w → process noise covariance in the mathematical model which is the variance of w_k

R_v → Measurement noise covariance matrix, which is the variance of v_k

Linear Quadratic Regulator (LQR)

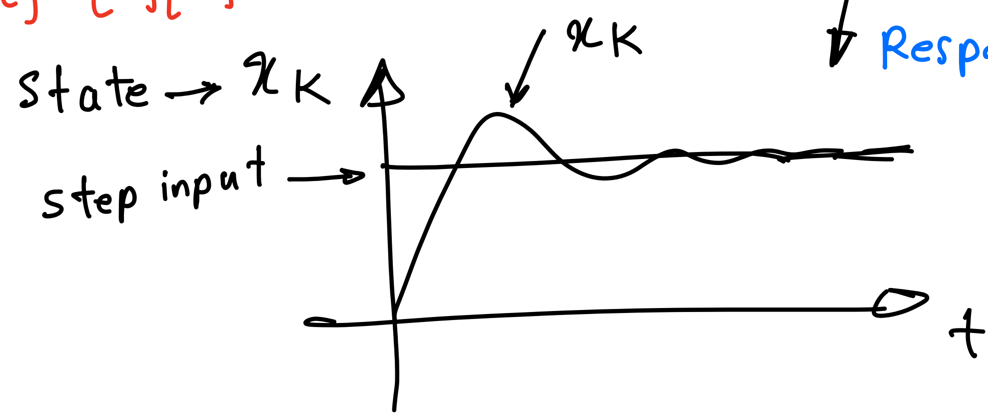
LQR assumes full state availability and minimizes the cost function:

$$J = \sum_{K=0}^{\infty} \left(x_K^T Q x_K + u_K^T R u_K \right)$$

Response

Control effort (from the actuator)

$[\dot{x}] = [A][x] + [B][u]$



Q : penalizes state deviation (error is state)

R : penalizes control effort

The optimal control law is:

$$u_K = -\mathbf{K} x_K$$

Contoller gain found from solving the cost function J

where K is obtained using Riccati equation.

Kalman Filter

The Kalman Filter estimates the states optimally under Gaussian noise:

$$\hat{x}_{K+1} = F\hat{x}_K + Bu_K + L_K(y_K - C\hat{x}_K)$$

where:

- $\hat{x}_K$ is the estimated state
- L_K is th Kalman gain
- $(y_K - C\hat{x}_K)$ is the difference between measurement value (from the sensor) and the estimated value (from the mathematical model). It is refered to as innovation.

LQG control law:

LQR gives:

$$u_K = -K x_K$$

we use the Kalman estimate:

$$u_K = -\underbrace{K}_{\text{from LQR}} \underbrace{\hat{x}_K}_{\text{from Kalman Filter}}$$

The LQR and Kalman Filter can be designed independently (separation principle)

- The LQR control law does not depend on noise
- The estimator (Kalman Filter) does not depend on the cost function (control cost)

Advantages of LQG

- Handles noise optimaly.
- works with unmeasured states.
- Optimal trade-off between accuracy and input energy.
- Modular design via separation principle

Limitations

- Assumes linear dynamics (LQG may not work well for highly nonlinear systems; use Extended Kalman Filter and nonlinear Model Predictive Control to account for nonlinearity)
- Sensitive to modeling errors (poor model → poor performance)

- For robustness use other techniques such as LQG with Loop Transfer Recovery or H_∞. (Robust control methods).

LQG is used in control systems for drones, robotics, mechanical systems, aerospace industrial control.