Quaternion part 1

Can be used to find the orientation of a vehicle in space without the gimbal lock problem.

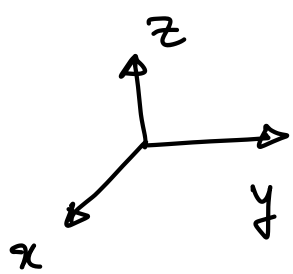
Gimbal lock problem

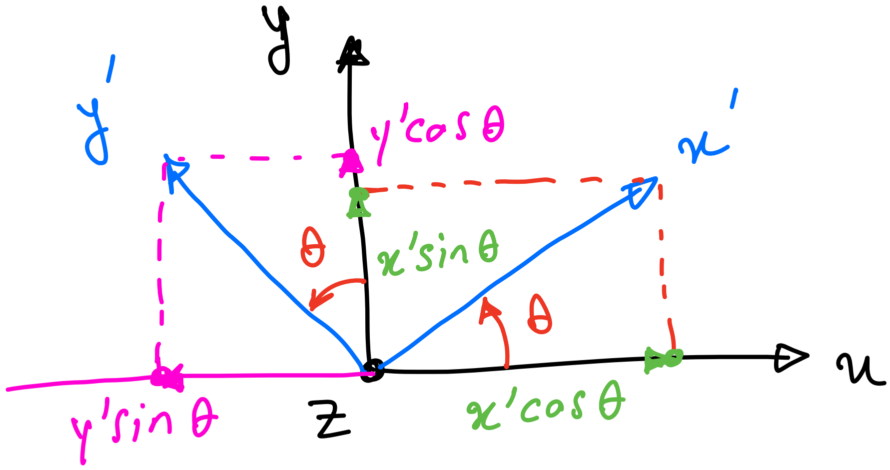
If we use Euler angle rotation matrices, there is a potentially a problem caused by Gimbal lock.

Euler angle rotations

Introduced in the dynamics lecture and briefly explained here.

Rotation about z axis:





$$x = x'\cos\theta - y'\sin\theta + (0)z$$

$$y = x'\sin\theta + y'\cos\theta + (0)z$$

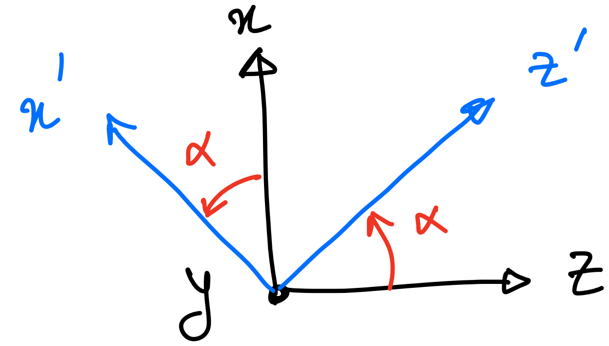
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

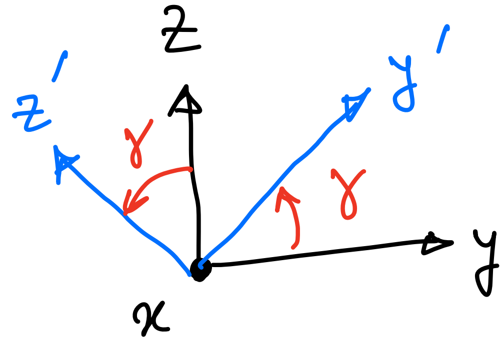
In 3D rotation about z-axis: $z = z'$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

Rotation about y-axis

Rotation about x-axis

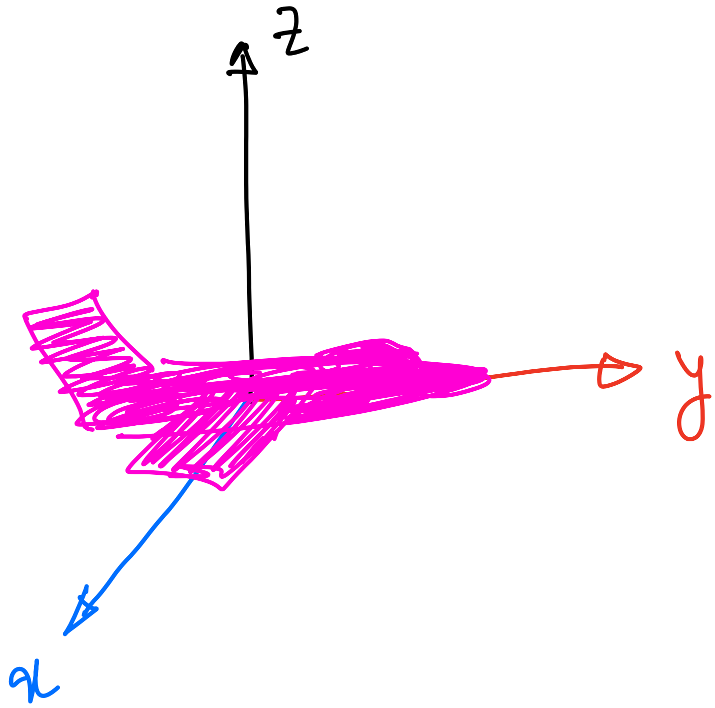




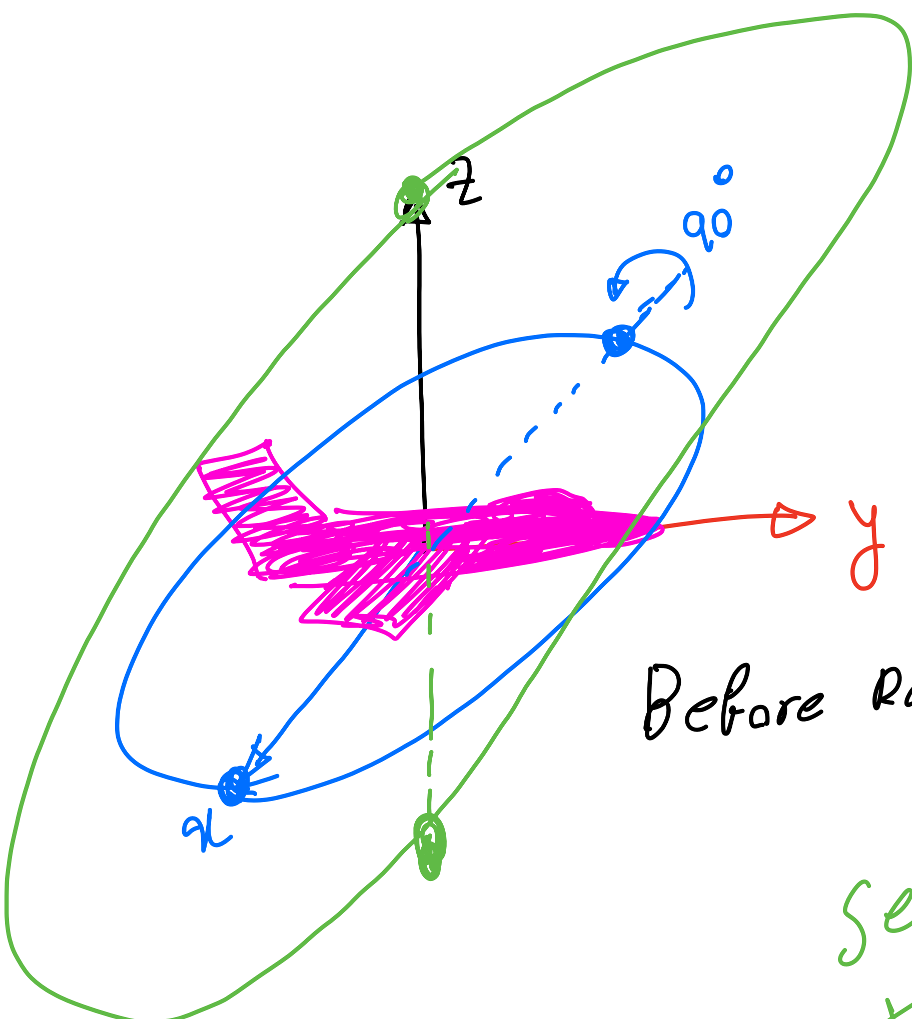
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

we can represent any rotation in space by multiplying these matrices. But there is a problem called gimbal lock that can cause issues with controllers which can make the control system of the vehicle unstable. see the following Rotations as an example:



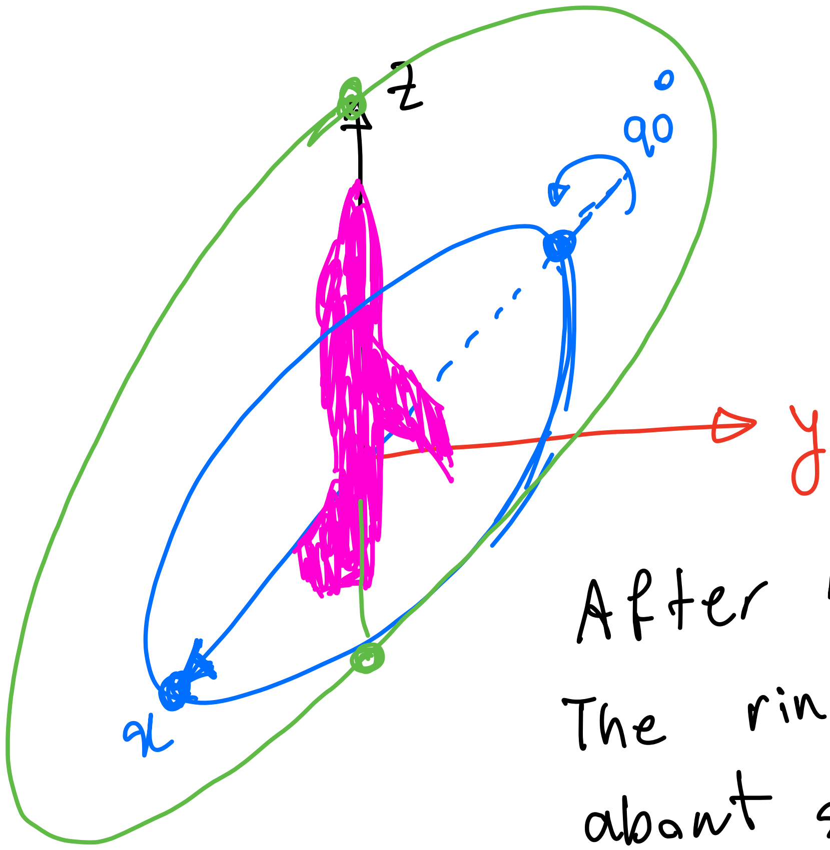
If I Rotate the vehicle 90° about the x-axis.



Before Rotation

See end of this lecture for

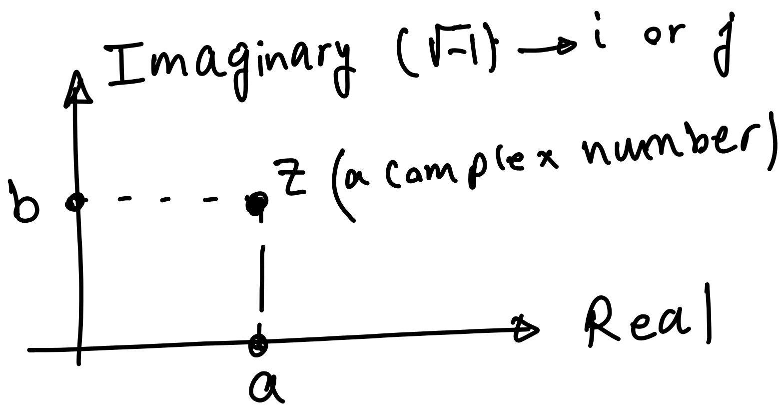
Note better figures



After rotation
The rings of rotation about x and Rotation about z are in the same plane.
This is gimbal lock.

I will show you animation for this problem and then the solution using quatrenion.

Camplex numbers are given by
Real and imaginary numbers in 2D:



i or j = $\sqrt{-1}$

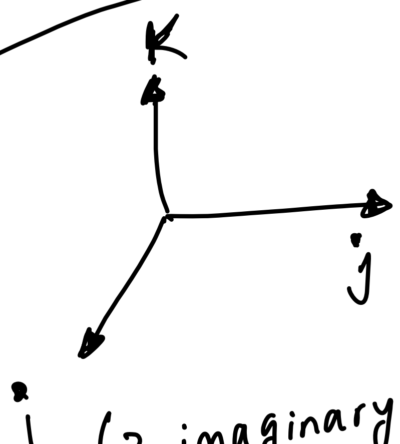
$\boxed{i^2 = -1}$

$z = a + ib$

Real

Imaginary

A quateranion can be thaught of a 4 Dimensional camplex number.

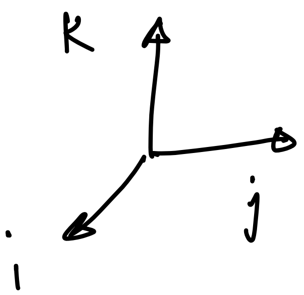


+ A Real number

(3 imaginary numbers)

Rules are similar to complex numbers such as:

$$i^2 = j^2 = k^2 = -1$$



Imagine i j k are given as unit vectors in a 3D space. If cross product of the unit vectors are calculated we have:

$$i \times j = k \qquad j \times k = i \qquad k \times i = j$$

$$j \times i = -k \qquad k \times j = -i \qquad i \times k = -j$$

The Quaternion rules are similar:

$$ij = k \qquad jk = i \qquad ki = j$$

$$ji = -k \qquad kj = -1 \qquad ik = -j$$

Quaternions are defined by

$$q = a + bi + cj + dk$$

Real part

$b, c, d, \rightarrow$ imaginary coefficients

i, j, k are imaginary basis units.

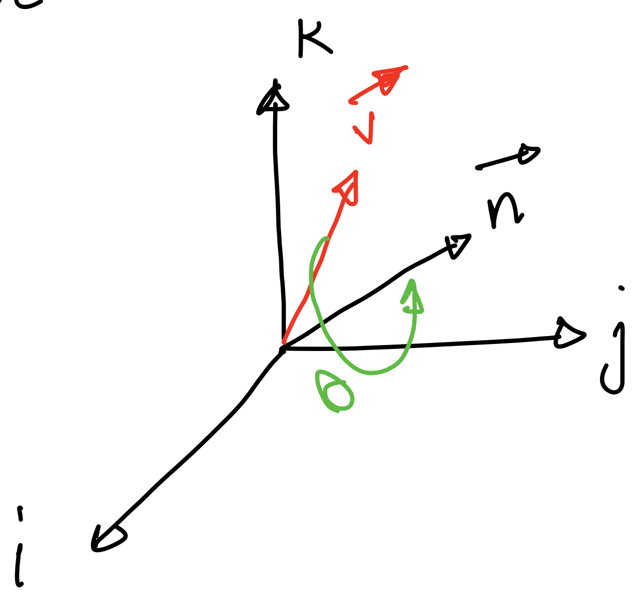
A unit quaternion has the form:

$$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\,\vec{n}$$

where θ is a rotation angle

$\vec{n}$ is a unit vector giving the axis of rotation.

Imgine that we have a vector $\vec{V}$ that we want to rotate.



$$\vec{V} = xi + yj + zK$$

To rotate a vector $\vec{V}$ we need to

perform the following calculation:

$$\vec{v}' = q\,\vec{v}\,q^{-1}$$

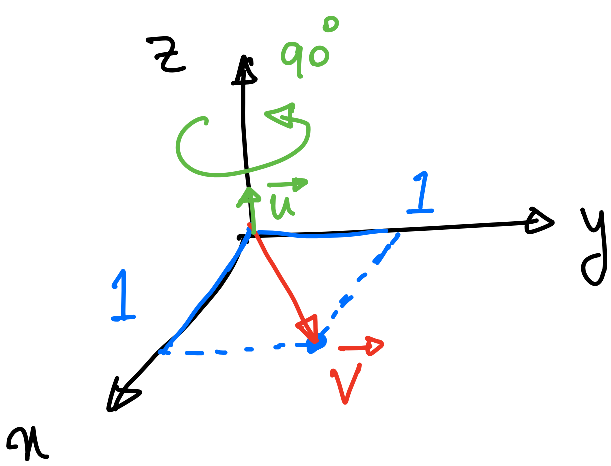
Some more quaternion operation rules:

quaternion product of two vectors $\vec{a}$ and $\vec{b}$:

$$(a\,b) = \underbrace{-\vec{a}\cdot\vec{b}}_{\text{Dot product}} + \underbrace{\vec{a}\times\vec{b}}_{\text{cross product}}$$

Example

Rotate $\vec{v} = (1, 1, 0)$ by $\theta = 90^\circ$ around the z-axis using a quaternion.



To rotate a 3D vector $\vec{v}$ around a unit axis $\vec{u}$ by an angle θ using a quaternion, we have:

$$q = \cos\frac{\theta}{2} + \vec{u}\,\sin\frac{\theta}{2}$$

The rotated vector can be calculated by:

$$v' = q v q^{-1}$$

Now, we introduce u, a_1, a_2, w as below:

$\vec{u}$ is the unit rotation axis.

In this example:

$$\vec{u} = 0\vec{i} + 0\vec{j} + \vec{k} \quad \text{or} \quad \vec{u} = (0,0,1)$$

In quaternion $\vec{u}$ is given by:

$$\vec{u} = 0 + 0\vec{i} + 0\vec{j} + \vec{k} \quad \text{or} \quad \vec{u} = (0,0,0,1)$$

Real part

Real part

a_1 and a_2 are two unit vector orthogonal to $\vec{u}$, and orthogonal to each other:

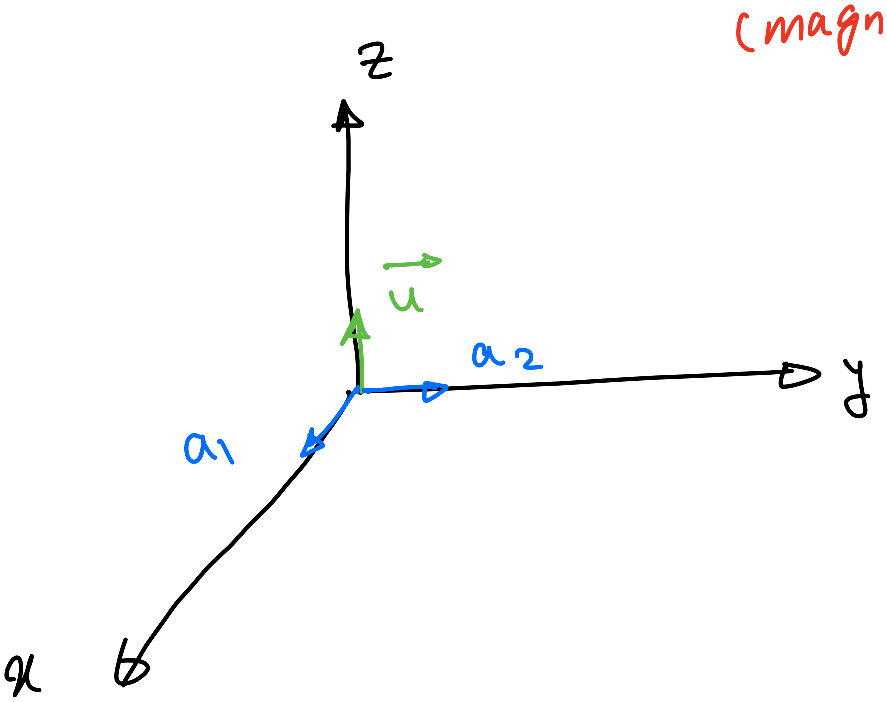
$a_1 \perp u \qquad a_2 \perp u \qquad a_1 \perp a_2$

perpendicular

$\|a_1\| = 1 \qquad \|a_2\| = 1$

unit vectors (magnitudes are 1)

In this example:

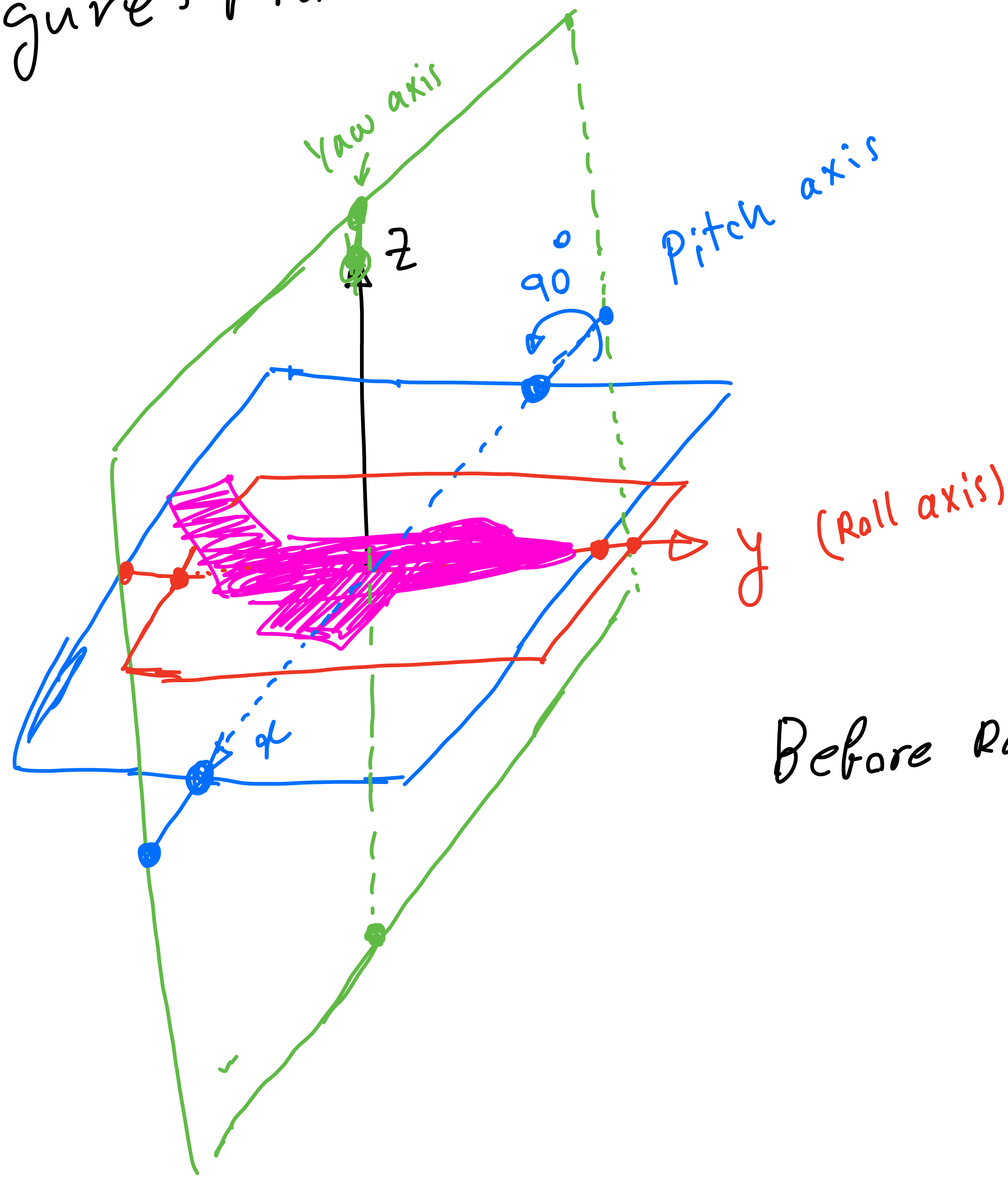


$a_1 = (1, 0, 0)$ or $a_1 = \hat{i}$

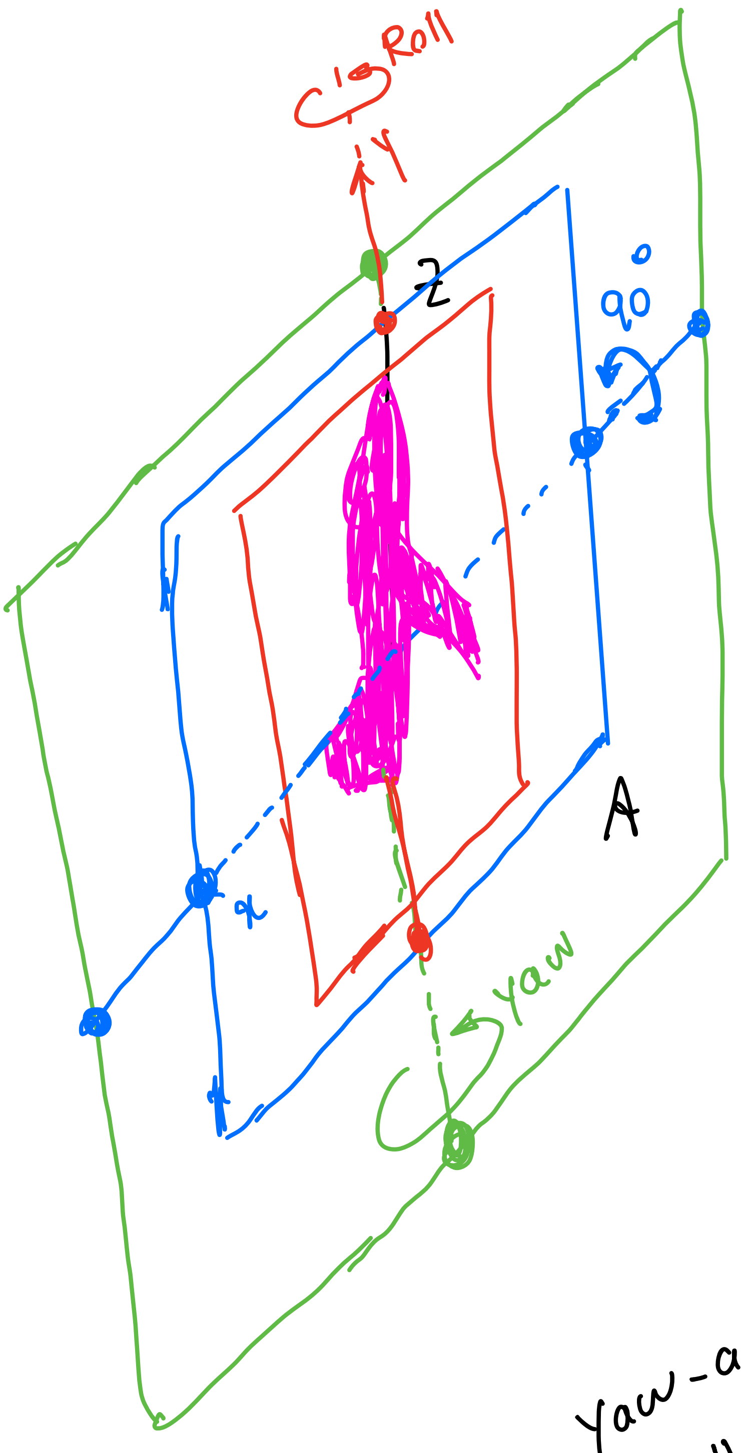
$a_2 = (0, 1, 0)$ or $a_2 = \hat{j}$

Calculation of the rotation using the quaternion approach will be given in the next lecture.

Figures from above:



Before Rotation



After 90° pitch

After rotation 90° about x-axis
Yaw-axis (about Z) and
Roll-axis (about Y) are
along the same axis now
it means yaw
causes Roll

This can cause instability in the cantral becouse
we loose the yaw degree of Freedam.