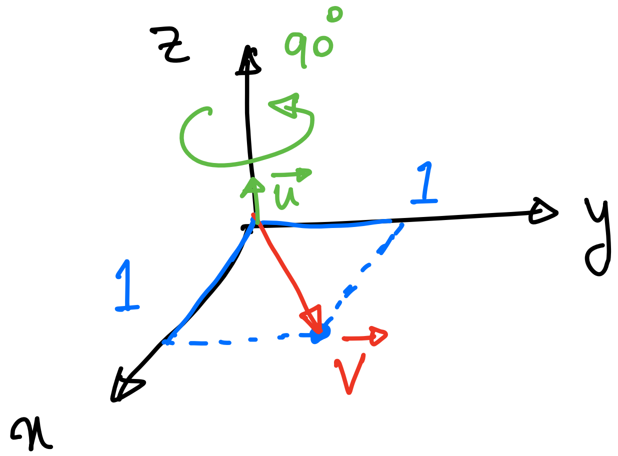
Quaternion

part2

Example

Rotate $\vec{v} = (1, 1, 0)$ by $\theta = 90^\circ$ around the z-axis using a quaternion.



To rotate a 3D vector $\vec{v}$ around a unit axis $\vec{u}$ by an angle θ using a quaternion, we have:

$$q = \underbrace{\cos\frac{\theta}{2}}_{\text{Representing Rotation}} + \underbrace{\vec{u}\ \sin\frac{\theta}{2}}_{\text{Representing the axis of rotation}}$$

The rotated vector can be calculated by:

$$\boxed{v' = q v q^{-1}}$$

Now, we introduce u, a_1, a_2, w as below:

$\vec{u}$ is the unit rotation axis.

In this example (u is along the z-axis):

$$\vec{u} = 0\vec{i} + 0\vec{j} + \vec{K} \quad \text{or} \quad \vec{u} = (0, 0, 1)$$

In quaternion $\vec{u}$ is given by:

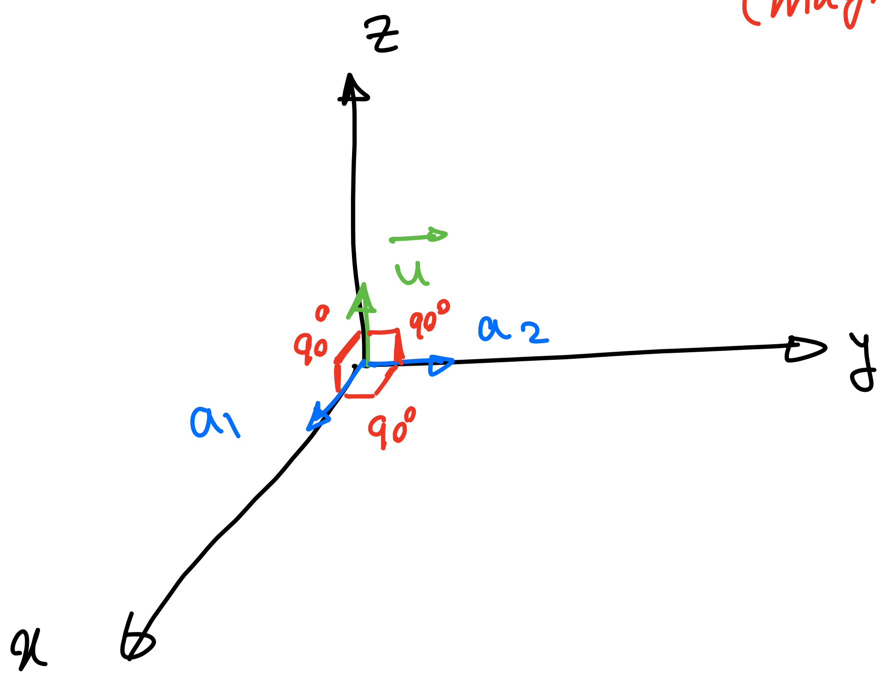
$$\vec{u} = \underbrace{0}_{\text{Real part}} + 0\vec{i} + 0\vec{j} + \vec{K} \quad \text{or} \quad \boxed{\vec{u} = (\underbrace{0}_{\text{Real part}}, 0, 0, 1)}$$

a_1 and a_2 are two unit vector orthogonal to $\vec{u}$, and orthogonal to each other:

$$\underbrace{a_1 \perp u}_{\perp\text{: perpendicular}} \qquad a_2 \perp u \qquad a_1 \perp a_2$$

$\|a_1\| = 1$ $\|a_2\| = 1$ ← unit vectors (magnitudes are 1)

In this example:



$a_1 = (1, 0, 0)$ or $a_1 = i + (0)j + (0)k = i$

$a_2 = (0, 1, 0)$ or $a_2 = (0)i + j + (0)k = j$

we can use DoT product to check if they are orthaganal:

$\vec{a}_1 \cdot \vec{u} = |\vec{a}_1| \, |\vec{u}| \cos(\widehat{\vec{a}_1, \vec{u}}) = (1)(1)\cos(90^\circ) = 0$

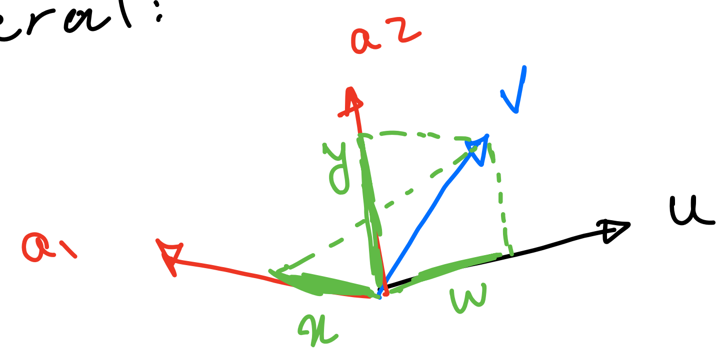
$\vec{a}_2 \cdot \vec{u} = 0$

$\vec{a}_1 \cdot \vec{a}_2 = 0$

writing vector $\vec{V}$ into this basis $(\vec{u}, \vec{a_1}, \vec{a_2})$:

$$\vec{V} = w\,\vec{u} + x\,\vec{a_1} + y\,\vec{a_2}$$

In general:

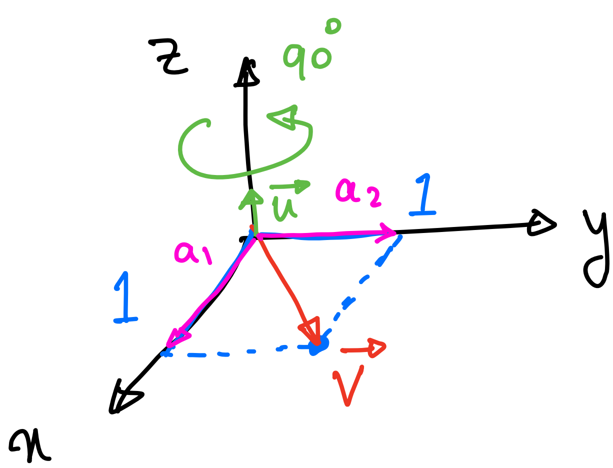


- w is the (projection) component of $\vec{V}$ along the $\vec{u}$ axis
- x, y are the (projections) components of $\vec{V}$ in the perpendicular plane (direction of a_1 and a_2)

In this example:

$\vec{V} = (1, 1, 0)$

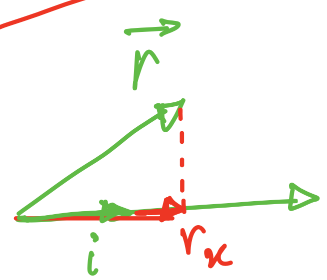
$\vec{u} = (0, 0, 1)$



$\vec{a}_1 = (1, 0, 0)$

$\vec{a}_2 = (0, 1, 0)$

In order to find w, x, and y we can use the DOT product. The DOT product of a vector by a unit vector gives the projection (or component) of the vector along the unit vector.



$\vec{r} \cdot \vec{i} = \vec{r}_x$

unit vector

In general

The projection of $\vec{r}$ along $\vec{i}$

Therefore,

$$w = \vec{v} \cdot \vec{u} = (1,1,0) \cdot (0, 0, 1) = 0$$

$$(i + j + 0) \cdot (0 + 0 + k) = 0 + 0 + 0$$

means the vector has no component along $\vec{u}$.

$$x = \vec{v} \cdot \vec{a_1} = (1,1,0) \cdot (1,0,0) = 1$$

$$(i + j + 0) \cdot (i + 0 + 0) = 1$$

$$y = \vec{v} \cdot \vec{a_2} = (1,1,0) \cdot (0,1,0) = 1$$

Therefore, $w = 0$, $x = 1$, $y = 1$

substituting in the vector $\vec{v}$, we have:

$$\vec{V} = w\vec{u} + x\vec{a_1} + y\vec{a_2}$$

$$\Rightarrow \vec{V} = 0 + (1)\vec{a_1} + (1)\vec{a_2}$$

$$\Rightarrow \boxed{\vec{V} = \vec{a_1} + \vec{a_2}}$$

Building the rotation quaternion q:

Rotation by angle θ around unit axis u corresponds to:

$$q = \cos\frac{\theta}{2} + \vec{u}\sin\frac{\theta}{2}$$

For example, for rotation of $\theta = 90^\circ = \pi/2$ around $\vec{u}$ axis (same as $\vec{z}$ axis in this example)

$$\frac{\theta}{2} = 45^\circ = \frac{\pi}{4}$$

$$\cos\frac{\theta}{2} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$\sin\frac{\theta}{2} = \frac{\sqrt{2}}{2}$$

Therefore, the quaternion is:

$$q = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\vec{u}$$

$\vec{u} = (0, 0, 1)$ or $\vec{u} = (0)\vec{i} + (0)\vec{j} + (1)\vec{k}$

$$q = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(0 + 0 + \vec{k})$$

$$\boxed{q = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\vec{k}}$$

or $$q = \left(\frac{\sqrt{2}}{2}, (0)\vec{i}, (0)\vec{j}, \frac{\sqrt{2}}{2}\vec{k}\right)$$

$$q = (\sqrt{2}/2, 0, 0, \sqrt{2}/2)$$

The rotated vector can be calculated by:

$$\boxed{v' = q v q^{-1}}$$

Let's find q^{-1}.

The inverse of a quaternion is found as below. For example for $q = a+bi+cj+dk$

First calculate conjugate of q:

The conjugate of q is simply the negative of the imaginary parts:

$$q = \underbrace{a}_{\text{Real part}} + \underbrace{bi + cj + dk}_{\text{Imaginary part}}$$

The conjugate is: $\boxed{\bar{q} = a - bi - cj - dk}$

Also calculate norm of q as follows:

Norm of q: $\|q\| = \sqrt{a^2+b^2+c^2+d^2}$

The inverse of q is found by:

$$q^{-1} = \frac{\bar{q}}{\|q\|}$$

Therefore, the inverse of q is:

$$q^{-1} = \frac{\bar{q}}{\|q\|} = \frac{\cos\frac{\theta}{2} - u\sin\frac{\theta}{2}}{\sqrt{\cos^2\frac{\theta}{2} + \sin^2\theta/2}}$$

$$q^{-1} = \cos\frac{\theta}{2} - u\sin\frac{\theta}{2}$$

$$q^{-1} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}u = \left(\frac{\sqrt{2}}{2}, 0, 0, -\frac{\sqrt{2}}{2}\right)$$

$$q v q^{-1} = ?$$

$$V = a_1 + a_2$$

$$q\,(a_1 + a_2)\,q^{-1}$$

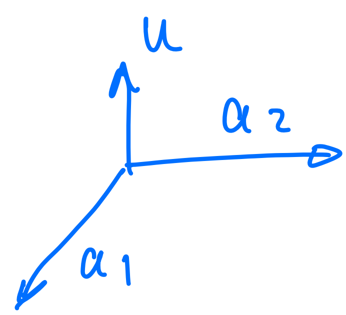
$$= q a_1 q^{-1} + q a_2 q^{-1}$$

$$q = c + s u \qquad \text{where} \begin{cases} c = \cos \theta/2 \\ s = \sin \theta/2 \end{cases}$$

Then multiply $q a_1$:

$$q a_1 = (c + s u) a_1 = c a_1 + s u a_1$$

$$= c a_1 + s a_2$$



$u a_1 = +a_2$

(using right hand rule similar to cross product)

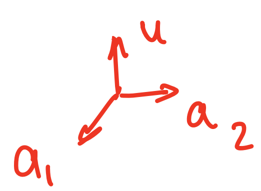
Now multiply by $q^{-1} = c - s u$:

$$q a_1 q^{-1} = (c a_1 + s a_2)(c - s u)$$

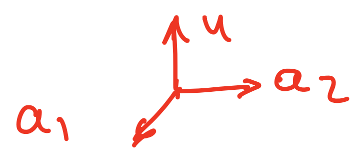
Expand

$$(c a_1 + s a_2)(c - s u) =$$

$$c^2 a_1 - cs\, a_1 u + cs\, a_2 - s^2 a_2 u$$



$a_1 u = -a_2$



$a_2 u = a_1$

$$= c^2 a_1 + cs\, a_2 + cs\, a_2 - s^2 a_1$$

$$= (c^2 - s^2) a_1 + (2cs) a_2$$

We know $c^2 - s^2 = \cos\theta$

$2cs = \sin\theta$

Therefore, $q a_1 q^{-1} = a_1 \cos\theta + a_2 \sin\theta$

$\Rightarrow$ Now similarly calculate $q a_2 q^{-1}$:

$$q a_2 = (c + su) a_2 = c a_2 + s(u a_2)$$

$(u a_2 = -a_1)$

$$q a_2 = c a_2 - s a_1$$

$$q^{-1} = c - su$$

$$\Rightarrow q a_2 q^{-1} = (c a_2 - s a_1)(c - su)$$

$$= c^2 a_2 - cs \underbrace{a_2 u}_{a_1} - cs a_1 + s^2 \underbrace{a_1 u}_{-a_2}$$

$$= c^2 a_2 - cs a_1 - cs a_1 - s^2 a_2$$

$$= (c^2 - s^2) a_2 - 2cs a_1$$

$$\Rightarrow q a_2 q^{-1} = \cos\theta \, a_2 - \sin\theta \, a_1$$

$$qvq^{-1} = q(wu + xa_1 + ya_2)q^{-1}$$

$$= w(quq^{-1}) + x(qa_1q^{-1}) + y(qa_2q^{-1})$$

quq^{-1}

$$\Rightarrow \begin{cases} qu = (c+su)u = cu + su^2 = cu - s \\ quq^{-1} = (cu-s)q^{-1} = (cu-s)(c-su) \\ \quad = c^2u - csu^2 - sc + s^2u \\ \quad = (c^2+s^2)u + cs - sc = u \end{cases}$$

$$qvq^{-1} = wu + x(a_1\cos\theta + a_2\sin\theta)$$
$$+ y(-a_1\sin\theta + a_2\cos\theta)$$

$$\Rightarrow v' = qvq^{-1} = wu + (x\cos\theta - y\sin\theta)a_1$$
$$+ (x\sin\theta + y\cos\theta)a_2$$

For this example $u = (0,0,1)$, $a_1 = (1,0,0)$, $a_2 = (0,1,0)$, $v = (1,1,0)$, $\theta = 90^\circ$

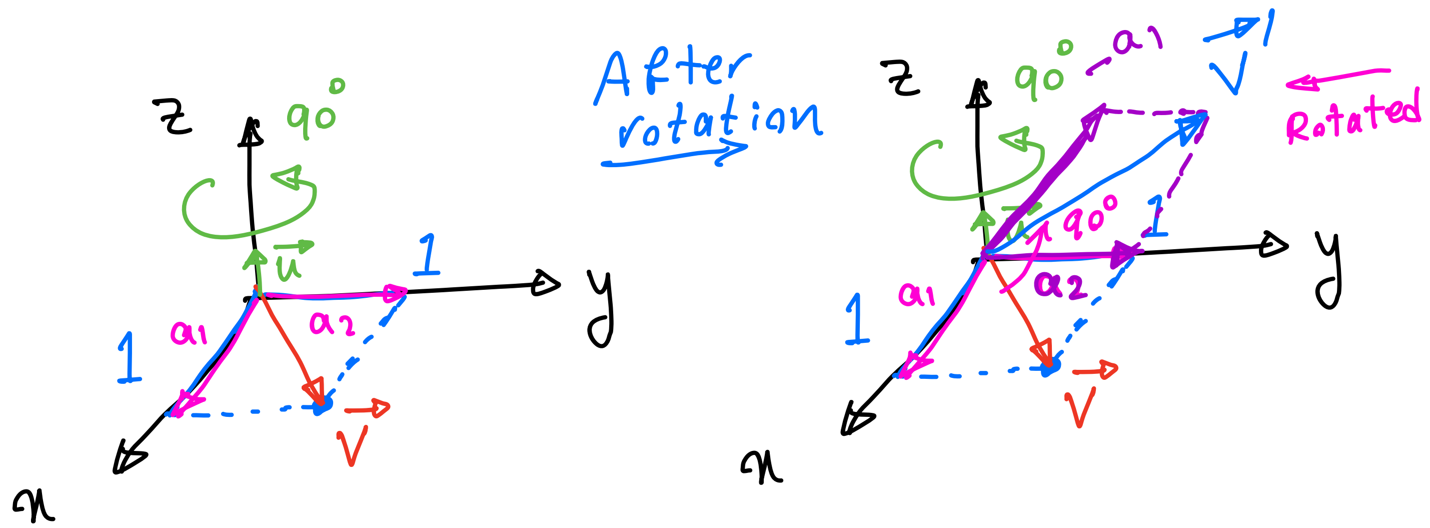
Therefore, $(V = wu + x\,a_1 + y\,a_2 = 0 + a_1 + a_2)$

$$v' = wu + (x\cos\theta - y\sin\theta)\,a_1 + (x\sin\theta + y\cos\theta)\,a_2$$

$$\begin{cases} \cos\theta = \cos 90^\circ = 0 \\ \sin\theta = \sin 90^\circ = 1 \end{cases}$$

$$\Rightarrow \quad v' = (0)\,u + (-1)\,a_1 + (1)\,a_2$$

$$v' = \underline{-a_1} + \underline{a_2}$$



MATLAB function:

rotationVector = [20, 30, 40]

Rotation about 3 axes example

quaternion(rotationVector)