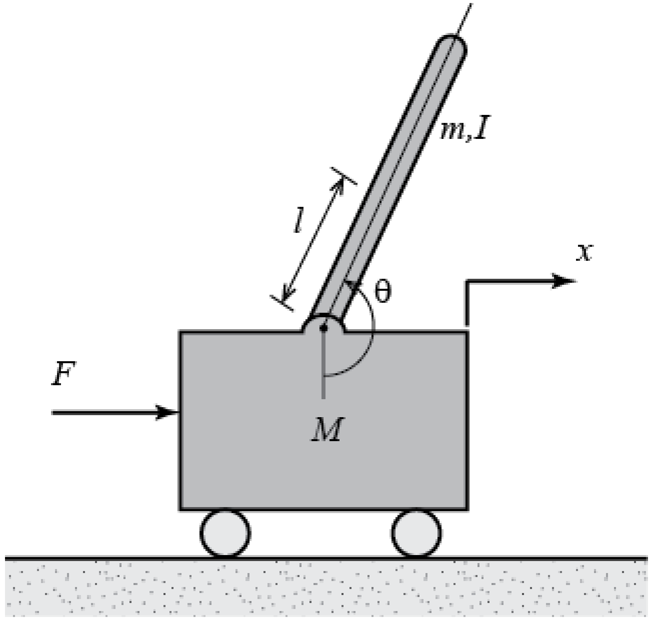
State-Space Model and LQR Example

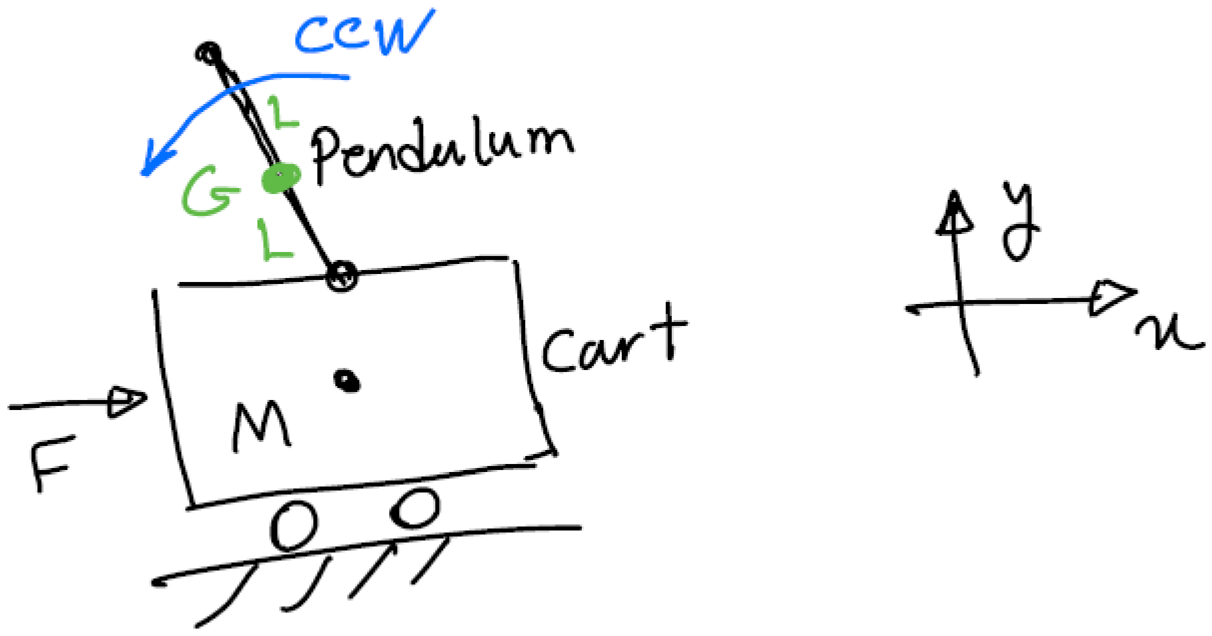
Inverted pendulum

- A real-world example that relates directly to this inverted pendulum system is the attitude control of a booster rocket at takeoff.

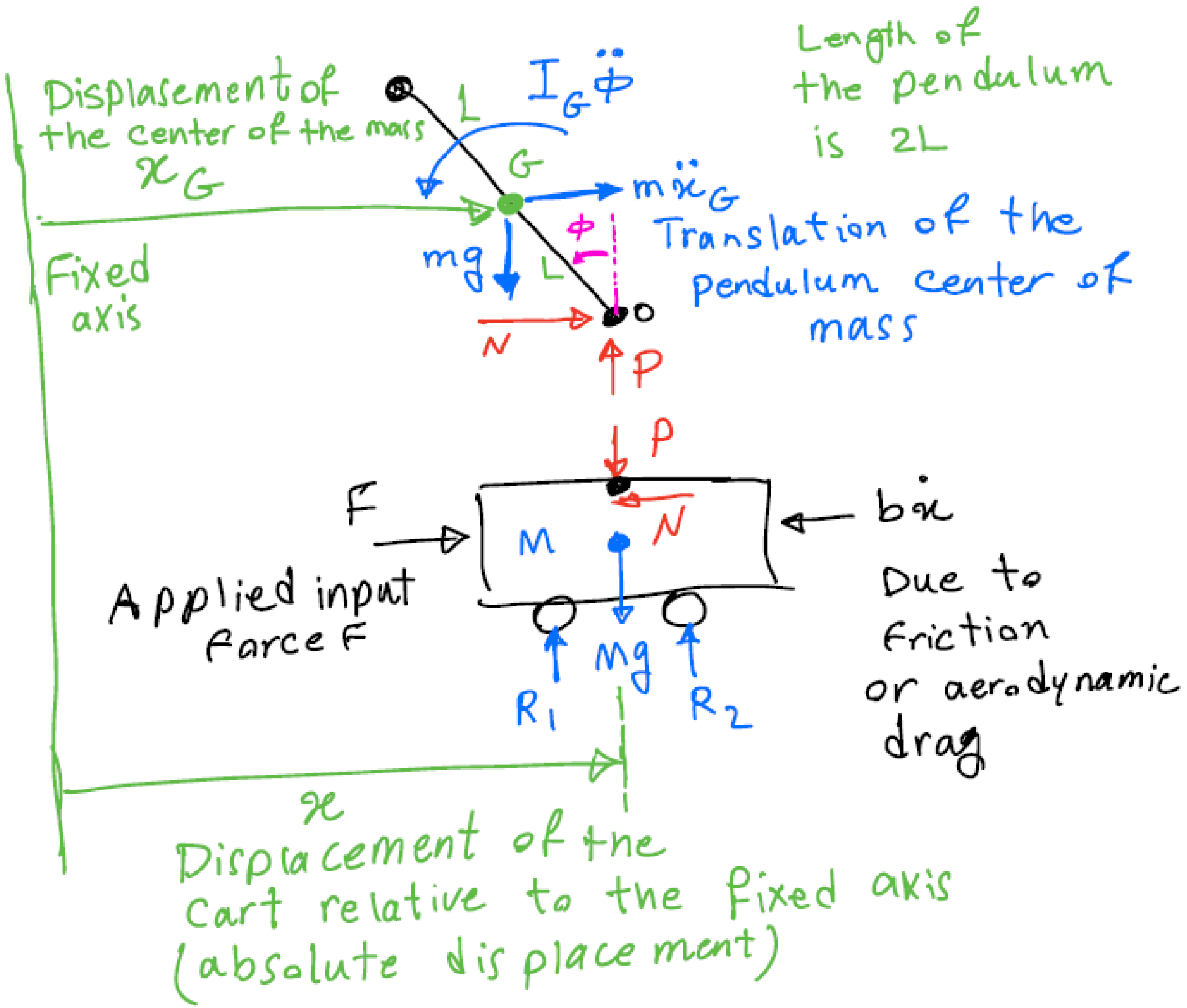


https://ctms.engin.umich.edu/CTMS/index.php?example=InvertedPendulum&section=SystemModeling

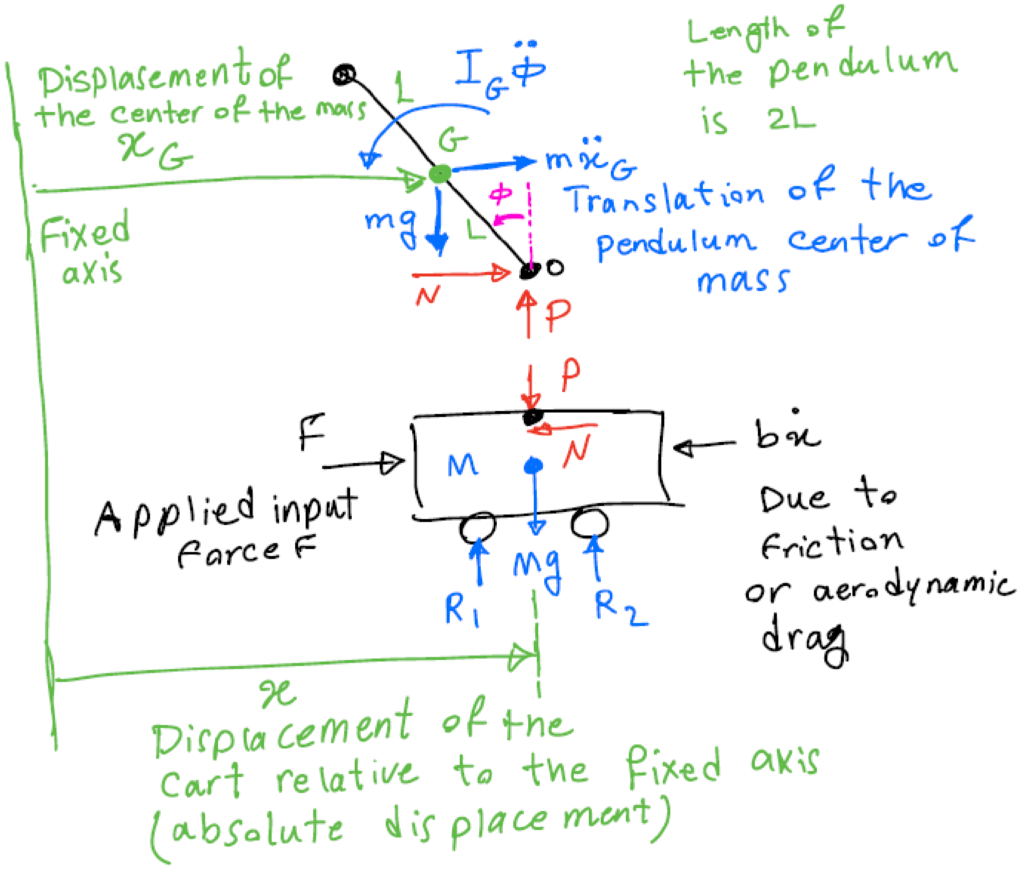
External force F is used to control the motion of the pendulum to keep it vertical



Free-body-diagram (F.B.D.)



Free-body-diagram (F.B.D.)



Equations of Motion:

The cart:

$$\xrightarrow{+} \sum F_x = M\ddot{x}$$

$$F - b\dot{x} - N = M\ddot{x} \qquad (1)$$

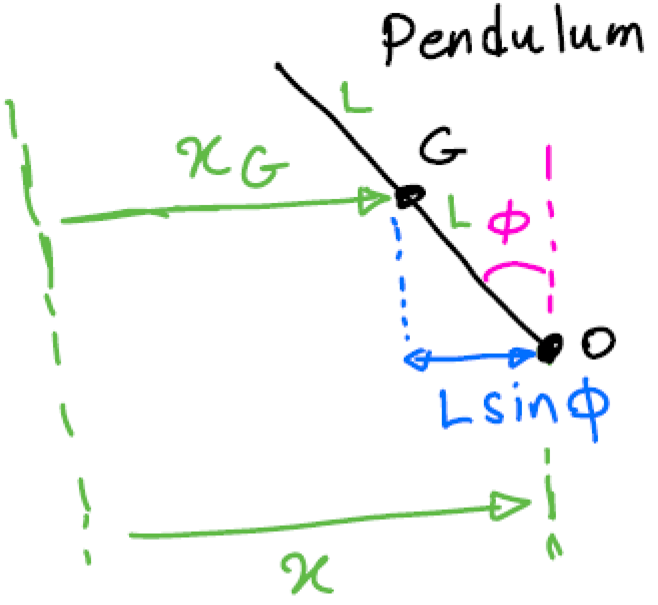
The pendulum:

Translation: (From FBD)

$$\xrightarrow{+} \sum F_x = m\ddot{x}_G$$

$$N = m\ddot{x}_G \qquad (2)$$

x_G in terms of x displacement:



$$x_G = x - L\sin\phi$$

$$\dot{x}_G = \frac{dx}{dt} - L\frac{d}{dt}\sin\phi$$

$$\dot{x}_G = \dot{x} - L\cos\phi\,\dot{\phi}$$

$$\ddot{x}_G = \ddot{x} - L\frac{d}{dt}(\cos\phi\,\dot{\phi})$$

or

$$\ddot{x}_G = \ddot{x} - L\left[-\sin\phi\,\dot{\phi}^2 + \cos\phi\,\ddot{\phi}\right] \qquad 3$$

Chain rule:

$$\frac{d}{dt}\sin\phi = \frac{d\sin\phi}{dt}\frac{d\phi}{d\phi}$$

$$= \frac{d\sin\phi}{d\phi}\frac{d\phi}{dt}$$

$$= \cos\phi\,\dot{\phi}$$

product rule:

$$\frac{d}{dt}(\cos\phi\,\dot\phi) = \frac{d\cos\phi}{dt}\,\dot\phi + \cos\phi\,\frac{d}{dt}\dot\phi$$

$$= \frac{d\cos\phi}{d\phi}\,\frac{d\phi}{dt}\,\dot\phi + \cos\phi\,\ddot\phi$$

$$= -\sin\phi\,\dot\phi^2 + \cos\phi\,\ddot\phi$$

The pendulum:

Translation: (From FBD)

$$\xrightarrow{+} \sum F_x = m\ddot{x}_G$$

$$N = m\ddot{x}_G \qquad (2)$$

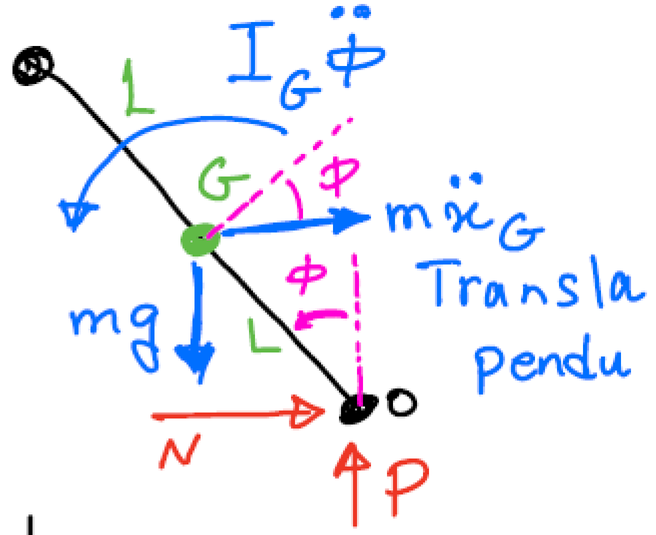
$$\ddot{x}_G = \ddot{x} - L\left[-\sin\phi\,\dot{\phi}^2 + \cos\phi\,\ddot{\phi}\right] \qquad (3)$$

(2) & (3) $\Rightarrow$

$$N = m\ddot{x}_G$$

$$N = m\left\{\ddot{x} - L\left[-\sin\phi\,\dot{\phi}^2 + \cos\phi\,\ddot{\phi}\right]\right\} \qquad (4)$$

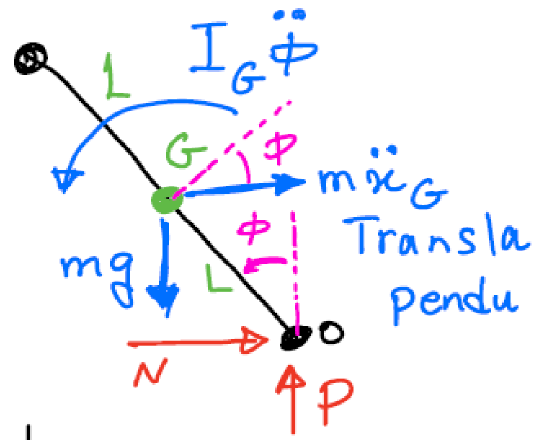
The pendulum rotation:
(from F.B.D)



$$\circlearrowleft^{+} \sum M_o = I_G \ddot{\phi} - m\ddot{x}_G \cos\phi \, L$$

$$\begin{cases} mgL\sin\phi = I_G\ddot{\phi} - m\cos\phi \, L \, \ddot{x}_G \\ ③ \rightarrow \ddot{x}_G = \ddot{x} - L\left[-\sin\phi \, \dot{\phi}^2 + \cos\phi \, \ddot{\phi}\right] \end{cases}$$

The pendulum rotation:
(from F.B.D)



$$\circlearrowleft + \quad \Sigma M_o = I_G\ddot{\phi} - m\ddot{x}_G \cos\phi \, L$$

$$mgL\sin\phi = I_G\ddot{\phi} - m\cos\phi L \, \ddot{x}_G$$

$$③ \rightarrow \quad \ddot{x}_G = \ddot{x} - L\left[-\sin\phi\,\dot{\phi}^2 + \cos\phi\,\ddot{\phi}\right]$$

$$\Rightarrow mgL\sin\phi =$$

$$I_G\ddot{\phi} - mL\cos\phi\left\{\ddot{x} - L\left[-\sin\phi\,\dot{\phi}^2 + \cos\phi\,\ddot{\phi}\right]\right\}$$

⑤

In a control problem we desire to have small deviation of ϕ.

For small ϕ we have : These will linearized the equations $\rightarrow$ $\begin{cases} \sin\phi = \phi \\ \cos\phi = 1 \\ \dot{\phi}^2 \cong 0 \end{cases}$

substitute in (5) $\Rightarrow$

$$mgL\phi = I_G\ddot{\phi} - mL\left[\ddot{x} - L\ddot{\phi}\right]$$

$$\Rightarrow \quad -I_G\ddot{\phi} - mL^2\ddot{\phi} + mgL\phi = -mL\ddot{x}$$

or

$$(I_G + mL^2)\ddot{\phi} - mgL\phi = mL\ddot{x} \qquad (a)$$

$$① \,\&\, ④ \Rightarrow \quad F - b\dot{x} - N = M\ddot{x} \quad ①$$

$$N = m\left\{\ddot{x} - L\left[-\sin\phi\,\dot{\phi}^2 + \cos\phi\,\ddot{\phi}\right]\right\} \quad ④$$

Small ϕ

$\left\{\begin{array}{l} \sin\phi = \phi \\ \cos\phi = 1 \\ \dot{\phi}^2 = 0 \end{array}\right\} \rightarrow F - b\dot{x} - m\left[\ddot{x} - L\ddot{\phi}\right] = M\ddot{x}$

Linearizing

$$F - b\dot{x} - m\ddot{x} + mL\ddot{\phi} = M\ddot{x}$$

Rearranging: $\quad -(M+m)\ddot{x} - b\dot{x} + mL\ddot{\phi} = -F$

or

$$(M+m)\ddot{x} + b\dot{x} - mL\ddot{\phi} = F \quad (b)$$

Therefor, the Equations of Motion are:

a & b →

$$\begin{cases} (I_G + mL^2)\ddot{\phi} - mgL\phi = mL\ddot{x} \\ (M+m)\ddot{x} + b\dot{x} - mL\ddot{\phi} = F \end{cases}$$

Next, we will find the transfer functions:

$$\frac{\Phi(s)}{F(s)} \quad \text{and} \quad \frac{X(s)}{F(s)}$$

Also, the equations of Motion can be written in state-space representation

$$[\dot{x}] = [A][x] + [B][u]$$

$$[Y] = [C][x] + [D][u]$$

$$[\dot{x}] = [A][x] + [B][u]$$

$$[Y] = [C][x] + [D][u]$$

---

or

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = [A] \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + [B] F$$

$$[Y] = [C] \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + [D] F$$

Inverted Pendulum - State-state model and LQR

The equations of motion of the inverted pendulum are given as follows (for the derivation of the equations, please see the equations of motion notes posted on canvas).

$$(I + ml^2)\ddot{\emptyset} - mgl\emptyset = ml\ddot{x} \tag{1}$$

$$(M + m)\ddot{x} + b\dot{x} - ml\ddot{\emptyset} = F \tag{2}$$

Step 1: Define the state variables:

The state variables can be given as follows.

$$\begin{bmatrix} x \\ \dot{x} \\ \emptyset \\ \dot{\emptyset} \end{bmatrix}$$

The state-space model is given by

$$[\dot{x}] = [A][x] + [B][u]$$

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The state variables can be given as follows.

$$\begin{bmatrix} x \\ \dot{x} \\ \emptyset \\ \dot{\emptyset} \end{bmatrix}$$

The state-space model is given by

$$[\dot{x}] = [A][x] + [B][u]$$

Therefore, the state-state model for the inverted pendulum can be defines as

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\emptyset} \\ \ddot{\emptyset} \end{bmatrix} = [A]\begin{bmatrix} x \\ \dot{x} \\ \emptyset \\ \dot{\emptyset} \end{bmatrix} + [B]F$$

$$(I + ml^2)\ddot{\emptyset} - mgl\emptyset = ml\ddot{x} \tag{1}$$

$$(M + m)\ddot{x} + b\dot{x} - ml\ddot{\emptyset} = F \tag{2}$$

Step 2: Find $[A]$ and $[B]$ for the state-space model above, using the equations of motions

Rearrange Equation (2) as:

$$\ddot{\emptyset} = \frac{F - (M + m)\ddot{x} - b\dot{x}}{-ml} \tag{3}$$

Rearrange Equation (1) as:

$$\ddot{x} = \frac{(I + ml^2)}{ml}\ddot{\emptyset} - g\emptyset \tag{4}$$

Rearrange Equation (2) as:

$$\ddot{\emptyset} = \frac{F - (M + m)\ddot{x} - b\dot{x}}{-ml} \tag{3}$$

Rearrange Equation (1) as:

$$\ddot{x} = \frac{(I + ml^2)}{ml}\ddot{\emptyset} - g\emptyset \tag{4}$$

Substitute Equation (3) into Equation (4):

$$\ddot{x} = \frac{(I + ml^2)}{ml} . \frac{F - (M + m)\ddot{x} - b\dot{x}}{-ml} - g\emptyset \tag{5}$$

Rearrange Equation (5) to obtain $\ddot{x}$ for matrices $[A]$ and $[B]$:

$$\ddot{x}(1+\frac{(I+ml^2)}{ml}\frac{(M+m)}{-ml})=\frac{(I+ml^2)}{ml}.\frac{(F-b\dot{x})}{-ml}-g\emptyset$$

$$\ddot{x}\left(1+\frac{(I+ml^2)}{ml}\frac{(M+m)}{-ml}\right)=\frac{(I+ml^2)}{ml}.\frac{b\dot{x}}{ml}-g\emptyset-\frac{(I+ml^2)F}{ml\,ml}$$

$$\ddot{x}\left(\frac{-(ml)^2+(I+ml^2)(M+m)}{-(ml)^2}\right)=\frac{(I+ml^2)}{ml}.\frac{b\dot{x}}{ml}-g\emptyset-\frac{(I+ml^2)F}{ml\,ml}$$

$$\ddot{x}\big(-(ml)^2+(I+ml^2)(M+m)\big)=\frac{(I+ml^2)}{ml}.\frac{b\dot{x}(-(ml)^2)}{ml}-g\emptyset(-(ml)^2)-\frac{(I+ml^2)F(-(ml)^2)}{ml\,ml}$$

$$\ddot{x}\big(-(ml)^2+(I+ml^2)(M+m)\big)=-(I+ml^2)b\dot{x}+g\emptyset(ml)^2+(I+ml^2)F$$

$$\ddot{x}\big(Mml^2+I(M+m)\big)=-(I+ml^2)b\dot{x}+g\emptyset(ml)^2+(I+ml^2)F$$

$$\ddot{x}=\frac{-(I+ml^2)b\dot{x}+g\emptyset(ml)^2+(I+ml^2)F}{\big(Mml^2+I(M+m)\big)}$$

$$\ddot{x} = \frac{-(I + ml^2)b\dot{x} + g\emptyset(ml)^2 + (I + ml^2)F}{\left(Mml^2 + I(M + m)\right)}$$

Show the first and second rows of the $[A]$ and $[B]$ matrices in the state-space model ($[\dot{x}] = [A][x] + [B][u]$):

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\emptyset} \\ \ddot{\emptyset} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \dfrac{-(I + ml^2)b}{Mml^2 + I(M + m)} & \dfrac{g(ml)^2}{Mml^2 + I(M + m)} & 0 \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \emptyset \\ \dot{\emptyset} \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{(I + ml^2)}{Mml^2 + I(M + m)} \\ 0 \\ \square \end{bmatrix} F$$

Now, find $\ddot{\emptyset}$, and complete the state-space model:

$(I + ml^2)\ddot{\emptyset} - mgl\emptyset = ml\ddot{x}$ (1)

$(M + m)\ddot{x} + b\dot{x} - ml\ddot{\emptyset} = F$ (2)

Rearrange Equation (1) as:

$$\frac{(I + ml^2)\ddot{\emptyset} - mgl\emptyset}{ml} = \ddot{x} \tag{6}$$

Substitute Equation (6) into Equation (2):

$$(M + m)\frac{(I + ml^2)\ddot{\emptyset} - mgl\emptyset}{ml} + b\dot{x} - ml\ddot{\emptyset} = F$$

$$(M+m)\frac{(I+ml^2)\ddot{\emptyset}-mgl\emptyset}{ml}+b\dot{x}-ml\ddot{\emptyset}=F$$

Rearrange to find $\ddot{\emptyset}$:

$$(M+m)\frac{(I+ml^2)\ddot{\emptyset}}{ml}-\frac{mgl(M+m)\emptyset}{ml}+b\dot{x}-ml\ddot{\emptyset}=F$$

$$(M+m)\frac{(I+ml^2)\ddot{\emptyset}}{ml}-ml\ddot{\emptyset}-\frac{mgl(M+m)\emptyset}{ml}+b\dot{x}=F$$

$$\left[(M+m)\frac{(I+ml^2)}{ml}-ml\right]\ddot{\emptyset}-\frac{mgl(M+m)\emptyset}{ml}+b\dot{x}=F$$

$$\left[\frac{(M+m)(I+ml^2)-(ml)^2}{ml}\right]\ddot{\emptyset}-\frac{mgl(M+m)\emptyset}{ml}+b\dot{x}=F$$

$$\left[\frac{(M+m)(I+ml^2)-(ml)^2}{ml}\right]\ddot{\emptyset}=\frac{mgl(M+m)\emptyset}{ml}-b\dot{x}+F$$

$$[(M+m)(I+ml^2)-(ml)^2]\ddot{\emptyset}=mgl(M+m)\emptyset-mlb\dot{x}+mlF$$

$$\ddot{\emptyset}=\frac{mgl(M+m)\emptyset-mlb\dot{x}}{[(M+m)(I+ml^2)-(ml)^2]}+\frac{mlF}{[(M+m)(I+ml^2)-(ml)^2]}$$

$$\ddot{\emptyset} = \frac{mgl(M+m)\emptyset - mlb\dot{x}}{[(M+m)(I+ml^2)-(ml)^2]} + \frac{mlF}{[(M+m)(I+ml^2)-(ml)^2]}$$

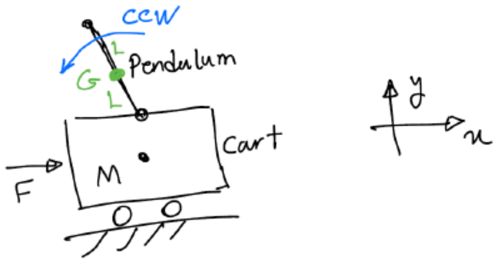
$$\ddot{\emptyset} = \frac{mgl(M+m)\emptyset - mlb\dot{x}}{[Mml^2+(M+m)(I+ml^2)]} + \frac{mlF}{[Mml^2+(M+m)(I+ml^2)]}$$

The state-space model will be given by:

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\emptyset} \\ \ddot{\emptyset} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-(I+ml^2)b}{Mml^2+I(M+m)} & \frac{g(ml)^2}{Mml^2+I(M+m)} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-mlb}{Mml^2+I(M+m)} & \frac{mgl(M+m)}{Mml^2+I(M+m)} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \emptyset \\ \dot{\emptyset} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{(I+ml^2)}{Mml^2+I(M+m)} \\ 0 \\ \frac{ml}{Mml^2+I(M+m)} \end{bmatrix} F$$

Lab 2

Inverted pendulum assignment:



1. Find the equations of motion of the inverted pendulum (show all the steps. Don't skip steps). Use the FBD given in the notes on canvas.
2. Find the transfer functions in the following forms:

$$\frac{\emptyset(s)}{F(s)} \text{ and } \frac{X(s)}{F(s)}$$

3. Find the state-state model in the following form:

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\emptyset} \\ \ddot{\emptyset} \end{bmatrix} = [A] \begin{bmatrix} x \\ \dot{x} \\ \emptyset \\ \dot{\emptyset} \end{bmatrix} + [B]F$$

$$y = [C] \begin{bmatrix} x \\ \dot{x} \\ \emptyset \\ \dot{\emptyset} \end{bmatrix} + [D]F$$

Use the following parameters:

M = .5;
m = 0.2;
b = 0.1;
I = 0.006;
g = 9.8;
l = 0.3;

4. Use the following commands in MATLAB to obtain the transfer function:

```
states = {'x' 'x_dot' 'phi' 'phi_dot'};
inputs = {'u'};
outputs = {'x'; 'phi'};
sys_ss = ss(A,B,C,D,'statename',states,'inputname',inputs,'outputname',outputs)
```

5. Use the following command to convert state-state model to transfer functions.

```
sys_tf = tf(sys_ss)
```

6. Examine how the system responds to an impulsive force applied to the cart employing the MATLAB command impulse. (Use the same parameters as in part 3 for all the calculations).

```
q = (M+m)*(I+m*l^2)-(m*l)^2;
s = tf('s');
P_cart = (((I+m*l^2)/q)*s^2 - (m*g*l/q))/(s^4 + (b*(I + m*l^2))*s^3/q - ((M + m)*m*g*l)*s^2/q - b*m*g*l*s/q);
P_pend = (m*l*s/q)/(s^3 + (b*(I + m*l^2))*s^2/q - ((M + m)*m*g*l)*s/q - b*m*g*l/q);
sys_tf = [P_cart ; P_pend];
inputs = {'u'};
outputs = {'x'; 'phi'};
set(sys_tf,'InputName',inputs)
set(sys_tf,'OutputName',outputs)
t=0:0.01:1;
impulse(sys_tf,t);
title('Open-Loop Impulse Response')
```

7. Find the zeros and poles of the system using the following commends:

```
[zeros poles] = zpkdata(P_pend,'v')
```

8. Find the Open-loop step response:

```
t = 0:0.05:10;
u = ones(size(t));
[y,t] = lsim(sys_tf,u,t);
plot(t,y)
title('Open-Loop Step Response')
axis([0 3 0 50])
legend('x','phi')
```

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9. Design a PID controller for the pendulum. Start with Kp = 1, Ki = 1, Kd = 1. Is the response acceptable:

```
Kp = 1;
Ki = 1;
Kd = 1;
C = pid(Kp,Ki,Kd);
T = feedback(P_pend,C);
t=0:0.01:10;
impulse(T,t)
title({'Response of Pendulum Position to an Impulse Disturbance';'under PID Control: Kp = 1, Ki = 1, Kd = 1'});
```

10. Change the parameters of the PID as below and explain if the response is acceptable.

```
Kp = 100;
Ki = 1;
Kd = 1;
C = pid(Kp,Ki,Kd);
T = feedback(P_pend,C);
t=0:0.01:10;
impulse(T,t)
axis([0, 2.5, -0.2, 0.2]);
title({'Response of Pendulum Position to an Impulse Disturbance';'under PID Control: Kp = 100, Ki = 1, Kd = 1'});
```

11. Change the parameters of the PID as below and explain if the response is acceptable.

```
Kp = 100;
Ki = 1;
Kd = 20;
C = pid(Kp,Ki,Kd);
T = feedback(P_pend,C);
t=0:0.01:10;
impulse(T,t)
axis([0, 2.5, -0.2, 0.2]);
title({'Response of Pendulum Position to an Impulse Disturbance';'under PID Control: Kp = 100, Ki = 1, Kd = 20'});
```

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Derivation of the LQR Algebraic Riccati Equation (ARE):

The Linear Quadratic Regulator (LQR) approach to solve the state-space model:

Finding the optimal values of the control parameters

Minimize a quadratic cost function, J:

$$J = \int_0^\infty (x^T Q x + u^T R u)\, dt$$

Where, Q minimizes the state of the system (to drive the state of the system to zero), and R controller effort (how much effort). What control value, k, produces the lowest cost J.

For this purpose, let's introduce a matrix P, where $P = P^T$.

Add and subtract:

$$J = {x_0}^T P x_0 - {x_0}^T P x_0 + \int_0^\infty (x^T Q x + u^T R u)\, dt$$

Move ${x_0}^T P x_0$ into the integral:

$$J = {x_0}^T P x_0 + \int_0^\infty \left[\frac{d}{dt}(x^T P x) + x^T Q x + u^T R u\right] dt$$

It is valid because:

$$\int_0^\infty \left[\frac{d}{dt}(x^T P x)\right] dt = \left[\frac{d}{dt}(x^T P x)\right]_0^\infty = 0 - {x_0}^T P x_0$$

Then, find the derivative below:

$$\frac{d}{dt}(x^T P x) = \dot{x}^T P x + x^T P \dot{x}$$

Substitute for $\dot{x}$, using $[\dot{x}] = [A][x] + [B][u]$ (or use this form: $\dot{x} = Ax + Bu$ for simplicity:

$$\frac{d}{dt}(x^T P x) = (Ax + Bu)^T P x + x^T P (Ax + Bu)$$

$$J = \int_0^\infty (x^T Q x + u^T R u)\, dt$$

Where, Q minimizes the state of the system (to drive the state of the system to zero), and R controller effort (how much effort). What control value, k, produces the lowest cost J.

For this purpose, let's introduce a matrix P, where $P=P^T$.

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$$J = {x_0}^T P x_0 - {x_0}^T P x_0 + \int_0^\infty (x^T Q x + u^T R u)\, dt$$

Move ${x_0}^T P x_0$ into the integral:

$$J = {x_0}^T P x_0 + \int_0^\infty \left[\frac{d}{dt}(x^T P x) + x^T Q x + u^T R u\right] dt$$

It is valid because:

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$$\frac{d}{dt}(x^T P x) = (Ax + Bu)^T P x + x^T P (Ax + Bu)$$

Substitute in J:

$$J = {x_0}^T P x_0 + \int_0^\infty [(Ax + Bu)^T P x + x^T P (Ax + Bu) + x^T Q x + u^T R u]\, dt$$

Substitute in J:

$$J = {x_0}^T P x_0 + \int_0^\infty [(Ax + Bu)^T Px + x^T P(Ax + Bu) + x^T Qx + u^T Ru]\, dt$$

Apply the transpose:

$$J = {x_0}^T P x_0 + \int_0^\infty [(x^T A^T + u^T B^T)Px + x^T P(Ax + Bu) + x^T Qx + u^T Ru]\, dt$$

Multiply the terms:

$$J = {x_0}^T P x_0 + \int_0^\infty [x^T A^T Px + u^T B^T Px + x^T PAx + x^T PBu + x^T Qx + u^T Ru]\, dt$$

Rearrange:

$$J = {x_0}^T P x_0 + \int_0^\infty [x^T A^T Px + x^T PAx + x^T PBu + x^T Qx + u^T B^T Px + u^T Ru]\, dt$$

Factor x^T:

$$J = {x_0}^T P x_0 + \int_0^\infty [x^T (A^T Px + PAx + Qx) + x^T PBu + u^T B^T Px + u^T Ru]\, dt$$

Factor x:

$$J = {x_0}^T P x_0 + \int_0^\infty [x^T (A^T P + PA + Q)x + u^T B^T Px + x^T PBu + u^T Ru]\, dt$$

Rearrange the terms:

$$J = {x_0}^T P x_0 + \int_0^\infty [x^T (A^T P + PA + Q)x + u^T Ru + x^T PBu + u^T B^T Px]\, dt$$

The goal is to minimize u. The term ${x_0}^T P x_0$ does not depend on u, so we can remove it from the cost function.

$$J = \int_0^\infty [x^T (A^T P + PA + Q)x + u^T Ru + x^T PBu + u^T B^T Px]\, dt \tag{7}$$

Rewrite $u^T Ru + x^T PBu + u^T B^T Px$ in the following form by adding the term $x^T (PBR^{-1}B^T P)x$ and subtracting the same term $x^T (PBR^{-1}B^T P)x$.

$$u^T Ru + x^T PBu + u^T B^T Px + [x^T (PBR^{-1}B^T P)x - x^T (PBR^{-1}B^T P)x] \tag{8}$$

We will rewrite the above equation in a different form, which can help in solving the problem.

Start by defining the following term:

$$(u + R^{-1}B^T Px)^T R(u + R^{-1}B^T Px) - x^T (PBR^{-1}B^T P)x \tag{9}$$

Apply the transpose to the first term (knowing that $P^T = P$, and $(R^{-1})^T = R^{-1}$):

$$(u^T + x^T PBR^{-1})R(u + R^{-1}B^T Px) - x^T (PBR^{-1}B^T P)x$$

Apply the transpose to the first term (knowing that $P^T = P$, and $(R^{-1})^T = R^{-1}$):

$$(u^T + x^T PBR^{-1})R(u + R^{-1}B^T Px) - x^T(PBR^{-1}B^T P)x$$

Multiply the first terms:

$$u^T Ru + u^T RR^{-1}B^T Px + x^T PBR^{-1}Ru + x^T PBR^{-1}RR^{-1}B^T Px - x^T(PBR^{-1}B^T P)x$$

$RR^{-1} = I$, therefore:

$$u^T Ru + u^T B^T Px + x^T PBu + x^T PBR^{-1}B^T Px - x^T(PBR^{-1}B^T P)x$$

Which is the same as the Equation (8):

$$(u + R^{-1}B^T Px)^T R(u + R^{-1}B^T Px) - x^T(PBR^{-1}B^T P)x$$

Therefore, Equations (8) and (9) are the same, as below:

$$\begin{aligned}(u + R^{-1}B^T Px)^T R(u + R^{-1}B^T Px) - x^T(PBR^{-1}B^T P)x \\ = u^T Ru + x^T PBu + u^T B^T Px + [x^T(PBR^{-1}B^T P)x - x^T(PBR^{-1}B^T P)x]\end{aligned}$$

We can now use Equation (9) instead of Equation (8) in Equation (7)

$$\begin{aligned}J = \int_0^\infty [x^T(A^T P + PA + Q)x + (u + R^{-1}B^T Px)^T R(u + R^{-1}B^T Px) \\ - x^T(PBR^{-1}B^T P)x]\, dt\end{aligned} \tag{10}$$

Move the term $PBR^{-1}B^T P$ into the first term:

$$J = \int_0^\infty [x^T(A^T P + PA + Q - PBR^{-1}B^T P)x + (u + R^{-1}B^T Px)^T R(u + R^{-1}B^T Px)]\, dt$$

$$J = \int_0^{\infty} [x^T(A^T P + PA + Q - PBR^{-1}B^T P)x + (u + R^{-1}B^T Px)^T R(u + R^{-1}B^T Px)]\, dt$$

In order to minimize the cost function J, we need to minimize both terms in the above equation.

The first term is:

$$A^T P + PA + Q - PBR^{-1}B^T P = 0$$

This equation is called the Algebraic Riccati Equation (ARE). The ARE can be solved numerically. The goal is to find P that solves the ARE.

The second term in J is:

$$(u + R^{-1}B^T Px)^T R(u + R^{-1}B^T Px) = 0$$

The input effort u is a function of the controller gain k as below:

$$u = -kx$$

If $k$is chosen as $k = R^{-1}B^T P$ then $u + R^{-1}B^T Px = 0$.

Therefore, P is obtained from ARE, and then substituted in $u = -kx$, where $k = R^{-1}B^T P$.

PID vs LQR

- PID:

https://ctms.engin.umich.edu/CTMS/index.php?example=InvertedPendulum&section=ControlPID

- LQR:

https://ctms.engin.umich.edu/CTMS/index.php?example=InvertedPendulum&section=ControlStateSpace

It is much easier to control multi-input, multi-output systems with the state-space method (LQR here) than with the other methods such as PID, Root Locus and Frequency analysis approaches.