Errata

December 23, 2025

Arbitrarily Close: An Introduction to Real Analysis, first edition 2025. Published by 619 Wreath.

This errata addresses mistakes and typos found in the 2025 print versions of *Arbitrarily Close* as well as the identical electronic versions acl_print_251005.pdf and acl_web_251005.pdf.

Deleted content is struck through, ~~like this~~, while added content appears between brackets, [like this].

1. (p.10) **Remark 0.7.1**, last bullet. "A continuous function on a closed and bounded interval is arbitrarily close [to] the set of polynomials."
2. (p.36) **Notation 1.2.23**. Above line (1.2.23), $0! = 1$, not 0.
3. (p.41) Above **Axiom 1.3.1**. "I am assuming properties therein are familiar and will work ~~them~~ without explicitly citing them."
4. (p.130) Above line (2.4.35). "…and proper[t]ies of inequalities,…"
5. (p.170) **Example 2.8.11**. "… since $\mathbb{Z}$ [is] the range of…"
6. (p.173) Below line (2.8.43)."…the nth digit of y is distinct from ~~from~~ [the] nth digit…"
7. (p.236) **Exercise 3.5.4**(ii). "sub~~t~~intervals" (delete the extra "t").
8. (p.289) **Example 4.4.2**. "…define $f : \mathbb{R} \to \mathbb{R}$ ~~define~~ by…"
9. (p.291) **Example 4.4.3**. "…define $g : \mathbb{R} \to \mathbb{R}$ ~~define~~ by…"
10. (p.293) **Example 4.4.4**. "…define $v : \mathbb{R} \to \mathbb{R}$ ~~define~~ by…"
11. (p.300) **Exercise 4.4.1**. "…a *level set* is [a] set of the form…"
12. (p.301) End of **Exercise 4.4.6**. "Thomae's function g is [dis]continuous on the rationals and ~~dis~~continuous on the irrationals."
13. (p.304) End of Proof of Thm. 4.5.5. "Therefore, αf is continuous [at $\mathbf{c}$] when $\alpha = 0$."
14. (p.309) Proof of Thm. 4.5.16. Above and below line (4.5.46), replace equation "$h(x) = 1/x$" with "$h(y) = 1/y$".
15. (p.313) Proof of thm. 4.6.7, first paragraph. "By the Heine-Borel Theorem 3.5.1, K is sequentially compact (Definition ~~4.4.5~~ [3.5.11])…"
16. (p.319) **Exercise 4.6.8**(iii). Line (4.6.40) should be "$\bigcap_{\delta>0} \overline{h(V_\delta(0)\setminus\{0\})} = \mathbb{R}$." Also, below line (4.6.40), "…the image of *every* δ-neighborhood of 0 [(excluding 0)] under h…"
17. (p.330) Above **Definition 5.1.2**. "Modifying continuity to allow the functions to *not* [be] defined at c is accomplished …"
18. (p.348) Proof of Thm. 5.2.15 between (5.2.40) and (5.2.41). "Since f and ~~g~~ h converge to ℓ at c,…"

19. (p.349) **Exercise 5.2.7**(ii) is false. Replace with "Prove $\lim_{x\to c} g(x) = 0$ at each $c \in \mathbb{R}$."

20. (p.354) Proof for Example 5.3.6. "Then by quotients and linearity of functional limits (Theorems 5.2.13 [and] 5.2.6) we have..."

21. (p.354) Between **Definitions 5.3.8** and **5.3.9**. "To codify [the] idea that differentiable functions..."

22. (p.369) **Scratch Work 5.5.4**, first paragraph. "..., or rather nonnegative and nonpositive numbers, through a manipulation [of] difference quotients and inequalities."

23. (p.370) Second-to-last paragraph of **Scratch Work 5.5.4**. "Combining these ~~denomators~~ [denominators] with our nonpositive numerator allows..."

24. (p.379) **Exercise 5.5.7**, line (5.5.60). Replace "$f(x) > f(c)$" with "$f(x_0) > f(c)$".

25. (p.405) **Theorem 6.2.12**(iii), line (6.2.56). "There is [a] sequence (P_n) of partitions..."

26. (p.430) Above **Theorem 6.4.16**. "Modifying the hypothesis of integrability of the integrand in Theorem 6.4.14 with ~~continat~~ [continuity at] a point yields..."

27. (p.430) Proof of Thm. 6.4.16, above line (6.4.86). "Next, to show the limit of the difference [quotient] of g at c is $f(c)$,..."

28. (p.433) Above **Example 7.1.1**. "Pointwise convergence is ~~and~~ [an] extension [of] componentwise convergence."

29. (p.434) **Remark 7.1.2**. "...vectors in a Euclidean space $\mathbb{R}^m$ to ~~a~~ the *pointwise convergence* of a sequence..." Also, add parentheses at the end of the paragraph to get "(Theorem 2.4.11)".

30. (p.449) Between lines (7.2.21) and (7.2.22). "By (1.2.33) and (7.2.21), for every ~~indx~~ [index] $n \geq n_\varepsilon$ ..."

31. (p.464) First sentence of Section **7.4**. "Polynomials are ~~one~~ [some] of...

32. (p.482) Proof for Example 8.1.18. A key mistake in line (8.1.62) leads to more mistakes throughout this proof.

    - Line (8.1.62). Replace the rightmost $\dfrac{1}{2^n}$ with $\dfrac{1}{2^{n-1}}$.
    - Above line (8.1.63). "The sequence of partial sums (s_k) is bounded above by ~~2~~ [4] since,..."
    - Replace line (8.1.63) with

    $$s_k = \sum_{n=0}^{k} \frac{1}{n!} \leq \sum_{n=0}^{k} \frac{1}{2^{n-1}} = \frac{2(1-(1/2)^{n+1})}{1-(1/2)} \leq \frac{2}{1/2} = 4,$$

    - Line (8.1.64). Replace the rightmost sum $\displaystyle\sum_{n=0}^{k+1} \frac{1}{2^n}$ with $\displaystyle\sum_{n=0}^{k+1} \frac{1}{n!}$.

    NOTE: For convenience, a copy of the revised page 482 is included at the end of this document.

33. (pp.492–494) **Theorem 8.2.13** and its proof. Numerous instances of the same mistake occur throughout these pages. Specifically, replace a_k with a_{2^k} so that $\sum_{k=0}^{\infty} 2^k a_k$ becomes $\sum_{k=0}^{\infty} 2^k a_{2^k}$.

NOTE: For convenience, a copy of the revised pages 492, 493, and 494 are included at the end of this document.

34. (p.493) Line (8.2.48). Replace the middle $\leq$ with an equal sign since $\dfrac{1-2^q}{1-2} = 2^q - 1$.

35. (p.493) Line (8.2.53). Replace the equal sign with $\leq$.

36. (p.494) Above line (8.2.65). Replace "$\sum_{k=0}^{\infty} 2^k a_k$" (the first of the two series in that line) with "$\sum_{n=1}^{\infty} a_n$".

37. (p.527) Above line (8.5.64). The letter "n" is missing: "...(a fact from calculus that is assumed but not prove[n] here)..."

38. (p.527) Line (8.5.65) is missing a period at the end.

Many thanks to the following mathematicians for helping me catch so many mistakes!

Grant Gleass
Max Morphew
Philip Nicoll
Kenzie Lam
June Nguyen
John Rodriguez

Copies of the revised pages 482, 492, 493, and 494 are included below.

Thus, the sequence of partial sums is unbounded and so, by the Divergence Criteria for Sequences 2.6.9, the harmonic series diverges.

The details of the induction argument and the proof are left as an exercise.

The number e from calculus is known as *Euler's number*, and one way to define it is as the sum of a special convergent series of positive numbers.

Example 8.1.18: Euler's number e

Euler's number e is the sum of the series whose terms are $1/(n!)$ for each index $n \in \mathbb{N} \cup \{0\}$. That is, this series converges and we define e as the sum. Hence,

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}. \tag{8.1.61}$$

Scratch Work 8.1.19: Comparing partial sums

The goals is to show Euler's number e is well-defined by showing its series converges. As with all the proofs in the section up to this point, the result is obtained by considering partial sums and taking advantage of properties of sequential limits. Here, the convergence of the given series follows from showing the partial sums form an increasing sequence which is bounded above and, therefore, converges by the Monotone and Bounded Convergence Theorem 2.4.9.

Proof for Example 8.1.18. Consider the series whose terms are $1/n!$ for each $n \in \mathbb{N} \cup \{0\}$. Since $1/n \leq 1/2$ when $n \geq 2$, for each $n \in \mathbb{N} \cup \{0\}$ we have

$$0 \leq \frac{1}{n!} = \frac{1}{1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n} \leq \frac{1}{\underbrace{1 \cdot 2 \cdot 2 \cdots 2 \cdot 2}_{n \text{ factors}}} = \frac{1}{2^{n-1}}. \tag{8.1.62}$$

The sequence of partial sums (s_k) is bounded above by 4 since, for each $k \in \mathbb{N} \cup \{0\}$,

$$s_k = \sum_{n=0}^{k} \frac{1}{n!} \leq \sum_{n=0}^{k} \frac{1}{2^{n-1}} = \frac{2(1-(1/2)^{n+1})}{1-(1/2)} \leq \frac{2}{1/2} = 4, \tag{8.1.63}$$

where the sum on the right is a geometric sum and the Geometric Sum Formula 2.7.2 applies. Also, the sequence of partial sums (s_k) is increasing since

$$s_k = \sum_{n=0}^{k} \frac{1}{n!} \leq \left(\sum_{n=0}^{k} \frac{1}{n!}\right) + \frac{1}{(k+1)!} = \sum_{n=0}^{k+1} \frac{1}{n!} = s_{k+1}. \tag{8.1.64}$$

Hence, by the Monotone and Bounded Convergence Theorem 2.4.9, the sequence of partial sums (s_k) converges. Therefore, the number e is well-defined by

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} \tag{8.1.65}$$

since this series converges by Definition 8.1.3. □

Theorem 8.2.13: Cauchy Condensation Test

Suppose (a_n) is a decreasing sequence of nonnegative terms ($0 \leq a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$). Then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \qquad \Longleftrightarrow \qquad \sum_{k=0}^{\infty} 2^k a_{2^k} \text{ converges.} \tag{8.2.42}$$

When these series converge we have

$$\frac{1}{2}\sum_{k=0}^{\infty} 2^k a_{2^k} \leq \sum_{n=1}^{\infty} a_n \leq \sum_{k=0}^{\infty} 2^k a_{2^k}. \tag{8.2.43}$$

Scratch Work 8.2.14: A subtle comparison of terms

The powers of 2 have shown up from time to time, like with one of the geometric series that sums to 1 in Example 8.1.12 and in Scratch Work 8.1.16 which leads to a proof of the divergence of the harmonic series in Example 8.1.16. Here, they give us a way to compare the partial sums of the related series in the statement of Theorem 8.2.13 that allow us to take advantage of the Comparison Test 8.2.11 to get the convergence of one of the series to yield the other. In particular, by the Geometric Sum Formula 2.7.2, the geometric sum of 2^n from $n = 0$ to k simplifies nicely to a positive integer:

$$\sum_{n=0}^{k} 2^n = 1 + 2 + 4 + \cdots + 2^k = \frac{1 - 2^{k+1}}{1 - 2} = 2^{k+1} - 1 \in \mathbb{N}. \tag{8.2.44}$$

This fortuitous result allows us to compare not just the terms but the *indices* of the series in question, facilitating the proof.

Also, since the terms in both series are nonnegative, their partial sums define increasing sequences which, if they converge, they converge to suprema as in the Monotone and Bounded Convergence Theorem 2.4.9.

Proof of Theorem 8.2.13. Suppose $(a_n) \subseteq \mathbb{R}$ where $0 \leq a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$, and consider the pair of series

$$\sum_{n=1}^{\infty} a_n \qquad \text{and} \qquad \sum_{k=0}^{\infty} 2^k a_{2^k}. \tag{8.2.45}$$

For each $j \in \mathbb{N}$ and $q \in \mathbb{N} \cup \{0\}$, let s_j denote the jth partial sum of $\sum_{n=1}^{\infty} a_n$ and let t_q denote the qth partial sum of $\sum_{k=0}^{\infty} 2^k a_{2^k}$ so that

$$s_j = a_1 + a_2 + a_3 + \cdots + a_j \qquad \text{and} \tag{8.2.46}$$

$$t_q = a_1 + 2a_2 + 4a_4 + \cdots + 2^q a_{2^q}. \tag{8.2.47}$$

Then both sequences of partial sums (s_j) and (t_q) are increasing. So, by the

From here, we split the argument into two parts depending on which series is assumed to converge and how the index j compares to the index 2^q.

To prove the forward implication, assume $\sum_{n=1}^{\infty} a_n$ converges and $j \geq 2^q$. So as in Scratch Work 8.2.14, we have

$$\sum_{n=0}^{q-1} 2^n = 1 + 2 + 4 + \cdots + 2^{q-1} = \frac{1-2^q}{1-2} = 2^q - 1 \leq 2^q \leq j. \tag{8.2.48}$$

Since $0 \leq a_{n+1} \leq a_n$ for every $n \in \mathbb{N}$, we have

$$\begin{aligned}
\frac{1}{2}t_q &= \frac{1}{2}\left(a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots + 2^q a_{2^q}\right) && (8.2.49)\\
&= \frac{1}{2}a_1 + a_2 + 2a_4 + 4a_8 + \cdots + 2^{q-1}a_{2^q} && (8.2.50)\\
&= \frac{1}{2}a_1 + a_2 + \underbrace{(a_4 + a_4)}_{2 \text{ terms}} + \underbrace{(a_8 + a_8 + a_8 + a_8)}_{4 \text{ terms}} + \cdots + \underbrace{(a_{2^q} + \cdots + a_{2^q})}_{2^{q-1} \text{ terms}} && (8.2.51)\\
&\leq a_1 + a_2 + \underbrace{(a_3 + a_4)}_{2 \text{ terms}} + \underbrace{(a_5 + a_6 + a_7 + a_8)}_{4 \text{ terms}} + \cdots + \underbrace{(a_{2^{q-1}+1} + \cdots + a_{2^q})}_{2^{q-1} \text{ terms}} && (8.2.52)\\
&\leq a_1 + a_2 + a_3 + a_4 + \cdots + a_{2^q} + \cdots + a_j && (8.2.53)\\
&= s_j. && (8.2.54)
\end{aligned}$$

Now, since $\sum_{n=1}^{\infty} a_n$ converges and (s_j) is increasing, the Monotone and Bounded Convergence Theorem 2.4.9 implies

$$\frac{1}{2}t_q \leq s_j \leq \sup\{s_j : j \in \mathbb{N}\} = \lim_{j \to \infty} s_j = \sum_{n=1}^{\infty} a_n. \tag{8.2.55}$$

Thus, $\sum_{n=1}^{\infty} a_n$ is an upper bound for $(t_q/2)$. Therefore, by another application of the Monotone and Bounded Convergence Theorem 2.4.9 along with the fact that a supremum is the *least* upper bound (Theorem 1.3.10), the series $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges and

$$\frac{1}{2}\sup\{t_q : q \in \mathbb{N}\} = \frac{1}{2}\lim_{q \to \infty} t_q = \frac{1}{2}\sum_{k=0}^{\infty} 2^k a_{2^k} \leq \sum_{n=1}^{\infty} a_n. \tag{8.2.56}$$

Therefore, the forward implication holds.

To prove the backward implication, assume $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges and $j \leq 2^q$. Again, as in is Scratch Work 8.2.14, we have

$$j \leq 2^q \leq \sum_{n=0}^{q} 2^n = 1 + 2 + 4 + \cdots + 2^q = \frac{1-2^{q+1}}{1-2} = 2^{q+1} - 1 \in \mathbb{N}. \tag{8.2.57}$$

Then, as done in Scratch Work 8.1.16 on the divergence harmonic series, by grouping the terms of the partial sum s_j by taking indices in successive chunks of powers of two, and keeping in mind

$0 \le a_{n+1} \le a_n$ for every $n \in \mathbb{N}$, we get

$$s_j = a_1 + a_2 + a_3 + \cdots + a_j \tag{8.2.58}$$
$$\le a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \cdots + (a_{2^q} + \cdots + a_{2^{q+1}-1}) \tag{8.2.59}$$
$$\le a_1 + \underbrace{(a_2 + a_2)}_{2 \text{ terms}} + \underbrace{(a_4 + a_4 + a_4 + a_4)}_{4 \text{ terms}} + \cdots + \underbrace{(a_{2^q} + \cdots + a_{2^q})}_{2^q \text{ terms}} \tag{8.2.60}$$
$$= a_1 + 2a_2 + 4a_4 + \cdots + 2^q a_{2^q} \tag{8.2.61}$$
$$= t_q. \tag{8.2.62}$$

Now, since $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges converges and (t_q) is increasing, the Monotone and Bounded Convergence Theorem 2.4.9 implies

$$s_j \le t_q \le \sup\{t_q : q \in \mathbb{N} \cup \{0\}\} = \lim_{q\to\infty} t_q = \sum_{k=0}^{\infty} 2^k a_{2^k}. \tag{8.2.63}$$

Thus, $\sum_{k=0}^{\infty} 2^k a_{2^k}$ is an upper bound for (s_j). Therefore, by another application of the Monotone and Bounded Convergence Theorem 2.4.9 and noting a supremum is the *least* upper bound (Theorem 1.3.10), $\sum_{n=1}^{\infty} a_n$ converges and

$$\sup\{s_j : j \in \mathbb{N}\} = \lim_{j\to\infty} s_j = \sum_{n=1}^{\infty} a_n \le \sum_{k=0}^{\infty} 2^k a_{2^k}. \tag{8.2.64}$$

Therefore, the backward implication holds.

Overall, if either $\sum_{n=1}^{\infty} a_n$ or $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges, then both converge and

$$\frac{1}{2}\sum_{k=0}^{\infty} 2^k a_{2^k} \le \sum_{n=1}^{\infty} a_n \le \sum_{k=0}^{\infty} 2^k a_{2^k}. \tag{8.2.65}$$

□

One payoff of the Cauchy Condensation Test 8.2.13 is the p-series test from calculus.

Theorem 8.2.15: p-series

Suppose $p \in \mathbb{R}$. Then the so-called p-series given by

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \tag{8.2.66}$$

converges if and only if $p > 1$.

Proof of Theorem 8.2.15. First, suppose $p \le 0$. Then $-p = \alpha \ge 0$ and for each $n \in \mathbb{N}$,

$$\frac{1}{n^p} = n^{-p} = n^{\alpha} \ge 1. \tag{8.2.67}$$

So $(1/n^p) = (n^\alpha)$ does not converge to zero. Hence, the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \sum_{n=1}^{\infty} n^{\alpha}. \tag{8.2.68}$$