Dr. John A. Rock
May 1, 2024

**Linearity in calculus**

**Overview.** *Linearity* is a property found throughout many branches of mathematics, including calculus. A process, procedure, function, or other mathematical concept $T$ is said to be *linear* if it (i) distributes across addition and allows constant multiples to be factored out, like this:

(i) $T(f+g) = Tf + Tg$ for any $f$ and $g$, and (ii) $T(cf) = cTf$ for any constant $c$ and any $f$.

- Linearity of limits: $\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$ and $\lim_{x\to a}(cf(x)) = c\lim_{x\to a} f(x)$.
- Linearity of continuity: If $c$ is a constant and $f$ and $g$ are continuous, then $f+g$ and $cf$ are continuous.
- Linearity of differentiation: $(f(x)+g(x))' = f'(x) + g'(x)$ and $(cf(x))' = cf'(x)$.
- Linearity of integration: $\int(f+g) = \int f + \int g$ and $\int(cf) = c\int f$.
- Linearity of sequential limits: $\lim_{n\to\infty}(a_n+b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$ and $\lim_{n\to\infty}(ca_n) = c\lim_{n\to\infty} a_n$.

**Vector spaces: The home of linearity.** A *vector space* is a set whose objects are called *vectors* where addition and *scalar multiplication* (i.e., multiplication by constants) make sense and play nicely together. For instance, the set of functions we see in calculus form a vector space! That is, we can think of the functions in calculus as vectors. Thanks to the facts listed above, the sets of continuous functions, differentiable functions, and integrable functions form their own vector spaces as well.

The fundamental objects in a vector space are its *linear combinations* given by sums of a finite number of scaled vectors, like this:

$$c_1f_1 + c_2f_2 \cdots + c_kf_k = \sum_{j=1}^{k} c_jf_j,$$

where $c_1, c_2, \ldots, c_k$ are scalars and $f_1, f_2, \ldots, f_k$ are vectors. *Polynomials* are examples of linear combinations:

$$p(x) = c_kx^k + c_{k-1}x^{k-1} + \cdots + c_1x + c_0 = \sum_{j=0}^{k} c_jx^j.$$

**Linearity and linear combinations.** Induction on the above concepts extend linearity to linear combinations.

- The limit of a linear combination is the linear combination of limits (functions):

$$\lim_{x\to a}(c_1f_1(x) + \cdots + c_kf_k(x)) = \lim_{x\to a}\left(\sum_{j=1}^{k} c_jf_j(x)\right)$$
$$= \sum_{j=1}^{k}\left(c_j\lim_{x\to a} f_j(x)\right) = c_1\lim_{x\to a} f_1(x) + \cdots + c_k\lim_{x\to a} f_k(x).$$

- A linear combination of continuous functions is continuous:

$$f_j \text{ continuous and } c_j \text{ constant for all } j = 1, \ldots, k \quad \Longrightarrow \quad \textstyle\sum_{j=1}^{k}(c_jf_j) \text{ is continuous.}$$

- The derivative of a linear combination is the linear combination of derivatives:

$$\left(\textstyle\sum_{j=1}^{k} c_jf_j\right)' = \textstyle\sum_{j=1}^{k}(c_jf_j').$$

- The integral of a linear combination is the linear combination of integrals:

$$\int\left(\textstyle\sum_{j=1}^{k} c_jf_j\right) = \textstyle\sum_{j=1}^{k}(c_j\int f_j).$$

- The limit of a linear combination is the linear combination of limits (sequences):

$$\lim_{n\to\infty}(\textstyle\sum_{j=1}^{k} c_ja_{n,j}) = \textstyle\sum_{j=1}^{k}(c_j\lim_{n\to\infty} a_{n,j}).$$