Power Spectral Estimation Using Nonuniform Fast Fourier Transform

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Abstract

The frequency content of an analog signal can be found by uniformly sampling the analog signal, and then applying discrete Fourier transform (DFT). The fast Fourier transform is an efficient algorithm to calculate DFT. The number of complex multiplications required by fast Fourier transform (FFT) algorithm is in the order of $Nlog(N)$ compared to N^2 for direct calculation. When the samples are taken at nonuniform intervals, the fast Fourier transform does not apply. For the nonuniformly sampled signals, nonuniform fast Fourier transform (NUFFT) is applied. Like FFT, the NUFFT reduces the number of complex multiplications to the order of $Nlog(N)$. In this research, different types of NUFFT will be analyzed and implemented. Then the NUFFT will be applied to power spectral estimation of various signals.

Objective

To utilize MATLAB, a programming language and numeric computing environment, to compare the resolutions from nonuniform FFT and uniform FFT with comparable complexities. The resolutions can be measured by applying an input signal consisting of the sum of two sinusoids with frequency difference df and finding the smallest frequency difference df that can be distinguished.

Background

Nonparametric Power Spectral Estimation:

- No assumptions are made on how the data was generated
- To reduce variance, averaging using various windows are applied
- This process decreases frequency resolution
- A periodogram is used to identify the dominant periods (or frequencies) of a time series
- Examples: Bartlett Method, Welch Method, Blackman and Tukey Method

Bartlett Method:

- $P_{xx}\left(\frac{n}{N}\right) = \frac{1}{N}\left|\sum_{k=0}^{N-1} x(k)e^{-\frac{jn2\pi}{N}k}\right|^2, n = 0,1,\dots,N-1$
- $P_{xx}^{(i)}(F) = \frac{1}{M}\left|\sum_{k=0}^{M-1} x_i(k)e^{-j2\pi Fk}\right|^2, i = 0,1,\dots,K-1 \rightarrow$ average of $P_{xx}^{(i)}(F)$ = Bartlett power spectral estimate
- Reducing the length of the data from N points to M=N/K results in a window whose spectral width has been increased by a factor of K → frequency resolution is decreased by a factor of K
- Smaller variance = more accurate estimation of the spectrum

Welch Method:

- Bartlett Method becomes Welch Method when data segments overlap, and the data segments prior to computing the periodogram are windowed
- $P_{xx}^{(i)}(F) = \frac{1}{MU}\left|\sum_{k=0}^{M-1} x_i(k)w(k)e^{-j2\pi Fk}\right|^2, i = 0,1,\dots,L-1$

Blackman and Tukey Method:

- The autocorrelation function is windowed before discrete-time Fourier transform is taken
- $P_{xx}^{BT}(F) = \sum_{m=-(M-1)}^{M-1} r_{xx}(m)w(m)e^{-j2\pi Fm}$
- The variance is reduced because the smoothing lowers the possible fluctuations (variations) of the spectrum

Applications include:

- Radar imaging
- Computing oriented wavelets via the Random transform
- X-ray Computed Tomography (CT)
- Magnetic Resonance Imaging (MRI)

Results and Discussion

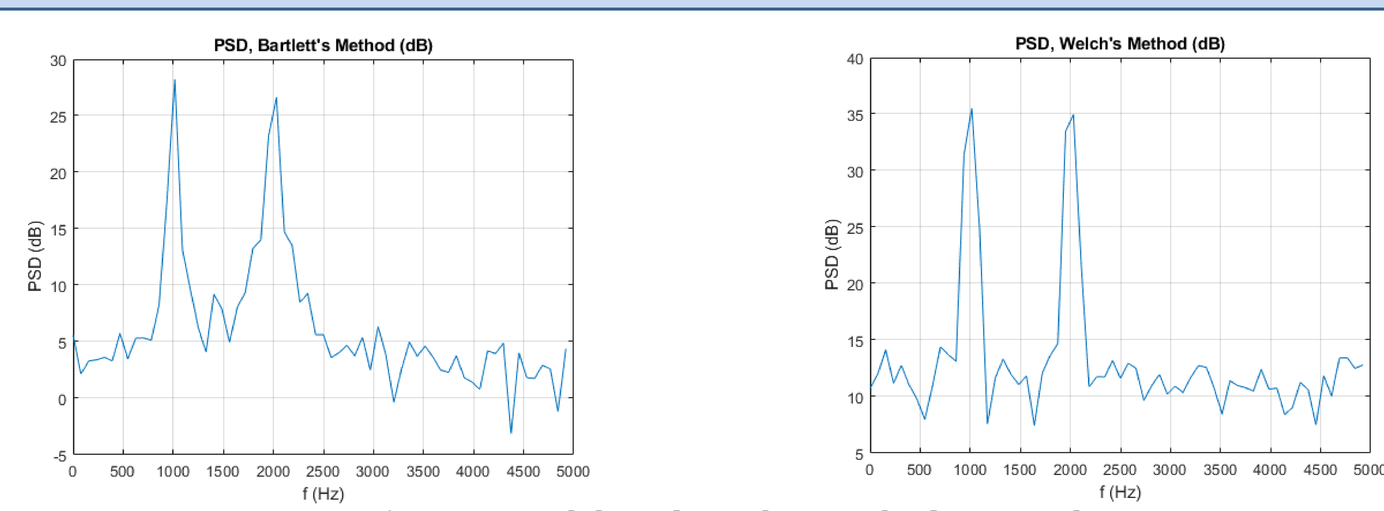

Figure 1: Welch and Bartlett Methods Example

- As shown above, we can see the difference between both methods when a sinusoid is applied
- Bartlett Method: Uses sequential, non-overlapped, windowless data segments
- Welch Method: Uses overlapped, windowed data segments

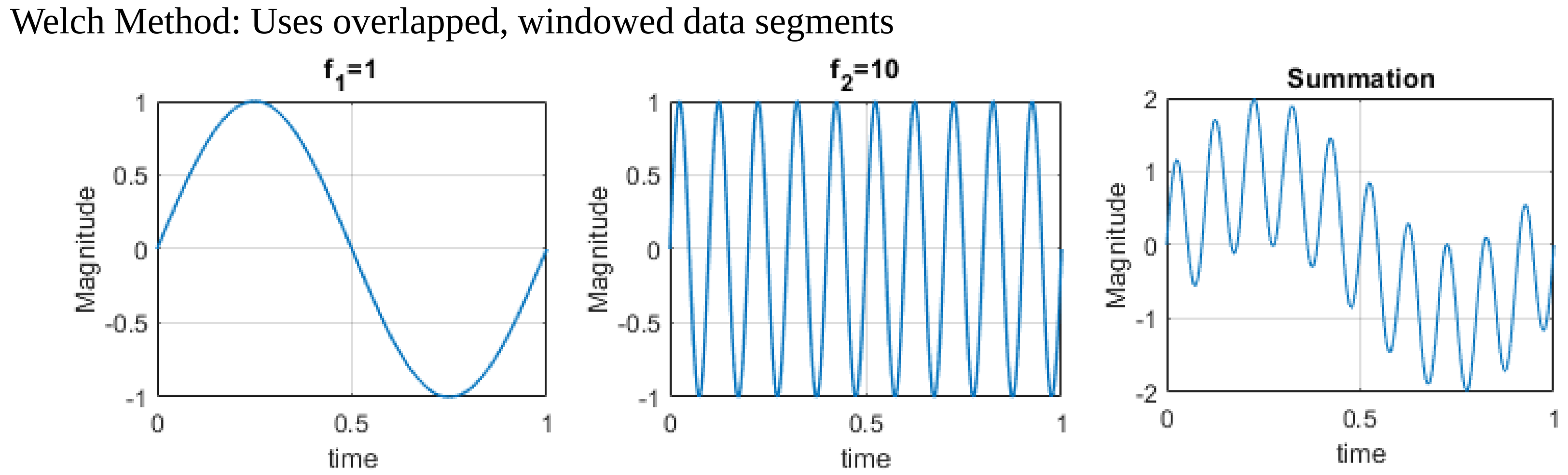

Figure 2: Summation of Two Sinusoids Example

- In this example, different frequencies ($f_1 = 1$ and $f_2 = 10$) are evaluated using the formula $\sin(2\pi ft)$
- The resulting summation is shown and we can note a frequency difference of 9 Hz

```
Fs = 150;  t = 0:1/Fs:1; % Sampling freq, Time vector of 1 sec
f = 5; % Create a sine wave of f Hz.
x = sin(2*pi*t*f); nfft = 1024; % Length of FFT
%Take fft, padding with zeros so that length(X)=nfft
X = fft(x,nfft);
%FFT is symmetric, throw away second half
X = X(1:nfft/2);
% Take the magnitude of fft of x
mx = abs(X);
% Frequency vector
f = (0:nfft/2-1)*Fs/nfft;
figure(1); plot(t,x); title('Sine Wave Signal');
xlabel('Time (s)'); ylabel('Amplitude');
figure(2); plot(f,mx); title('Power Spectrum of a Sine Wave'); xlabel('Frequency (Hz)'); ylabel('Power');
```

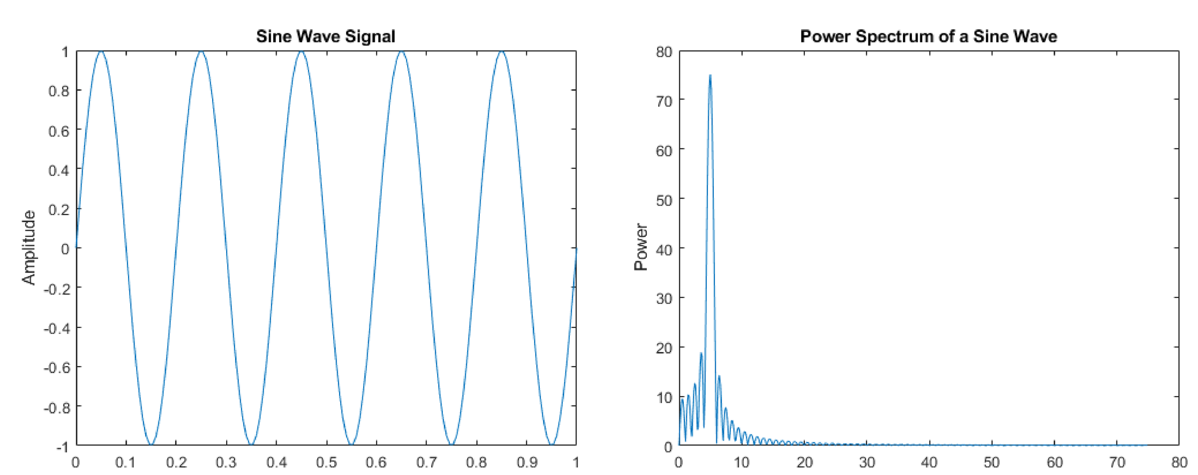

Figure 3: Power Spectrum of a Sine Wave

- In MATLAB sample code above, the FFT was taken, padding with zeros to make the length(x) equal to NFFT
- FFT is symmetric so NFFT is divided by a factor of two

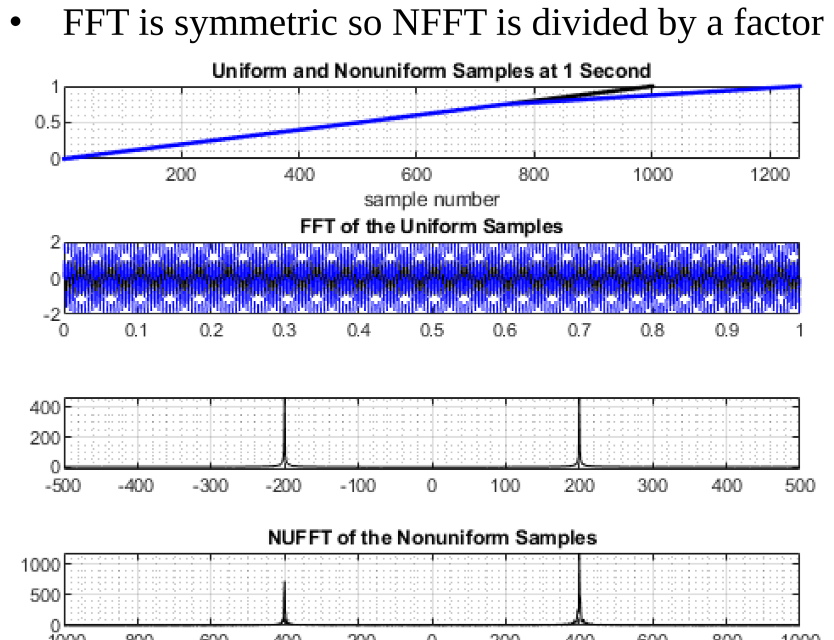

Figure 4: FFT and NUFFT Compared

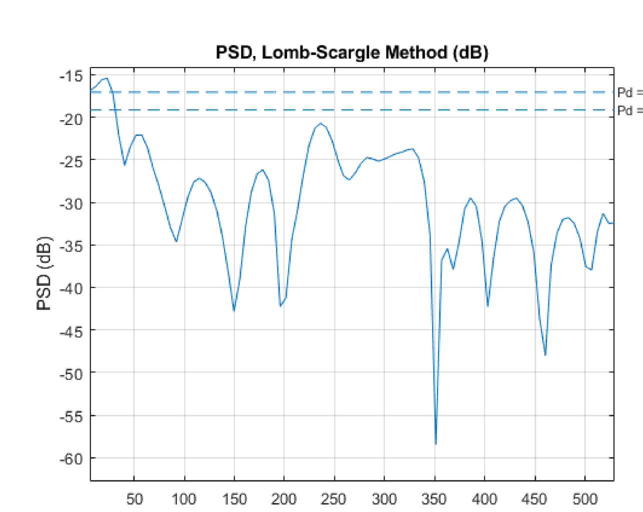

Figure 5: Lomb-Scargle Method

- The dashed lines denote 95% and 50% detection probabilities of the evaluated sinusoids peaks

Conclusion

- NUFFT is a fast algorithm for computing discrete sums of a certain type
- FFT is fast but fails to process many nonuniform data found in the real world
- NUFFT allows users to compute the FFT of nonuniformly sampled signals
- Nonuniform sampling techniques generate fewer samples → less data to process and lower power consumption
- The power spectrum is related to the correlation function through the Fourier transform
- Power spectrums reveal the repetitive and correlated patterns of a signal → important for detection, estimation, and decision making system
- Conventional spectral analysis techniques like the periodogram and the Welch method require the input signal to be uniformly sampled
- When the sampling is nonuniform, one can resample or interpolate the signal onto a uniform sample grid; however, this can add undesired artifacts to the spectrum and might lead to analysis errors.
- The Lomb-Scargle method works directly with nonuniform samples and makes it unnecessary to resample or interpolate
- The Lomb-Scargle method can handle signals that have been sampled unevenly or have missing samples
- Originally in this project, MATLAB nufft/nfft functions were solely going to be used but the Lomb-Scargle method was also included upon further research of the advantages it possessed
- In the future, I would like to conduct a more in depth analysis into NUFFT when it comes to medical applications, and I would utilize the pynufft function within Google Colab to implement the results in python

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