SUMMARY:

A- State of Stress

B- Stress Transformation Equations

C- Mohr's Circle, Equations

D- Mohr's Circle, Graphical Method

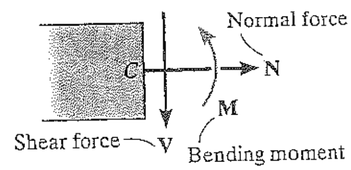
E- Examples and Class Exercises

F- Homework Assignments

A- State of Stress

Plane State of Stress on the surface of any structural member can be found using principals and concepts presented in Chapters 1 to 8. Examples are followed.

Internal Actions
From Statics Equilibrium



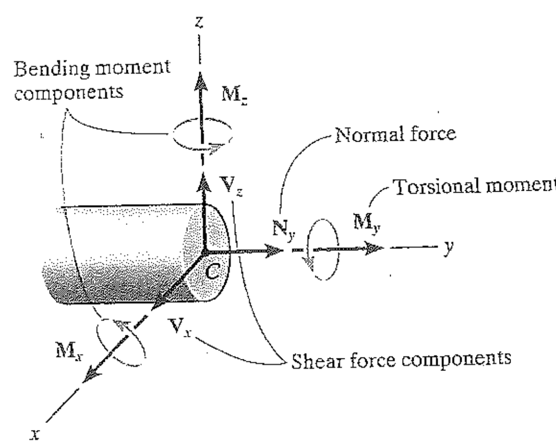
Possible State of Stress in 2-D:

$\sigma = \pm N/A \pm M_z C / I_z$

$\tau = \pm V\,Q/I\,t$

Where N is Axial Force, M_z is Bending Moment, and V is Transverse Shear Force.

Internal Actions
From Statics Equilibrium

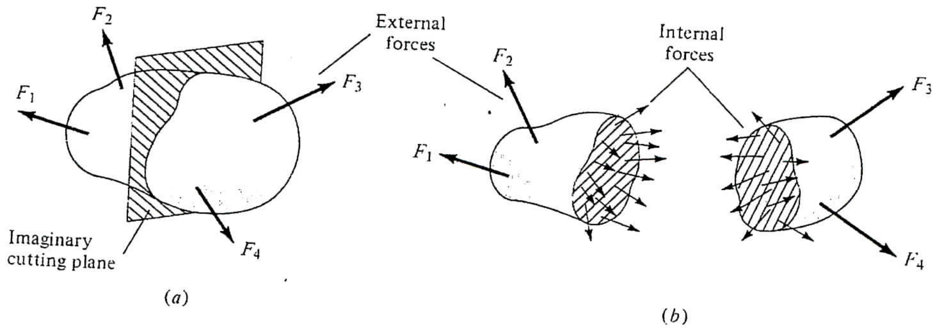


Possible State of Stress in 3-D:

$\sigma = \pm N/A \pm M_z C / I_z \pm M_x C / I_x$

$\tau = \pm TC / J \pm V_z Q / I_z\,t \pm V_x Q / I_x\,t$

Where N is Axial Force, M_x and M_z are Bending Moments, $T = M_y$ is Torque, and V_z and V_x are Transverse Shear Forces.



Internal forces exposed by an imaginary cutting plane.

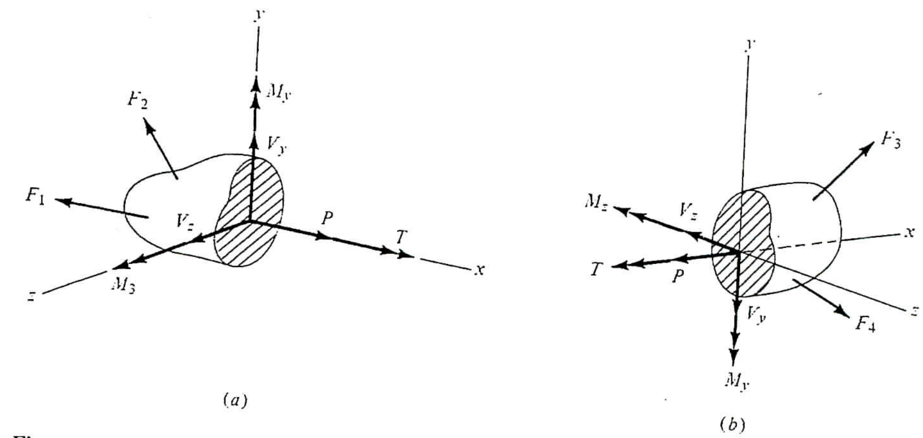


Figure 2-2 Internal actions on either side of an imaginary cutting plane.

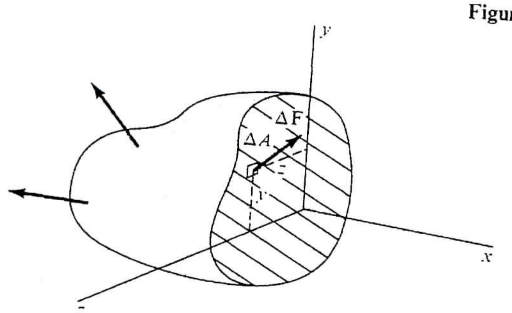


Figure 2-5 Force $\Delta \mathbf{F}$ acting on area ΔA of a cut section.

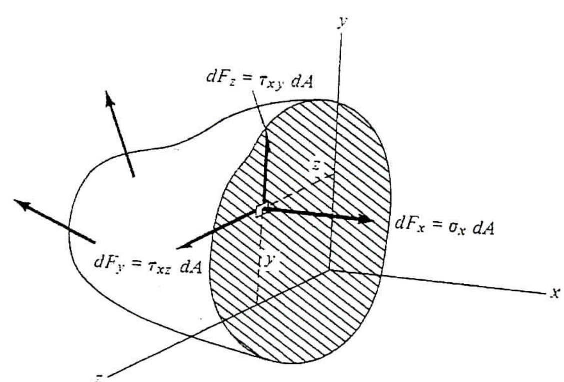


Figure 2-7 Forces on a typical infinitesimal area dA in terms of stress components.

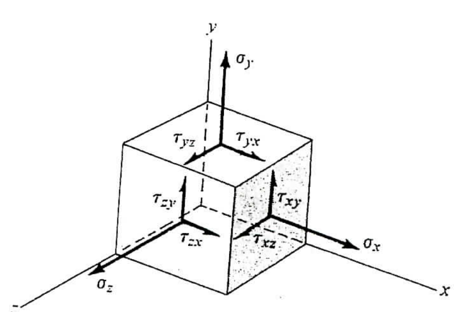
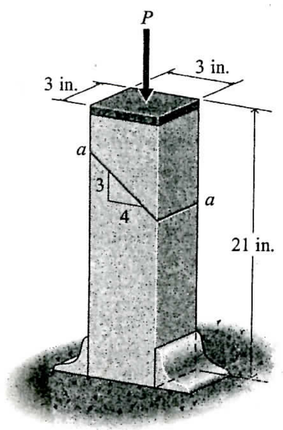
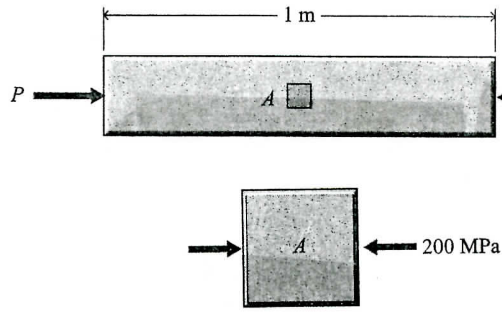
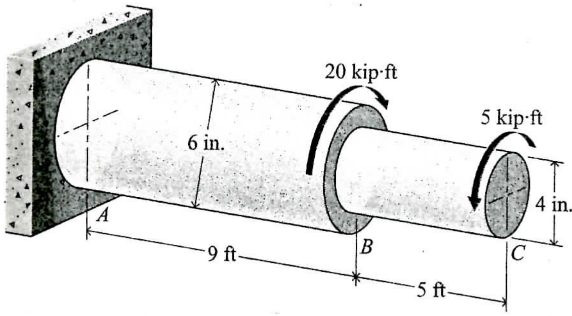


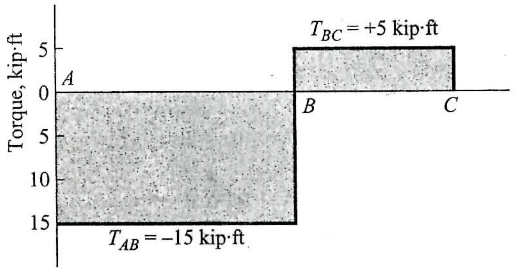
Figure 2-9 Stress components on an infinitesimal cubical element at a point.

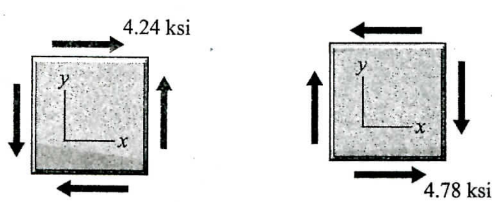




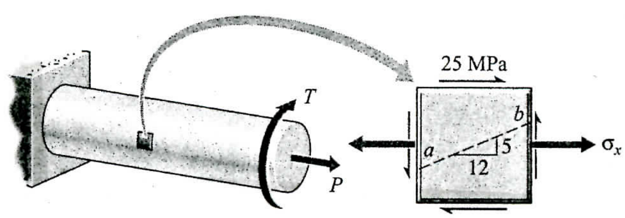
Axial



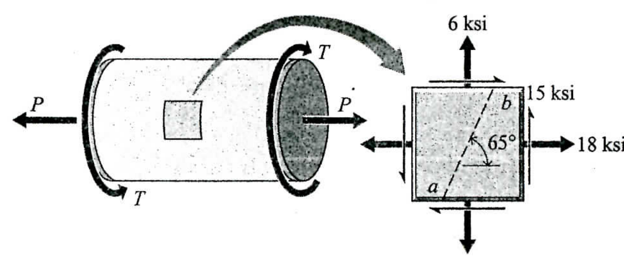




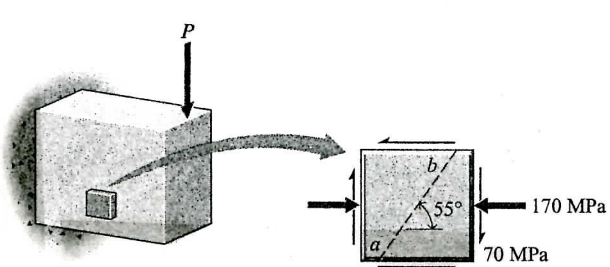
Torque/Torsion



Axial and Torsion Combined

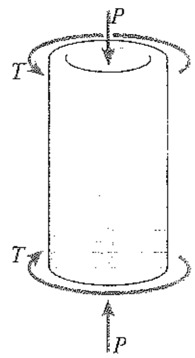


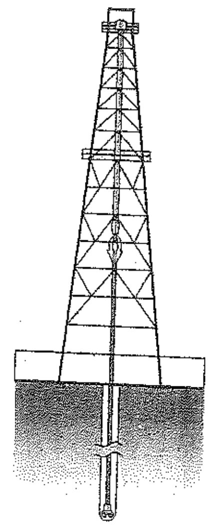
Plane Stress



Bending

Question 5:
The hollow drill pipe for a soil auger has 2.5-in outer diameter and ¼ -in thickness. The compressive force P is 5,000-lb and the drilling torque T is 500-lb-ft. Determine the State of Stress on a point on the outside surface of the pipe.



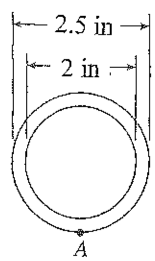


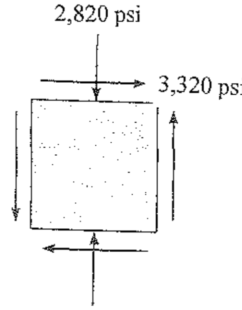
$A = \pi\,(1.25)^2 - \pi\,(1.00)^2 = 1.77\text{-in}^2$

$J = \pi\,(1.25)^4 - \pi\,(1.00)^4 = 2.26\text{-in}^4$

$\sigma_y = -\,(5{,}000) / 1.77 = -\,2{,}820\text{-psi}$

$\tau_{xy} = (\,500 \times 12) \times 1.25 / 2.26 = 3{,}320\text{-psi}$

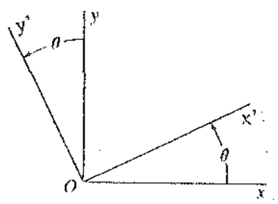


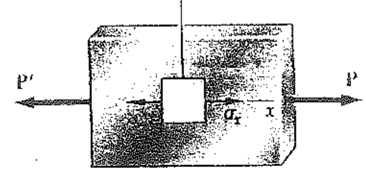


B- Stress Transformation Equations

1- The state of Stress on an inclined surface for an axially loaded member, when the load is along it's x-axis, was driven to be:

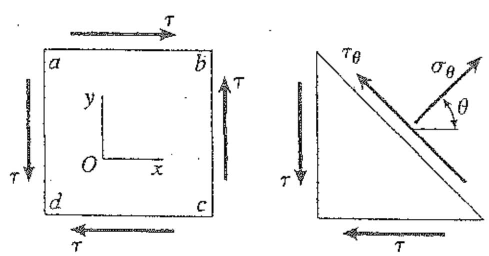
$$\sigma_{x'} = \sigma_\theta = \sigma_x \cos^2 \theta \quad \text{and} \quad \tau_{x'y'} = \tau_\theta = \sigma_x \sin\theta \cos\theta$$

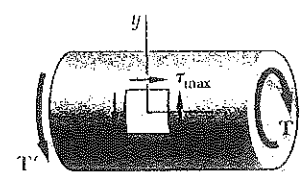




2- Similarly the state of stress on an inclined surface for a circular member under torsion or pure shear stress can be obtained as:

$$\sigma_{x'} = \sigma_\theta = \tau_{xy} \sin 2\theta \quad \text{and} \quad \tau_{x'y'} = \tau_\theta = \tau_{xy} \cos 2\theta$$

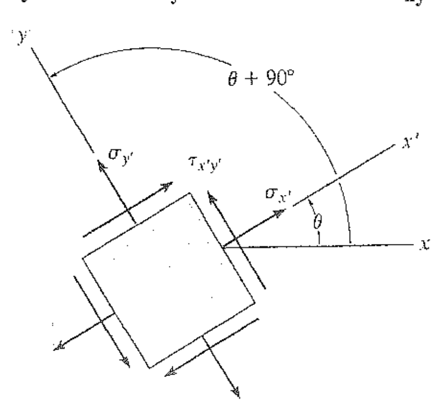


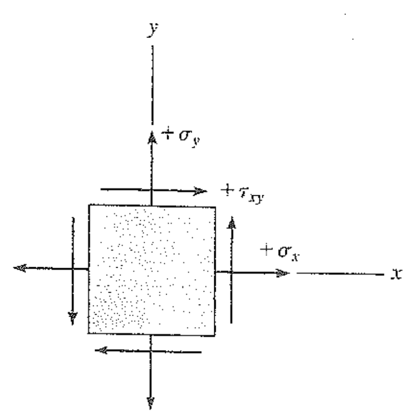


3- Finally for a General Plane Stress using similar approach, the state of stresses along planes perpendicular to the x' and y' axes will be driven to be:

$$\sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta \operatorname{con}\theta \quad \text{and}$$

$$\tau_{x'y'} = -(\sigma_x - \sigma_y) \sin\theta \operatorname{con}\theta + \tau_{xy} (\cos^2\theta - \sin^2\theta)$$





4- By using trigonometric formulas changing all θ angles to 2θ, the Stress Transformation Equations for a general Plane Stress will be changed to the following format (Equations 7-5, 7-6 and 7-7 in the text). These equations can be used for finding stresses on any inclined surface perpendicular to x' and y' axes at an angle of θ° measured positively from the x-axis

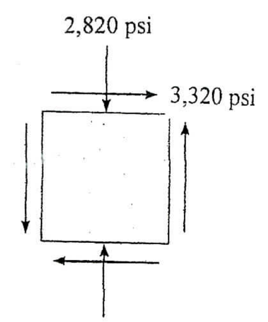
$$\sigma_{x'} = (\sigma_x + \sigma_y)/2 + (\sigma_x - \sigma_y)/2 \ \text{con}\ 2\theta + \tau_{xy} \sin 2\theta \qquad (7\text{-}5)$$

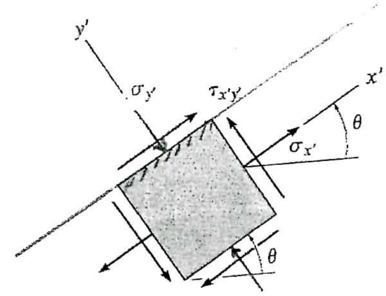
$$\tau_{x'y'} = -(\sigma_x - \sigma_y)/2 \ \sin 2\theta + \tau_{xy} \cos 2\theta \qquad (7\text{-}6)$$

and

$$\sigma_{y'} = (\sigma_x + \sigma_y)/2 - (\sigma_x - \sigma_y)/2 \ \text{con}\ 2\theta - \tau_{xy} \sin 2\theta \qquad (7\text{-}7)$$

5- Example: Find stresses along the weld line for the auger pipe given θ = 22.5° C.C.W. from the x-axis.





$\sigma_x = 0$
$\sigma_y = -2{,}820$ **psi**
$\tau_{xy} = 3{,}320$ **psi**

$\sigma_{x'} = 1{,}934$ **psi (increased)**
$\sigma_{y'} = -4{,}754$ **psi (increased)**
$\tau_{x'y'} = 1{,}350$ **psi (decreased)**

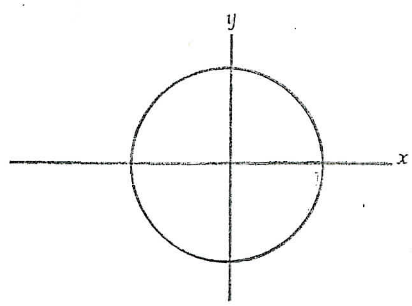
The results show the normal stress along the y' direction or normal to the weld line has increased compared to the original normal stress along the y-axis, therefore it is more critical for the pipe. At the same time the shear stress has decreased. Since the state of stress will change for different value of θ:

<u>We must find the highest value of normal stress σ and highest value of shear stressτ, and the planes associated with those values.</u>

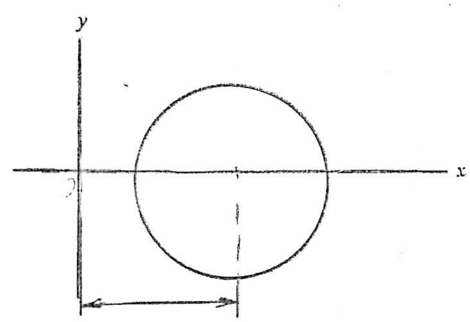
C- Mohr's Circle, Equations

If we are interested to find the highest values of normal and shear stresses in previous example and their planes, we can plot Equations 7-5 and 7-6 for θ from 0 to 180° with increments of 1° or less. Not a practical approach.

Equation of a circle in x-y co-ordinate system is:



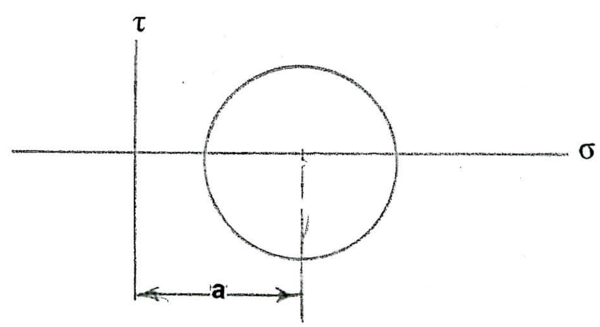
$$x^2 + y^2 = R^2$$



$$(x - a)^2 + y^2 = R^2$$

If we change the x and y axes to σ and τ respectively, the equation for the second circle would be:

$$(\sigma - a)^2 + \tau^2 = R^2$$



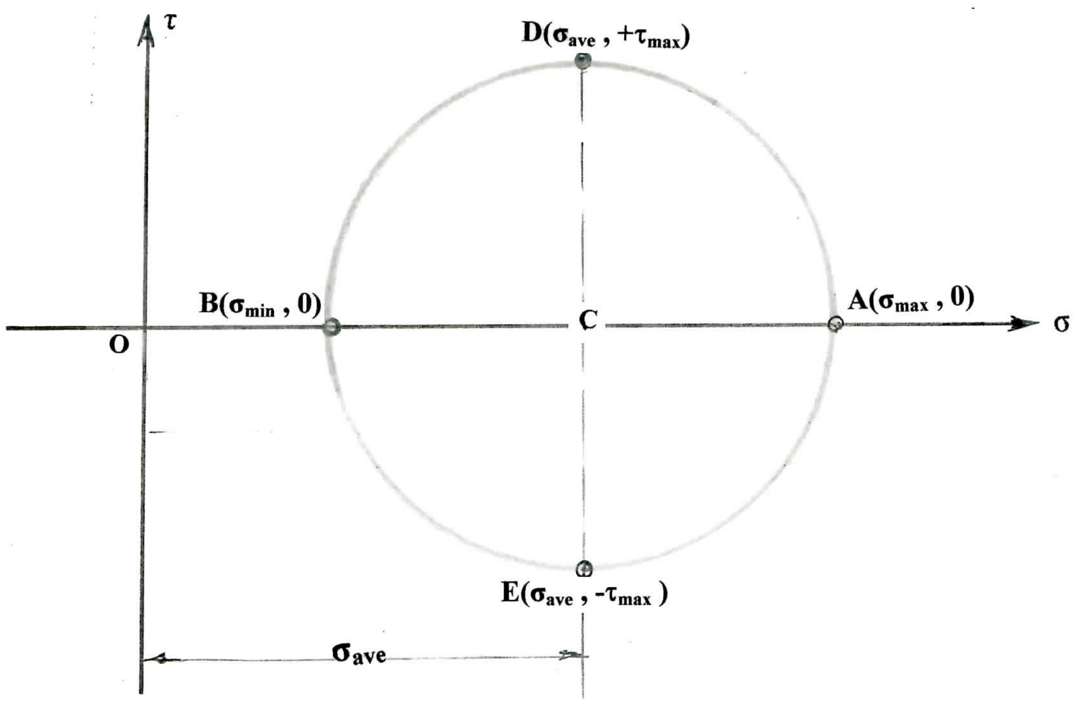
Equations 7-5 and 7-6 represent such a circle. To eliminate the angle θ from these equations we need to square both equations and add them together. The result is:

$$(\sigma - \sigma_{ave})^2 + \tau^2 = R^2$$

where

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} \quad \text{and} \quad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

This circle represents all the stresses at a point on the surface of a structure for every possible inclined surface or planes specified by a set of axes such as x' and y'. To plot the circle we need only the center of the circle at $\sigma = \sigma_{ave}$ and the radius R. Points A and B show two perpendicular planes with the maximum and minimum values of normal stress σ, which are called Principal Plans. Points D and E represents two perpendicular planes with the highest possible numerical value for the in-plane shear stress τ. Since points A and B correspond to zero shear stress, the direction of the principal planes or θ_P can be determine by setting Equation 7-6 equal to zero. Similarly for points D and E we need to set Equation 7-5 equal to $\sigma_{ave} = (\sigma x + \sigma y)/2$, which will give us the θ_S or the directions of the maximum shear planes.



$$\sigma_A = \sigma_{max} = \sigma_{ave} + R$$

$$\sigma_B = \sigma_{min} = \sigma_{ave} - R$$

$$\tan 2\theta_P = 2\tau_{xy} / (\sigma_x - \sigma_y)$$

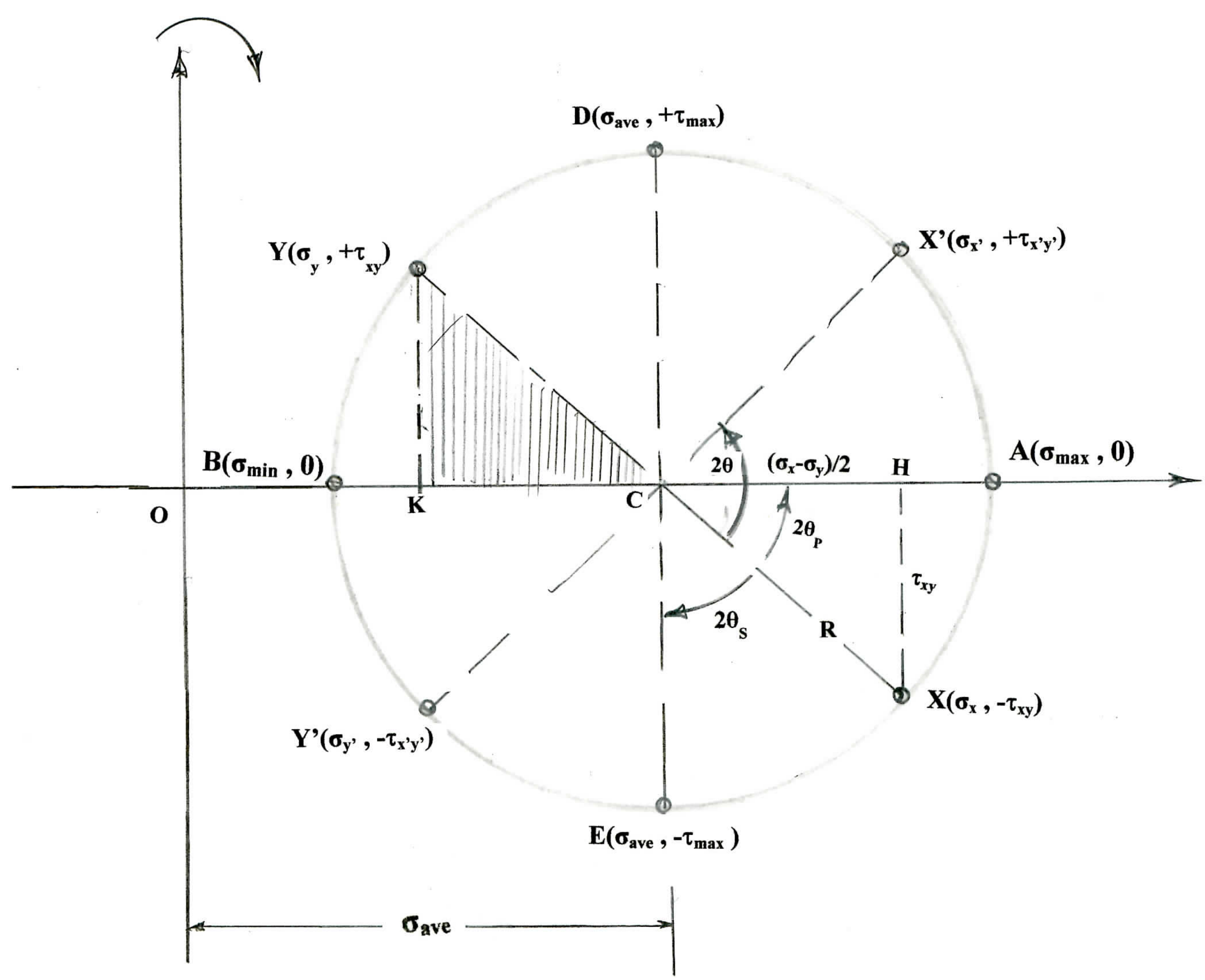
$$\tan 2\theta_S = -(\sigma_x - \sigma_y) / 2\tau_{xy}$$

Mohr's Circle – Graphical Method:

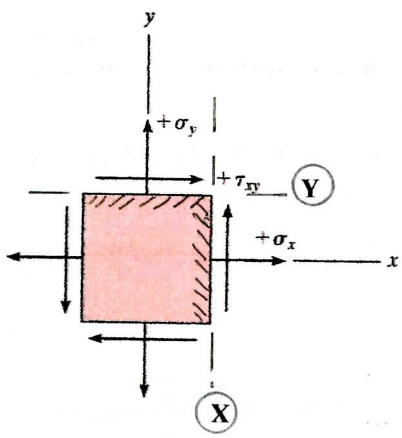
To plot the Mohr's Circle we can use equations given previously or use the following graphical method.

First we plot the stresses on the plane perpendicular to the x-axis on the circle(x-plane), and do the same for the plane perpendicular to the y-axis(y-plane). For the shear stress we need to consider the direction of arrow as positive and oppose to that as negative. We connect these two points to find the center of the circle C. The property of triangles CXH or CYK will give us values of R, θ_P , θ_S , σ_{max} , σ_{min} and τ_{max} without need for any equations.

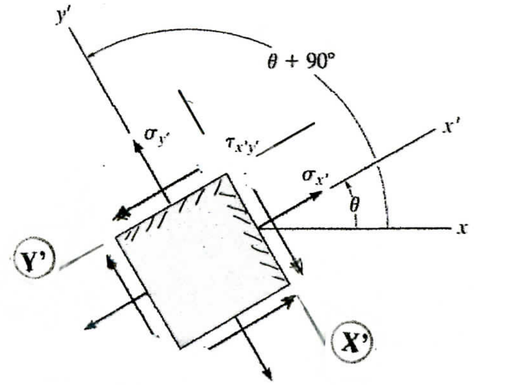
From the circle we can find stresses on any other plane perpendicular to x' and y' axis, when θ is given, by calculating σ and τ for points X' and Y' in place of Equations 7-5 to 7-7.



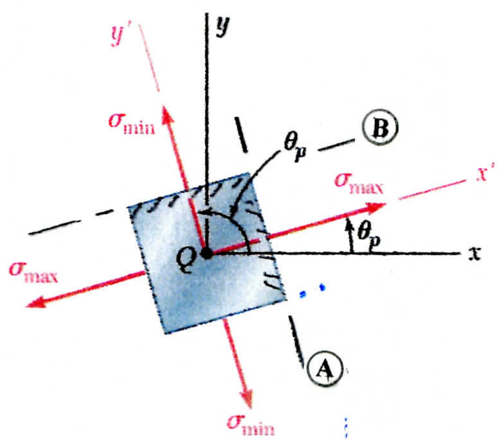
Stresses on different planes from the Mohr's Circle:



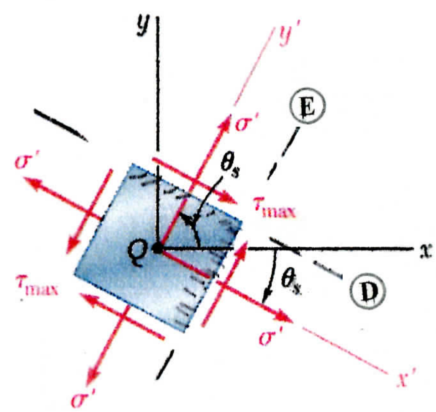
Original State of Stress
x-y Plane



State of Stress at θ° ccw
x'-y' Plane

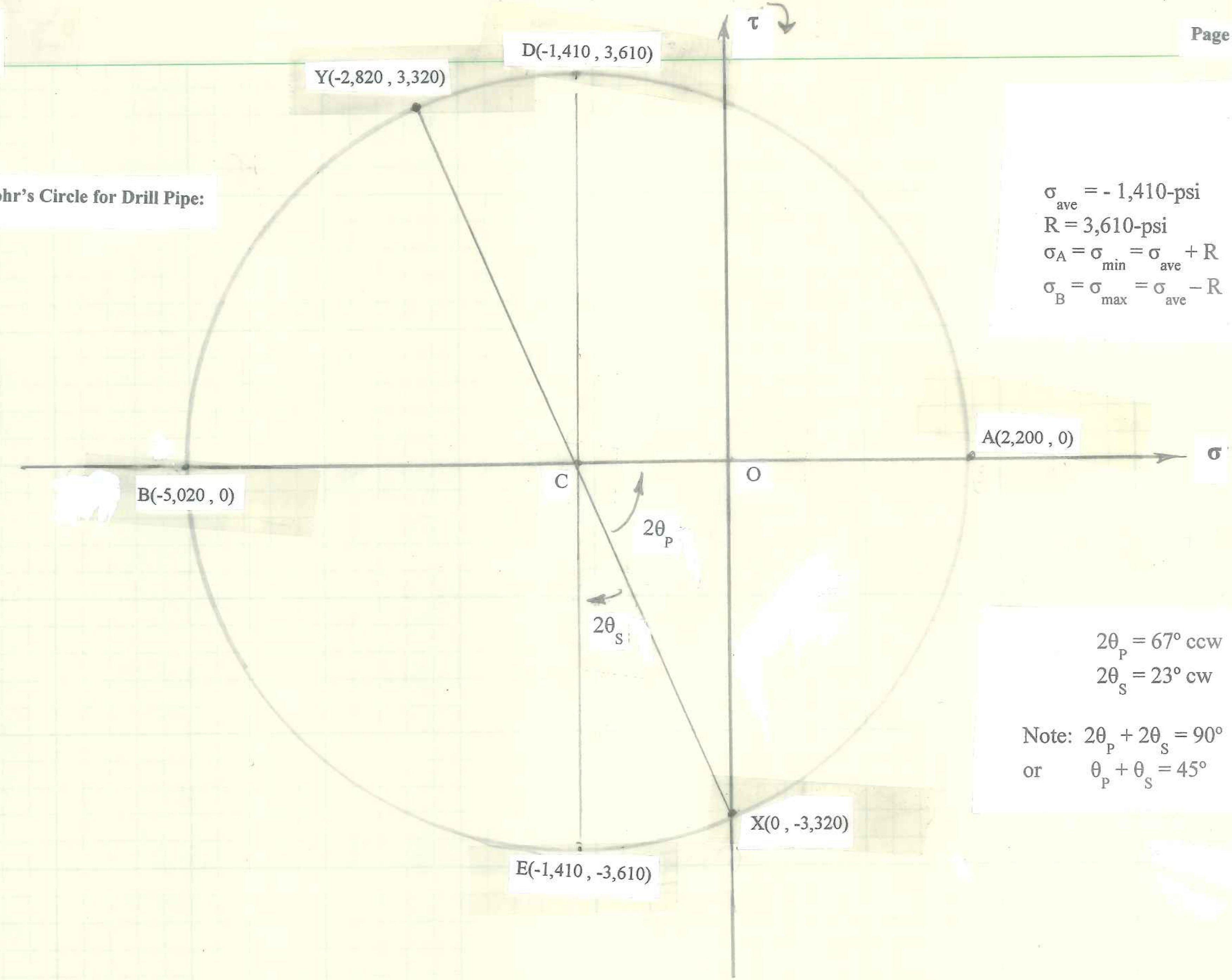


Principal Stresses
x'-y' Plane at θ_p
A-B Plane



Maximum in-Plane Shear
x'-y' Plane at θ_s
D-E Plane

Mohr's Circle for Drill Pipe:

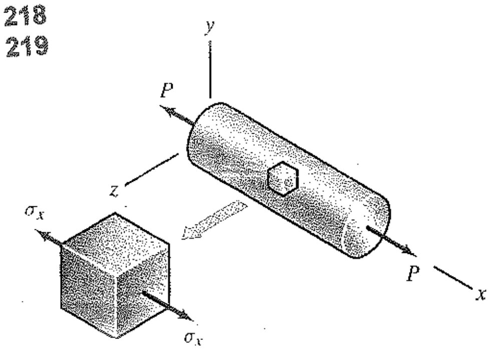


$\sigma_{ave} = -\ 1{,}410\text{-psi}$
$R = 3{,}610\text{-psi}$
$\sigma_A = \sigma_{min} = \sigma_{ave} + R$
$\sigma_B = \sigma_{max} = \sigma_{ave} - R$

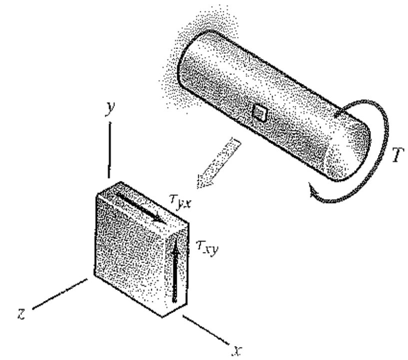
$2\theta_P = 67°$ ccw
$2\theta_S = 23°$ cw

Note: $2\theta_P + 2\theta_S = 90°$
or $\theta_P + \theta_S = 45°$

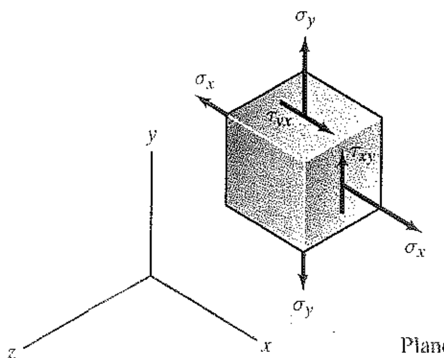
SUMMARY

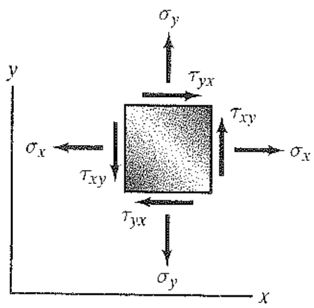


Stresses on an element of a bar subjected to axial forces.

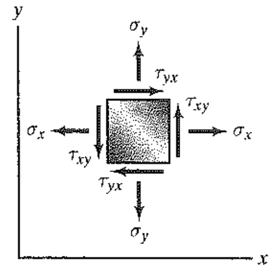


Stresses on an element of a bar subjected to torsion.

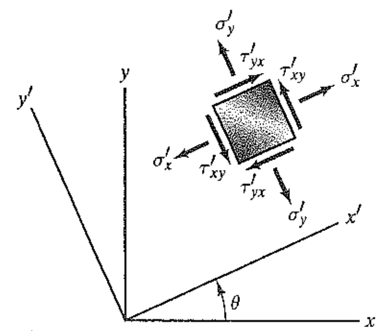




Plane stress.

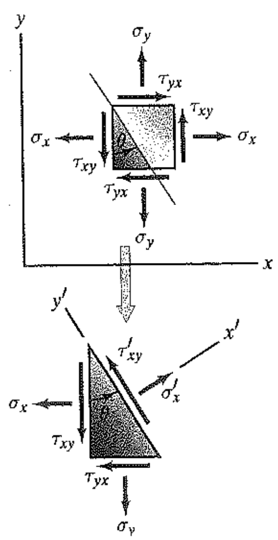


(a)



(b)

Stress components (a) in terms of the xyz coordinate system; (b) in terms of the $x'y'z'$ coordinate system.



Free-body diagram for determining σ_x' and τ_{xy}'.

$$\sigma_x' = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta,$$

$$\tau_{xy}' = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta.$$

$$\sigma_y' = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta.$$

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Question 1 :

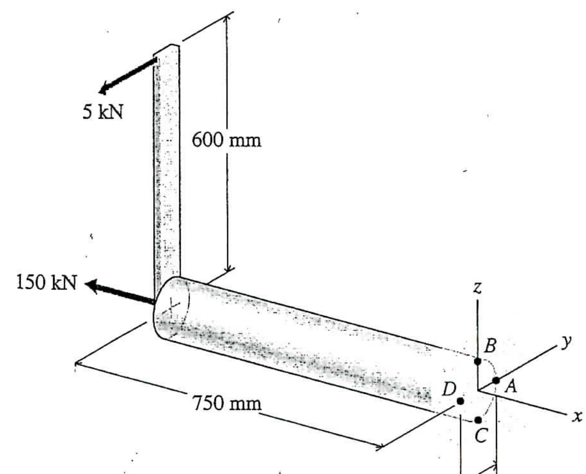
A 100-mm diameter shaft is loaded and is fixed to the wall as shown. Figure (a)

a- Determine the equivalent bonding forces and moments acting on a cross section near the support.

b- Find the state of stress for points A, B, C and D near the wall and show them on properly oriented 3D differential elements. Show two-dimensional state of stresses also (**Plane Stress**).

c- Find principal stresses and maximum shear stress for each point.

d- Locate the most critically stressed point.



(a)

A free-body diagram of the part of the shaft to the left of section *ABCD* is shown in Figure (b) The internal forces acting on section *ABCD* are shear forces V_y and V_z, an axial force P, a torque $T = M_x$, and bending moments M_y and M_z. The six equations of equilibrium used to determine these internal forces are

$$\Sigma F_x = 0: \quad P - 150 = 0 \quad P = 150 \text{ kN}$$

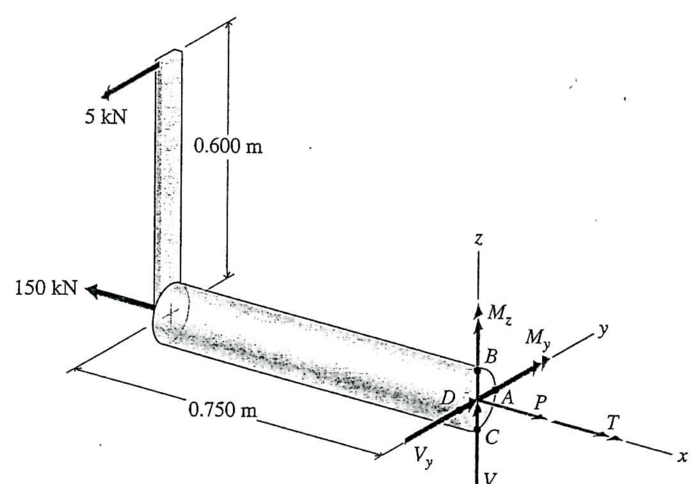
$$\Sigma F_y = 0: \quad V_y - 5 = 0 \quad V_y = 5 \text{ kN}$$

$$\Sigma F_z = 0: \quad V_z = 0 \quad V_z = 0$$

$$\Sigma M_x = 0: \quad T + 5(0.600) = 0 \quad T = -3 \text{ kN} \cdot \text{m}$$

$$\Sigma M_y = 0: \quad M_y = 0 \quad M_y = 0$$

$$\Sigma M_z = 0: \quad M_z + 5(0.750) = 0 \quad M_z = -3.75 \text{ kN} \cdot \text{m}$$



(b)

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Thus, there are four internal forces on the section: (1) an axial force P that produces a constant tensile stress over the section, (2) a shear force V_y that produces shearing stresses at points B and C of the section but zero shearing stresses at points A and D, (3) a torque T that produces the same shearing stress at all surface points of the shaft, and (4) a bending moment M_z that produces a tensile stress at point A and an equal compressive stress at point D. The cross-sectional area A, the first moment Q, the second moment I, and the polar second moment J are

$$A = \frac{\pi}{4}(d)^2 = \frac{\pi}{4}(100)^2 = 7854 \text{ mm}^2 = 7854(10^{-6}) \text{ m}^2$$

$$Q = \frac{\pi}{2}(r)^2\left(\frac{4r}{3\pi}\right) = \frac{2}{3}(r)^3 = \frac{2}{3}(50)^3 = 83.33(10^3) \text{ mm}^3 = 83.33(10^{-6}) \text{ m}^3$$

$$I = \frac{\pi}{4}(r)^4 = \frac{\pi}{4}(50)^4 = 4.909(10^6) \text{ mm}^4 = 4.909(10^{-6}) \text{ m}^4$$

$$J = \frac{\pi}{2}(r)^4 = \frac{\pi}{2}(50)^4 = 9.817(10^6) \text{ mm}^4 = 9.817(10^{-6}) \text{ m}^4$$

(a) The magnitude of the normal stress produced by axial force P is

$$\sigma_1 = \frac{P}{A} = \frac{150(10^3)}{7854(10^{-6})} = 19.099(10^6)\text{N/m}^2 = 19.099 \text{ MPa}$$

The magnitude of the normal stress produced by moment M_z is

$$\sigma_2 = \frac{M_z c}{I} = \frac{3.75(10^3)(50)(10^{-3})}{4.909(10^{-6})} = 38.195(10^6) \text{ N/m}^2 = 38.195 \text{ MPa}$$

The magnitude of the shearing stress produced by torque T is

$$\tau_1 = \frac{Tc}{J} = \frac{3(10^3)(50)(10^{-3})}{9.817(10^{-6})} = 15.280(10^6) \text{ N/m}^2 = 15.280 \text{ MPa}$$

The magnitude of the shearing stress produced by shear force V_y is

$$\tau_2 = \frac{V_y Q}{It} = \frac{5(10^3)(83.33)(10^{-6})}{4.909(10^{-6})(100)(10^{-3})} = 0.8487(10^6) \text{ N/m}^2 = 0.8487 \text{ MPa}$$

These component stresses are shown next page for points A, B, C, and D. The state of stress at each point is plane stress. Therefore, the superimposed states of stress for points A, B, C, and D can be represented on two-dimensional stress elements, as shown next page for each point, respectively.

(b) The principal stresses at each of the points is obtained by using Eq. (9-5) Thus, for point A shown :

$$\sigma_{p1,\,p2} = \frac{\sigma_x + \sigma_z}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_z}{2}\right)^2 + \tau_{xz}^2}$$

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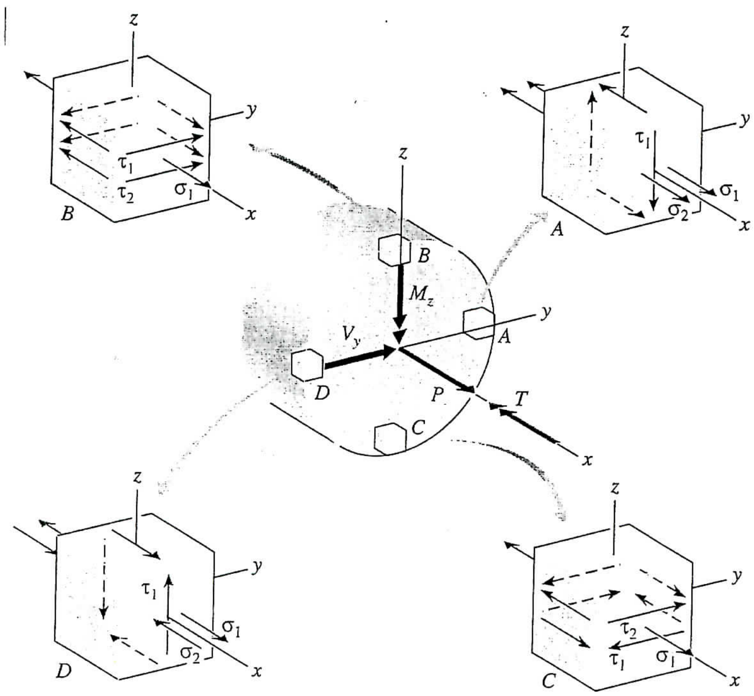
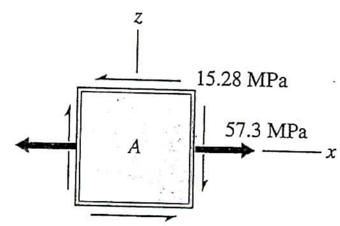
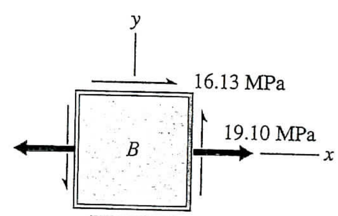




(a)



(b)

$$= \frac{57.3 + 0}{2} \pm \sqrt{\left(\frac{57.3 - 0}{2}\right)^2 + (-15.28)^2}$$

$$= 28.65 \pm 32.47$$

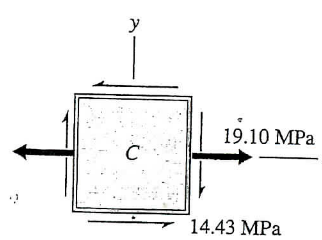
$$\sigma_{p1} = 28.65 + 32.47 = +61.12 \text{ MPa} \cong 61.1 \text{ MPa T}$$

$$\sigma_{p2} = 28.65 - 32.47 = -3.820 \text{ MPa} \cong 3.82 \text{ MPa C}$$

$$\sigma_{p3} = \sigma_y = 0$$

Since σ_{p1} and σ_{p2} have opposite signs, the maximum shearing stress is given
$\tau_{max} = R$ or

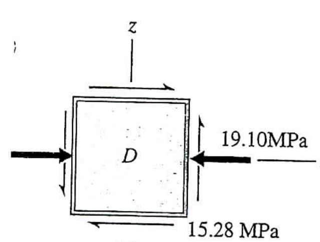
$$\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{61.12 - (-3.820)}{2} = 32.47 \text{ MPa} \cong 32.5 \text{ MPa}$$



(c)

Proceeding in a similar fashion for the remaining points yields the principal stresses and the maximum shearing stress as

Point	σ_{p1}	σ_{p2}	τ_{max}	
A	61.6 MPa T	3.82 MPa C	32.5 MPa	Ans.
B	28.3 MPa T	9.20 MPa C	18.75 MPa	Ans.
C	26.9 MPa T	7.75 MPa C	17.30 MPa	Ans.
D	8.47 MPa T	27.6 MPa C	18.02 MPa	Ans.



(d)

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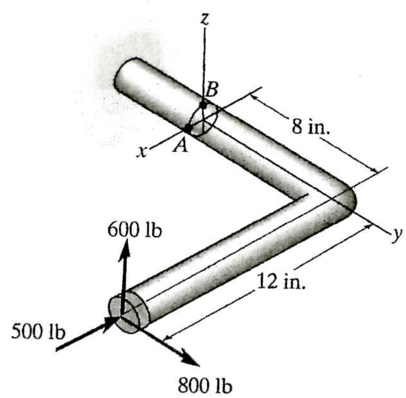
Question (1) :
Four forces are applied to the pipe assembly shown. Knowing that the pipe has a diameter of d =2.0-in:
a- Determine the equivalent bonding forces and moments acting on a cross section 4-in from support.
b- Find stresses at points A and B and sketch the result on differential element for each point. Show axes of the elements.
c- Use **Mohr's Circle** to find the principal stresses and maximum shear for the most critically stressed point, and sketch the results on differential elements with proper orientation with respect to the original x, y, z coordinates.

Answers:

b: $(\sigma_y)_A = 17.6$-ksi Tension
$(\tau_{yz})_A = 4.84$-ksi
$(\sigma_y)_B = -5.86$-ksi Compression
$(\tau_{yx})_B = 4.80$-ksi



c: $\sigma_{max} = 18.8$-ksi
$\sigma_{min} = -1.20$-ksi
$\tau_{max} = 10.0$-ksi

$\theta_P = 14.4°$ or CCW

$\theta_S = -30.6°$ or CW

Question (2) :
A state of stress on a machine part given as: $\sigma_x = 30$-MPa, $\sigma_y = 100$-MPa and $\tau_{xy} = 84$-MPa. Determine the **Absolute Maximum Shear Stress** when:

a- $\sigma_z = 0.0$
b- $\sigma_z = 60$-MPa
c- $\sigma_z = -60$-MPa

Answers:

a: $(\tau_{max})_{xy} = 91$-MPa
b: $(\tau_{max})_{xy} = 91$-MPa
c: $(\tau_{max})_{xz} = 108$-MPa

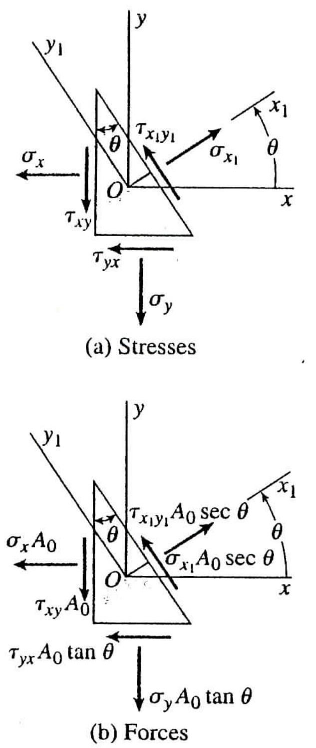


Fig. 7-2 Wedge-shaped stress element in plane stress: (a) stresses acting on the element, and (b) forces acting on the element

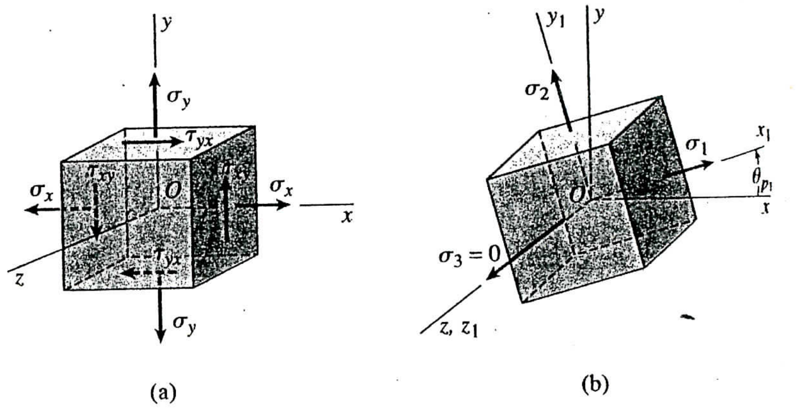


Fig. 7-12 Elements in plane stress: (a) original element, and (b) element oriented to the three principal planes and three principal stresses

A plane-stress condition exists at a point on the surface of a loaded structure, where the stresses have the magnitudes and directions shown on the stress element of Fig. 7-8a. Determine the stresses acting on an element that is oriented at a clockwise angle of 15° with respect to the original element.

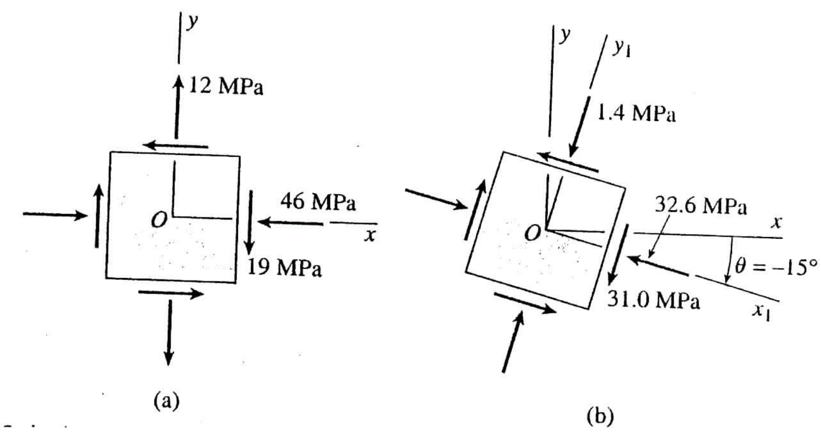


Fig. 7-8 Example 7-2. (a) Element in plane stress, and (b) element inclined at an angle $\theta = -15°$

An element in plane stress is subjected to stresses $\sigma_x = 12{,}300$ psi, $\sigma_y = -4{,}200$ psi, and $\tau_{xy} = -4{,}700$ psi, as shown in Fig. 7-13a. (a) Determine the principal stresses and show them on a sketch of a properly oriented element.

Fig. 7-13 Example 7-3. (a) Element in plane stress, (b) principal stresses, and (c) maximum shear stresses