Applications of the degree theory and mountain pass lemma to s-fractional p-Laplacian problems

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SEDE MANIZALES

Outline

1. A brief Overview of critical point theory
2. One result about existence of solutions for a semipositone problem
3. Overview of degree theory
4. An alternative proof for our existence result

Definitions

Let $(X, \| \ \|)$ be a normed space, $x_0 \in U \subseteq X$ and $J : U \to \mathbb{R}$ a functional.

- X', the dual of X, is the space of linear continuous functions $L : X \to \mathbb{R}$, with norm $\|L\| := \sup_{\|x\| \leq 1} \|Lx\|$.

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- If L exists, it is unique and $L = J'(x_0)$.
- $J \in C^1(U)$ if

$$J' : U \to X'$$

  is continuous.

An example

For a function $u : \Omega \subseteq \mathbb{R}^N \to \mathbb{R}$

$$\nabla u = (\partial_1 u, \ldots, \partial_N u), \qquad \text{the gradient of } u$$

$$\Delta u = \partial_{1,1} u + \cdots + \partial_{N,N} u, \qquad \text{the Laplacian of } u$$

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- a differentiable function $F : \mathbb{R} \to \mathbb{R}$ with $F' = f$, and
- $J : X \to \mathbb{R}$ defined by

$$J(u) = \int_\Omega (\frac{1}{2}|\nabla u|^2 - F(u)) \, \mathrm{d}\, x.$$

Then

$$\langle J'(u), \phi \rangle = \int_\Omega (\nabla u \cdot \nabla \phi - F'(u)\phi) \, \mathrm{d}\, x, \quad \phi \in X.$$

Integration by parts yields us to ($\phi \in X = C_0^1(\Omega)$)

$$\begin{aligned} \langle J'(u), \phi \rangle &= \int_{\partial\Omega} (\nabla u \cdot \eta)\phi \, \mathrm{d}\, S - \int_{\Omega} (\phi \Delta u + f(u)\phi) \, \mathrm{d}\, x \\ &= -\int_{\Omega} (\Delta u + f(u))\phi \, \mathrm{d}\, x \end{aligned}$$

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If $J'(u) = 0$ (u is a critical point of J) then

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Therefore

$$\Delta u + f(u) = 0$$

There is a relation between the critical points of J and the solutions of

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

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- In 1900 Hilbert presented 23 problems in the ICM. The 20th has to be with Riemann's ideas

First contribution: a definition

We say that u is a solution of

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How to find critical points of J?

- points of extreme value (minimum of maximum)
- saddle points
- mountain pass points

Mountain pass theorem

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- J satisfies the PS condition: Every sequence (x_n) that satisfies
  - $|J(x_n)| \leqslant C$ (is bounded) and
  - $J'(x_n) \to 0$, as $n \to \infty$.

  admits a convergent subsequence.

Mountain pass structure

Then,

$$c = \inf_{g \in \Gamma} \max_{0 \leqslant t \leqslant 1} J[g(t)]$$

is a critical value of J.
($\Gamma := \{g : [0,1] \to X| \ \ g \text{ continuous } g(0) = 0, g(1) = \phi\}$)

Fractional Sobolev Spaces

- $0 < s < 1$
- $1 \leqslant p < +\infty$

For any measurable $u : U \to \mathbb{R}$, let us define

$$[u]_{s,p}^{p} := \int_U \int_U \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, \mathrm{d}\, x \, \mathrm{d}\, y$$

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$$\|u\|_{s,p}^p := \|u\|_{L^p(U)}^p + [u]_{s,p}^p.$$

The closed subspace

$$W_0^{s,p}(\Omega) := \left\{ u \in W^{s,p}(\mathbb{R}^N) | \quad u = 0 \quad \text{a.e. in } \mathbb{R}^N \setminus \Omega \right\},$$

equivalently renormed by setting $\|u\| = [u]_{s,p}$.

Definition: solution to fractional problem

$$\begin{cases} (-\Delta)_p^s(u) = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1}$$

We say that $u \in W_0^{s,p}(\Omega)$ is a solution of this problem, if for all ϕ

$$\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}(\phi(x)-\phi(y))\,\mathrm{d}\,x\,\mathrm{d}\,y = \int_\Omega f(u)\phi\,\mathrm{d}\,x$$

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or

$$\int_{\mathbb{R}^{2N}} \frac{\psi_p(u(x)-u(y))}{|x-y|^{N+sp}}(\phi(x)-\phi(y))\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega f(u)\phi\,\mathrm{d}x$$

where

$$\psi_p(s) = |s|^{p-2}s, \qquad s \in \mathbb{R}$$

An existence result

We want to study the existence of positive solutions to the problem

$$\begin{cases} (-\Delta)_p^s(u) = \lambda(u^q - 1) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N - \Omega, \end{cases} \tag{2}$$

where

- $s \in (0, 1)$, $2 \leqslant p$ and $sp < N$ and $\lambda > 0$.
- In this case, $f(s) = s^q - 1$, $p - 1 < q < p_s^* - 1$.

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We are looking for functions u such that for every ϕ

$$\int_{\mathbb{R}^{2N}} \frac{\psi_p(u(x) - u(y))}{|x - y|^{N+sp}} (\phi(x) - \phi(y)) \,\mathrm{d}\,x \,\mathrm{d}\,y - \lambda \int_{\Omega} (u^q - 1)\phi \,\mathrm{d}\,x = 0$$

As a reminder

$$\psi_p(s) = |s|^{p-2} s$$

Since the left hand side is the derivative of

$$J_\lambda(u) = \frac{1}{p}\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y - \lambda\int_\Omega F(u)\,\mathrm{d}x \tag{3}$$

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Rewrite J_λ as

$$J_\lambda(u) = \frac{1}{p}\|u\|^p - \lambda\int_\Omega F(u)\,\mathrm{d}\,x$$

Our result

Theorem

Let us assume that Ω *is a bounded domain with* $C^{1,1}$ *boundary. Then there is* $\lambda_0 > 0$ *such that for all* $\lambda \in (0, \lambda_0)$ *problem* (2)

$$\begin{cases} (-\Delta)_p^s(u) = \lambda(u^q - 1) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N - \Omega, \end{cases}$$

has at least one positive weak solution $u_\lambda \in C^\alpha(\overline{\Omega})$, *for some* $\alpha \in (0, 1)$.

Lopera, E., López, C., & Vidal, R. E. (2023). Existence of positive solutions for a parameter fractional p-Laplacian problem with semipositone nonlinearity. Journal of Mathematical Analysis and Applications, 526(2), 127350.

Checking the mountain pass structure of J

- There exist $\tau > 0$, $c_1 > 0$ and $0 < \lambda_2 < 1$ such that if $\|u\| = \tau\lambda^{-r}$ then $J_\lambda(u) \geq c_1(\tau\lambda^{-r})^p$ for all $\lambda \in (0, \lambda_2)$.
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  $\varphi > 0$ s.t. $\|\varphi\| = c$.
- Let $\lambda_3 = \min\{\lambda_1, \lambda_2\}$. Then for all $\lambda \in (0, \lambda_3)$ the functional J_λ has a critical point u_λ.

The only missing part is to prove that J_λ satisfies PS.
Let (u_n) be a sequence s.t.

$$|J_\lambda(u_n)| \leqslant M \quad \text{and} \quad J'_\lambda(u_n) \to 0$$

$\|u_n\|^p$ can be written in terms of $J(u_n)$ and $\langle J'(u_n), u_n\rangle$.

$$\langle J'_\lambda u, \phi\rangle = \int_{\mathbb{R}^{2N}} \frac{\psi_p(u(x)-u(y))}{|x-y|^{N+sp}}(\phi(x)-\phi(y))\,\mathrm{d}\,x\,\mathrm{d}\,y - \int_\Omega f(u)\phi\,\mathrm{d}\,x$$

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$$\langle J'(u_n), u_n\rangle = \|u_n\|^p - \int_\Omega f(u_n)u_n \, \mathrm{d}\, x$$

and

$$J(u_n) = \frac{1}{p}\|u_n\|^p - \int_\Omega F(u_n) \, \mathrm{d}\, x$$

Thus (u_n) is bounded in $W_0^{1,p}(\Omega)$, which is reflexive. Therefore

$$u_n \rightharpoonup u.$$

Then using standard inequalities we prove that

$$\lim_{n\to\infty} \|u_n\| = \|u\|.$$

Consequently

$$u_n \to u.$$

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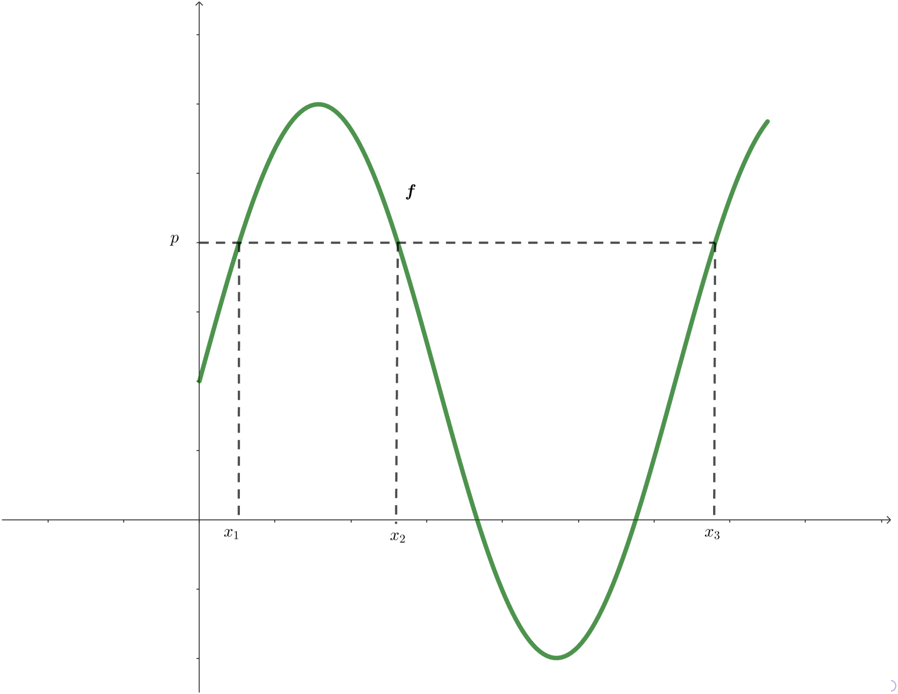
- For such regular values p of f

$$\deg(f, D, p) := \sum_{x \in f^{-1}(p) \cap D} \operatorname{sign}(J_f(x)).$$


f
p
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- $p \in X \setminus \phi(\partial D)$.
- Take $\hat{\phi} = I - \hat{F}$ where $\hat{F}$ is a continuous mapping with finite dimensional range and approximates F. Define

  $$\deg(\phi, D, p) = d(\hat{\phi}, \hat{D}, p)$$

  where $\hat{D}$ is contained in an appropriate finite dimensional subspace of X.

Theorems

- Suppose that $\phi = I - F$, with $F : \overline{D} \to X$ continuous and compact, $p \notin \phi(\partial D)$ and $d(\phi, D, p) \neq 0$, then there exists $x \in D$ s.t.

$$\phi(x) = p.$$

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- (Invariance under homotopy) Let $h(t)$ be a homotopy of compact transformations on D such that if $\phi_t = I - h(t)$, $p \notin \phi_t(\partial D)$ for all $0 \leqslant t \leqslant 1$. Then

$$\deg(\phi_t, D, p) \qquad \text{is independent of } t$$

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- If $D = D_1 \dot{\cup} D_2$ then

$$\deg(\phi, D, p) = \deg(\phi, D_1, p) + \deg(\phi, D_2, p)$$

Second proof

Using degree theory is proved the existence of positive solutions for

$$\begin{cases} (-\Delta)_p^s(u) = \lambda(u^q - 1) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{4}$$

Dhanya, R., Jana, R., Kumar, U., & Tiwari, S. (2023). Positive Solutions for Fractional p-Laplace Semipositone Problem with Superlinear Growth. arXiv preprint arXiv:2304.10887.

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$w = \gamma u$ where $\gamma^{q+1-p} = \lambda$

$$\begin{cases} (-\Delta)_p^s(w) = w^q - \gamma^q & \text{in } \Omega \\ w = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{5}$$

$$F_\gamma(w) = w^q - \gamma^q$$

Dhanya, R., Jana, R., Kumar, U., & Tiwari, S. (2023). Positive Solutions for Fractional p-Laplace Semipositone Problem with Superlinear Growth. arXiv preprint arXiv:2304.10887.

Classical theorem: for each $f \in C(\overline{\Omega})$ there exists a unique $u \in W_0^{s,p}(\Omega) \cap C(\overline{\Omega})$ such that $(-\Delta)_p^s(u) = f$.

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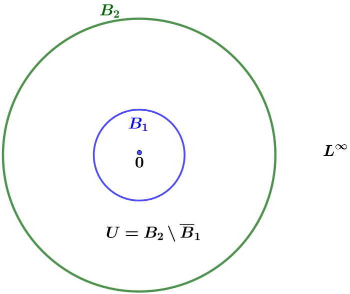
- There exists $0 < R_1 < R_2$ s.t. $S_0(w) \neq 0$ for all $w \in \partial U$ and

$$\deg(S_0, U, 0) = -1,$$

where $U = B_{R_2} \setminus \overline{B}_{R_1}$.

$$1 + \deg(S_0, U, 0) = 0.$$



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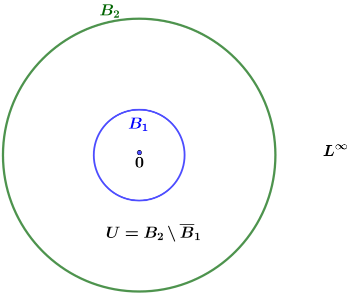

- There is γ_0 s.t. for if $0 < \gamma < \gamma_0$, then $0 \notin S_\gamma[\partial U]$.

- By the invariance of the degree under homotopy, for every $0 < \gamma < \gamma_0$, since $\deg(S_0, U, 0) = -1$, then

$$\deg(S_\gamma, U, 0) = -1.$$

which implies that there is $w \in U$ such that $S_\gamma(w) = 0$.

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Fonseca, I., & Gangbo, W. (1995). Degree theory in analysis and applications (Vol. 2). Oxford University Press.

Thanks

El conocimiento es de todos

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