Moduli Spaces in Geometry

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What is a Moduli Space?

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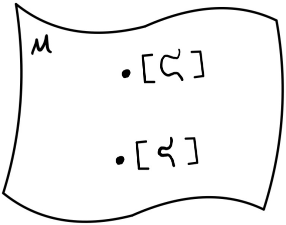

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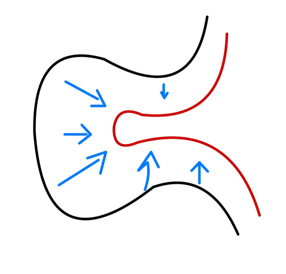

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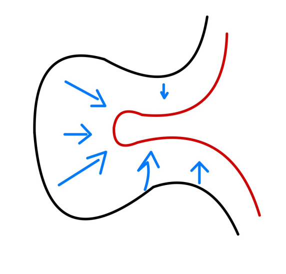

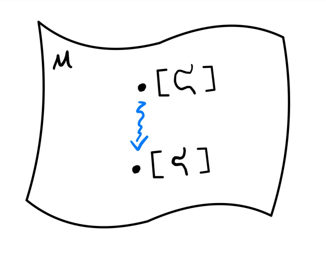

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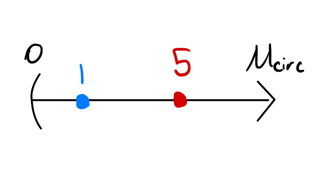

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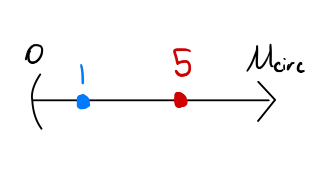

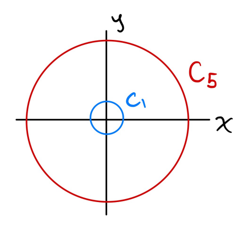

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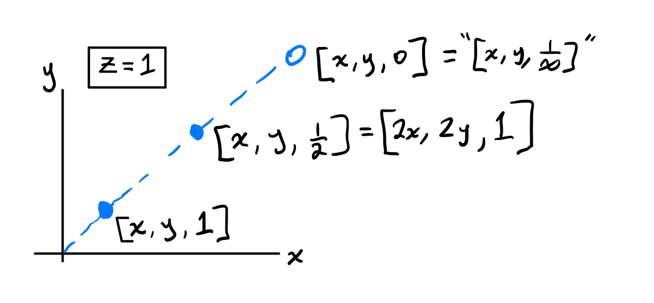

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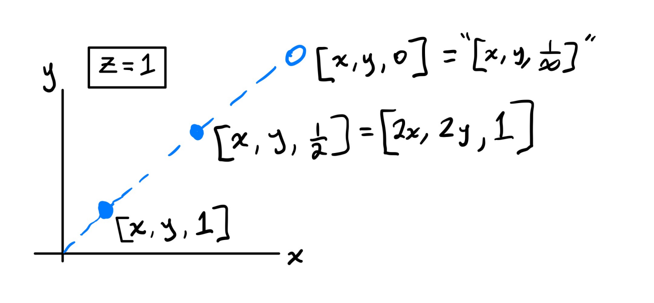

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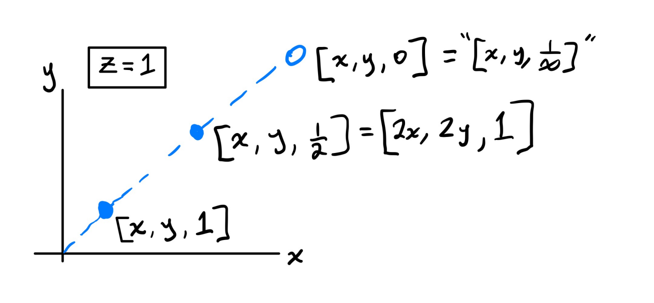

In essence:

$$\mathbb{P}^2_{\mathbb{R}} = \mathbb{R}^2 \cup \{\text{infinity points}\}$$

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- Fully proven in 1958-1965 by Jean-Pierre Serre in *Algèbre locale et multiplicités*.

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- C is a degenerate conic if and only if $\det A = 0$. In this case, C is a union of lines.

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Main Examples

1. Circles in $\mathbb{R}^2$ centered at the origin - $\mathcal{M}_{circ}$
2. Planes in k^3 centered at origin - Projective plane $\mathbb{P}^2$
3. Conics in $\mathbb{P}^2$ - $\mathcal{M}_{conics}$

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What does the geometry of a moduli space tell us about families in moduli problem?

Topics in Moduli

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1. Dimension
2. Compact Moduli
3. Deformation Theory

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The dimension of a moduli space $\mathcal{M}$ is equal to the degrees of freedom of the moduli problem.

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- The dimension of $\mathcal{M}$ is the number of local coordinates.

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$$\dim \mathcal{M}_{conics} = 5.$$

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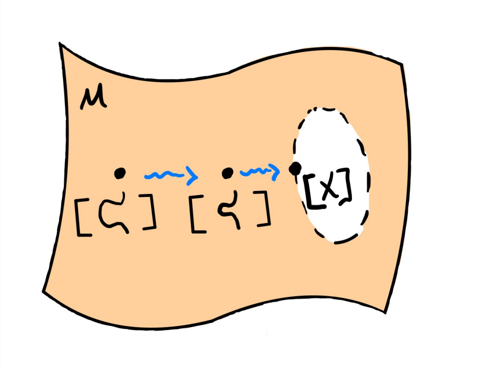

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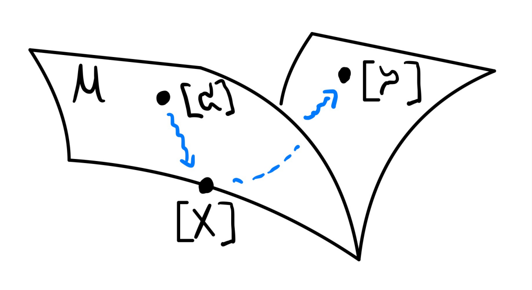

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Formal Definition

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  What we'll need: representability of functors.

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4. Dual: $Vect_k \to Vect_k^{op}$.

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A *natural transformation* $\eta : F \to G$ between two functors $F, G : \mathcal{C} \to \mathcal{D}$ is an assignment of morphisms: for $X \in \mathcal{C}$,

$$\eta_X : F(X) \to G(X)$$

such that for each $f : X \to Y$ in $\mathcal{C}$ the following diagram commutes:

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"] & G(Y). \end{tikzcd}$$

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A natural transformation $\eta : F \to G$ is a *natural isomorphism* if there exists a natural transformation $\mu : G \to F$ such that $\mu \circ \eta = 1_F$ and $\eta \circ \mu = 1_G$.

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A functor $F : \mathcal{C}^{op} \to Sets$ is *representable* by $X \in \mathcal{C}$ if there exists a natural isomorphism $Hom_{\mathcal{C}}(-,X) \to F$.

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- Representable functors are (up to natural isomorphism) of the form $Hom_{\mathcal{C}}(-, \mathcal{M})$.

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Not the only (or even best) way to study moduli.

Representable Moduli Functors

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Problem: Describe an “interesting” moduli functor for concentric circles.

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Generalize to integral finite-type k-algebras A:

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Moduli Problem: $F : \mathit{Var}_k^{op} \to \mathit{Sets}$ with k-variety V with coordinate ring A,

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Representability: An element of $F(A)$ corresponds to a morphism $V \to \mathbb{P}^2_k$.

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The functor H_2 is usually called the Hilbert functor of degree 2, and denoted

$$H_2 = Hilb^{\phi_2}_{\mathbb{P}^2_k}.$$

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The hilbert functor $Hilb_{\mathbb{P}^2_k}^{\phi_d}$ is representable by $\mathbb{P}^N$, with

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1. A moduli space is a "space" whose points parameterize a geometric phenomenon.
    - Compact moduli is equivalent to existence of limits in the moduli problem.
    - Wiggling a point of the moduli space amounts to deformation.
2. Rigorously, a moduli space is a representing object of a moduli-problem functor $F : \mathcal{C}^{op} \to \mathit{Sets}$

Summary: Examples

1. $\mathcal{M}_{circ}$ - Analytic
2. $\mathbb{P}^n$ - Linear algebraic
3. Conics in $\mathbb{P}^2$ - Algebraic

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4. Can we relate $\mathcal{M}$ with another moduli space?

Questions?