

Moduli Spaces in Geometry

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What is a Moduli Space?

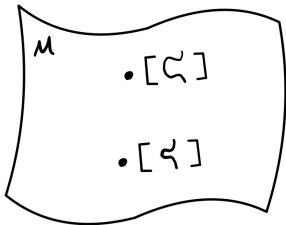
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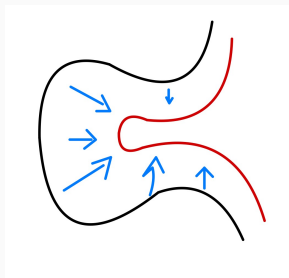
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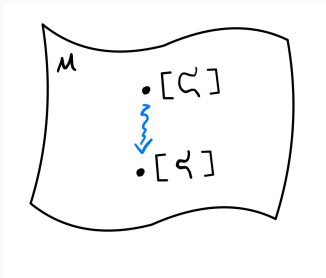
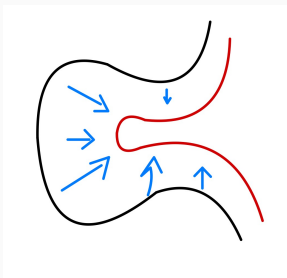


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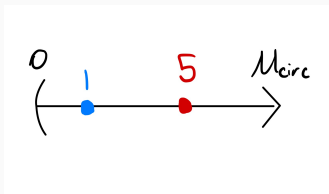
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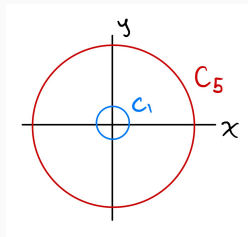
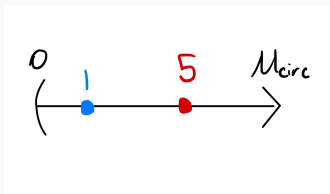


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$$L : a_0x + a_1y + a_2z = 0 \quad \text{and} \quad L^\theta : a_0^\theta x + a_1^\theta y + a_2^\theta z = 0$$

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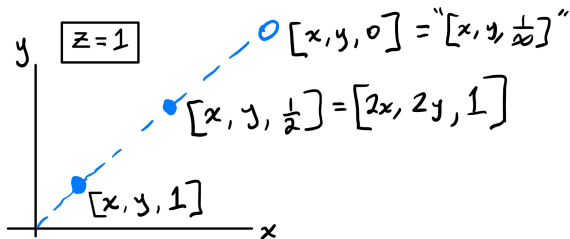
- Theoretical definition: $\mathbb{P}_k^2 = (k^3 \setminus \{0\})/k$.

2. Projective Plane: Picture

Over \mathbb{R} , the projective plane \mathbb{P}^2 is "locally-euclidean".

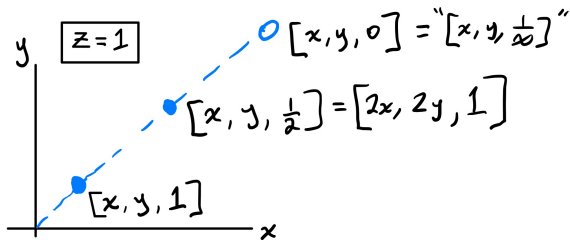
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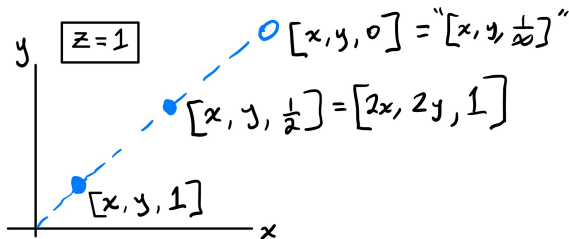
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$$\mathbb{P}_{\mathbb{R}}^2 = \mathbb{R}^2 \cup \{\text{infinity points}\}$$

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- Fully proven in 1958-1965 by Jean-Pierre Serre in *Algebre locale et multiplicites*.

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- C is a degenerate conic if and only if $\det A = 0$. In this case, C is a union of lines.

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Main Examples

1. Circles in \mathbb{R}^2 centered at the origin - \mathcal{M}_{circ}
2. Planes in k^3 centered at origin - Projective plane \mathbb{P}^2
3. Conics in \mathbb{P}^2 - \mathcal{M}_{conics}

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What does the geometry of a moduli space tell us about families in moduli problem?

Topics in Moduli

1. Dimension
2. Compact Moduli
3. Deformation Theory

1. Dimension

The dimension of a moduli space \mathcal{M} is equal to the degrees of freedom of the moduli problem.

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- The dimension of \mathcal{M} is the number of local coordinates.

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Examples:

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Coordinates are $[x, y, z]$, modulo scaling.

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3. Space of planar conics $\mathcal{M}_{conics} \quad \mathbb{P}^5$.

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$$\dim \mathcal{M}_{conics} = 5.$$

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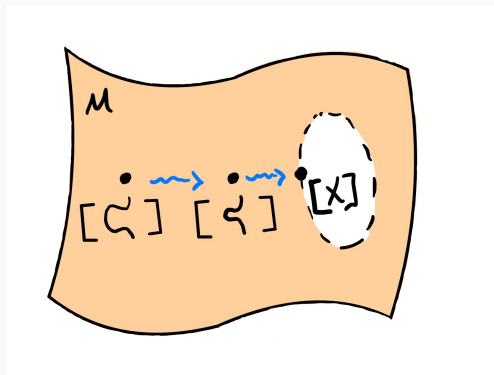
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Degenerate cases: $r = 0$ and $r = \infty$.

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Is \mathbb{P}^2 compact?

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Is \mathbb{P}^2 compact? **Yes!**

3. Space of planar conics $\mathcal{M}_{conics} \subset \mathbb{P}^5$.

Is \mathcal{M}_{conics} compact? **No!**

Degenerate locus $\det A = 0$ in \mathbb{P}^5 includes the case of lines.

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Degenerate cases: $r = 0$ and $r = \infty$. $\overline{\mathcal{M}}_{circ} = \mathbb{R}_0 [\infty]$

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Is \mathbb{P}^2 compact? **Yes!**

3. Space of planar conics $\mathcal{M}_{conics} \subset \mathbb{P}^5$.

Is \mathcal{M}_{conics} compact? **No!**

Degenerate locus $\det A = 0$ \mathbb{P}^5 includes the case of lines. $\overline{\mathcal{M}}_{conics} = \mathbb{P}^5$

3. Deformation Theory

What is the moduli view of deforming an object?

3. Deformation Theory

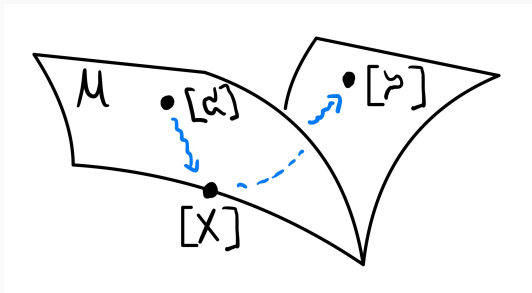
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Check $\mathcal{M}_{lines} = \mathbb{P}^5 \times \mathcal{M}_{conics}$, described by $\det A = 0$ in \mathbb{P}^5 , is singular at the locus of double lines.

Formal Definition

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What we'll need: representability of functors.

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- Identity: for each $X \in \mathcal{C}$, there exists $id_X : X \rightarrow X$ such that for any $f : X \rightarrow Y$,

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$$\mathcal{C} \xrightarrow{F} \mathcal{D}$$

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4. Dual: $\text{Vect}_k \rightarrow \text{Vect}_k^{op}$.

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A natural transformation $\eta : F \rightarrow G$ between two functors $F, G : \mathcal{C} \rightarrow \mathcal{D}$ is an assignment of morphisms: for $X \in \mathcal{C}$,

$$\eta_X : F(X) \rightarrow G(X)$$

such that for each $f : X \rightarrow Y$ in \mathcal{C} the following diagram commutes:

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A natural transformation $\eta : F \rightarrow G$ is a *natural isomorphism* if there exists a natural transformation $\mu : G \rightarrow F$ such that $\mu \circ \eta = 1_F$ and $\eta \circ \mu = 1_G$.

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Yoneda's lemma tells that these are $\text{Hom}_C(X, _): C \rightarrow \text{Sets}$ and $\text{Hom}_C(_, X): C^{op} \rightarrow \text{Sets}$.

Representable Functors

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Yoneda's lemma tells that these are $\text{Hom}_C(X, -) : C \rightarrow \text{Sets}$ and $\text{Hom}_C(-, X) : C^{op} \rightarrow \text{Sets}$.

A functor $F : C^{op} \rightarrow \text{Sets}$ is *representable* by $X \in C$ if there exists a natural isomorphism $\text{Hom}_C(-, X) \cong F$.

Recap

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- Functors relate categories $F : \mathcal{C} \rightarrow \mathcal{D}$.
- Natural transformations relate functors $\eta : F \rightarrow G$.
- Representable functors are (up to natural isomorphism) of the form $\text{Hom}_{\mathcal{C}}(-, \mathcal{M})$.

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Not the only (or even best) way to study moduli.

Representable Moduli Functors

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Problem: Describe an "interesting" moduli functor for concentric circles.

Examples: Projective Plane \mathbb{P}_k^2

Generalize to integral finite-type k -algebras A :

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Moduli Problem: $F : \text{Var}_k^{\text{op}} \rightarrow \mathbb{P}_k^2$! Sets with k -variety V with coordinate ring A ,

$$F(A) = f(s_0, s_1, s_2) \in (A)^3 \subset A^3 \quad ! \quad A \text{ with } e_i \nmid s_i \text{ is surjective } g/A \text{ .}$$

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Representability: An element of $F(A)$ corresponds to a morphism $V \rightarrow \mathbb{P}_k^2$.

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Moduli Problem: $H_2 : \text{Var}_k^{\text{op}}$! Sets with variety X with coordinate ring A , $H_2(A) =$

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The functor H_2 is usually called the Hilbert functor of degree 2, and denoted

$$H_2 = \text{Hilb}_{\mathbb{P}_k^2}^2.$$

Examples: Other degree d curves in \mathbb{P}^2

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Moduli Problem: $\text{Hilb}_{\mathbb{P}_k^2}^d : \text{Var}_k^{\text{op}} \rightarrow \text{Sets}$ with k -variety X with coordinate ring A , then $\text{Hilb}_{\mathbb{P}_k^2}^d(A) =$

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Theorem (Grothendieck, 1961)

The hilbert functor $\text{Hilb}_{\mathbb{P}_k^2}^d$ is representable by \mathbb{P}^N , with

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1. A moduli space is a "space" whose points parameterize a geometric phenomenon.
 - Compact moduli is equivalent to existence of limits in the moduli problem.
 - Wiggling a point of the moduli space amounts to deformation.
2. Rigorously, a moduli space is a representing object of a moduli-problem functor $F : \mathcal{C}^{op} \rightarrow \mathbf{Sets}$

Summary: Examples

1. \mathcal{M}_{circ} - Analytic
2. \mathbb{P}^n - Linear algebraic
3. Conics in \mathbb{P}^2 - Algebraic

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Homework

Define a moduli problem, and construct (if possible) a moduli space \mathcal{M} . Describe \mathcal{M} geometrically:

1. What is the dimension of \mathcal{M} ?
2. Is \mathcal{M} smooth?
3. Is \mathcal{M} compact?
4. Can we relate \mathcal{M} with another moduli space?

Questions?