From Finite to Infinite Dimensions: Morse Theory and Applications to Partial Differential Equations

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Agenda

1. Morse Theory: From Finite to Infinite Dimension.
2. Application of Morse Theory to a Boundary Value Problem.

Marston Morse - Biography

Born in 1892, Waterville, Maine. Completed his undergraduate studies at Colby College in 1914 and obtained his Ph.D. in 1917 at Harvard University under the supervision of George David Birkhoff. He advised 14 Ph.D. students and has 1614 descendants per The Mathematics Genealogy Project.

Figure: Marston Morse: 1892-1977. [15, Page 11]

Career interrupted during World War I and resumed his academic life in 1920 at Cornel University until 1925. He spent the academic year $1925 - 1926$ at Brown University before his appointment at Harvard University (see Bott [3]). In 1935, Morse accepted a position at the Institute of Advanced Studies, Princeton, NJ. He retired as a professor emmeritus at 1962. He has 156 publications according to MathScinet. President of the AMS during 1940-1942, played piano, and received twenty honorary degrees.

Marston Morse - Main works

In [3], Raul Bott described the main subjects that Morse studied throughout his career:

- Dynamics, geodesic flow (1917-1940's)
- Morse Theory (1921-1977)
- Minimal Surfaces (1940's)
- Topological Methods in a single complex variable (1940's).
- Integral Representations (1940's, 1950's).
- Pseudoharmonic functions (1950's).
- Differential topology (1960's - 1970's)

In the next slides, we will present a brief overview of the Morse Theory.

Morse Theory

In 1925, Morse published his first paper about the distribution of critical points of a function entitled *Relations Between the Critical Points of a Real Function of n Independent Variables*, Transactions of AMS, Volume 27, Number 3, Pages $345 - 396$. [12]. The Morse inequalities played a major role which we describe next.

Definition

Let M be a smooth manifold and $f : M \to \mathbb{R}$ be a smooth function on M. A **critical point of** f is a point p at which

$$\left.\frac{\partial f}{\partial x^i}\right|_p = 0 \quad \text{for } i = 1, 2, \ldots, n,$$

with respect to some local system of coordinates $(x^1, \ldots, x^n)$.

Morse Theory

If p is a critical point of f, the Hessian matrix of f at p defined by

$$H_p f = \left(\frac{\partial^2 f}{\partial x^i \partial x^j}\Big|_p \right)_{i,j=1}^{n} \tag{1}$$

has two intrinsic invariants:

- **Rank**.
- **Number Negative Eigenvalues:** Invariant called the *index* of p. Represents the largest subspace on which the quadratic form associated to $H_p f$ is negative definite.

Morse calls the *nullity of* p by $n(p)$ as the dimension of the null space of $H_p f$.

A critical point of f is **nondegenerate** if the Hessian matrix $H_p f$ is invertible at p. Then, Morse presented the following definition:

Definition

Let M be a smooth manifold and $f : M \to \mathbb{R}$ be a smooth function on M. f is a nondegenerate function if all of its critical points are nondegenerate.

Morse Theory

An interesting fact about the nondegenerate functions is the following:

Proposition

On a compact manifold M, a nondegenerate function can only have a finite number of critical points.

Based on the above proposition, we can count the critical points with a fixed index k. We will call this number, $m_k(f)$, as the **k-th type number of f,** and introduce the **Morse polynomial of** f

$$\mathcal{M}_t(f) = \sum_p t^k m_k. \tag{2}$$

Recall the Poincaré polynomial

$$P_t(M) = \sum_i \dim H_i(M) t^i, \tag{3}$$

where H_i denotes the *i-th* homology group of M with coefficients in in some field. Morse proved how the polynomials (2) and (3) are related.

Morse Theory

Theorem

[3, page 920] If f is any non-degenerate function on the compact $n-$manifold M, then its type numbers m_k and the Betti numbers $b_k = \dim H^k(M)$ of M satisfy the inequalities

$$\begin{aligned} m_0 &\geqslant b_0, \\ m_0 - m_1 &\leqslant b_0 - b_1, \\ m_0 - m_1 + m_2 &\geqslant b_0 - b_1 + b_2, \\ &\dots \\ m_0 - m_1 + \dots + (-1)^n m_n &= b_0 - b_1 + \dots + (-1)^n b_n. \end{aligned}$$

As pointed it out by Bott in [3], page 921, the equality was discovered independently abot the same time by H.Hopf, M. Morse, and S. Lefschetz.

Morse Theory

The method of proof used by Morse culminated into two main theorems: A deformation of the level sets when M_t does not pass through a critical point and when it crosses one.

Theorem

[3, Theorem A] Suppose that our knowledge of f has no critical points in the region $a \leqslant f \leqslant b$. Then if M_t denotes the "half-space" where $f \leqslant t$ on M, there is a diffeomorphism of M_a with M_b

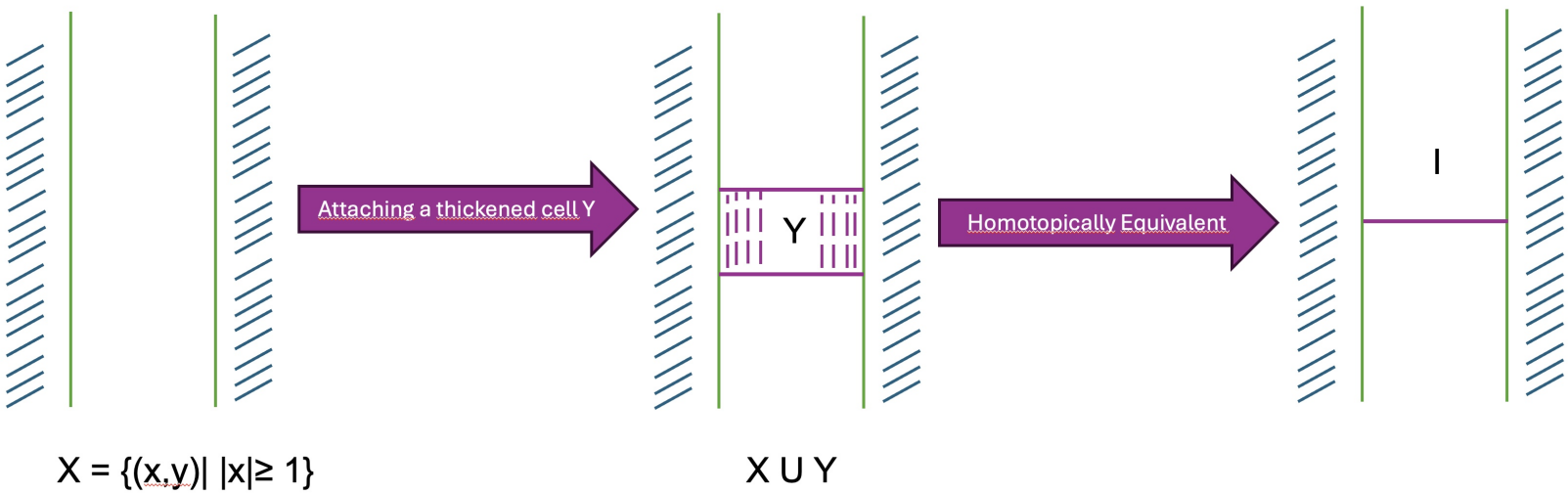
$$M_a \cong M_b.$$

The proof of this result follows directly from *pushing* the level set M_b along the **gradient** of f and using the **compactness** of the level sets $M_t's$.

Morse Theory

What happens when M_t as t passes a critical value of f?

First, let's recall the concept of attaching a *thickened cell* to a manifold. Consider, the following diagram:



where we have attached the "thickened 1-cell Y" to X. Here $Y = I \times I = \{(x, y) || x| \leqslant 1, |y| \leqslant 1\}$, with $I = [-1, 1]$.

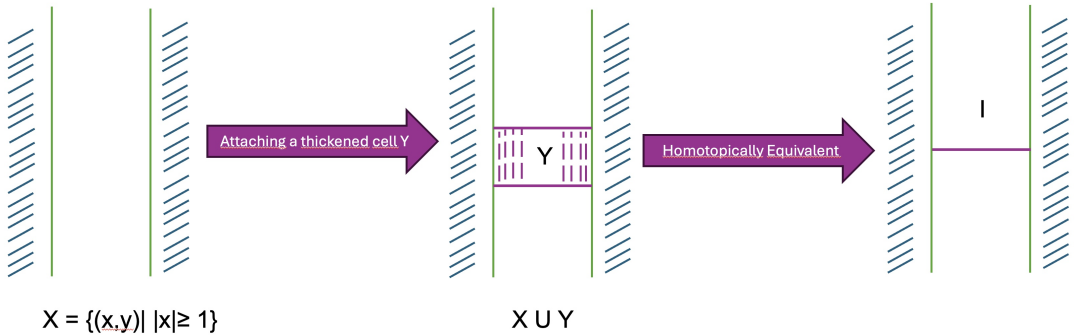
Morse Theory

In general, we say that the manifold X' is obtained from X by attaching a k-cell, if X' is obtained from the distinct union

$$X' = X \sqcup e^k \times e^{n-k},$$

by glueing "half" the boundary, $\partial e^k \times e^{n-k}$, of the cell $e^k \times e^{n-k}$ into the boundary ∂X by a diffeomorphism

$$\alpha : \partial e^k \times e^{n-k} \to \partial X.$$



The resulting manifold is often denoted by

$$X \cup_\alpha e^k \times e^{n-k}.$$

Morse Theory

The second main result from Morse is presented next:

Theorem

[3, Theorem B] Suppose that f has only one nondegenerate critical point p, of index λ in the range $a \leqslant f \leqslant b$ and that $a < f(p) < b$. Then

$$M_b = M_a \cup_\alpha e^\lambda \times e^{n-\lambda};$$

that is, M_b is diffeomorphic to M_a with a thickened $\lambda-$ cell attached.

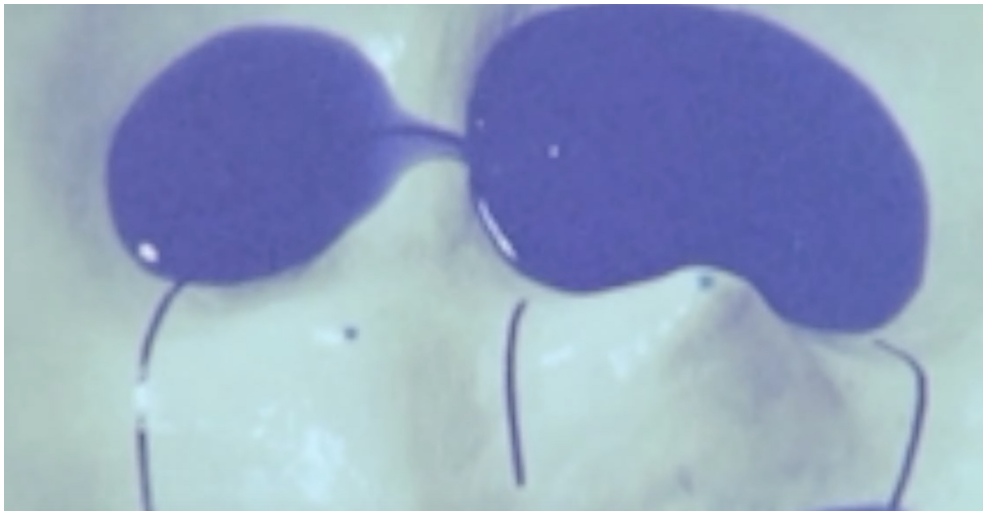
In the 50's, René Thom, E. Pitcher, and Raul Bott formulated the above theorem in purely homotopy-theoretic terms, that is in the form

$$M_b = M_a \cup_\alpha e_\lambda,$$

where one only needs to attach λ-cell as one passes through a critical point of index λ. Fore more details, see [3, pages 924,925].

Morse Theory

Marston Morse recorded a demonstration of how level sets change when passing through a critical point and showing the attachment of the 2-cells.



Link: Marston Morse demonstration of 1966.

Infinite-dimensional Morse Theory

In [3, Page 926], Raoul Bott mentioned that Morse's real goal was a corresponding theory in the Calculus of Variations.

- Ω : Space of piecewise differentiable paths $u : [a, b] \to M$ on an underlying manifold M.
- $F : TM \to \mathbb{R}$ a function on the tangent bundle of M and define a functional $J : \Omega \to \mathbb{R}$

$$J(u) = \int_a^b F(u, \dot{u})\, dt.$$

- It also assumed a nondegeneracy condition; that is, the Hessian of F is positive definite.
- Morse restricts to the space Ω of those paths which satisfy some "admissible boundary conditions" and he succeeds in defining the notion of *index* and *nullity* for any extremal in Ω.
- In proving that, he also shows that the Morse inequalities persist for any nondegenerate J.

Infinite-dimensional Morse Theory: Challenges

- **Lack of Compactness:** In infinite dimensions, functionals may fail to be proper, leading to difficulties in obtaining critical points and controlling their behavior.
- **Infinite-Dimensional Critical Point Theory:** In finite-dimensions, critical points are isolated generically due to non-degeneracy conditions. In infinite-dimensions, critical points can form non-discrete sets, and verifying non-degeneracy requires additional assumptions.
- **Fredholm Theory and Index Theory:** In infinite-dimensions, the Hessian at a critical point is replaced by a bounded linear operator between Banach or Hilbert spaces. Ensuring that it is a Fredholm operator with well-defined index is a nontrivial task.
- **Gradient Flows and Palais-Smale Condition:** Ensuring that bounded sequences with certain properties have convergent subsequences is very challenging.

Infinite-dimensional Morse Theory

- **John Milnor** (1950s-1960s): Contributed to Morse theory by studying exotic spheres and introducing the concept of Morse functions on manifolds, deeply impacting the study of differential topology.
- **Richard Palais** (1950s-1960s): Extended Morse theory to infinite-dimensional spaces, introduced the Palais-Smale condition, and applied it to Hilbert manifolds [13].
- **Steve Smale** (1960s): Applied Morse theory to dynamical systems, infinite-dimensional spaces, and the h-cobordism theorem for manifold classification.
- **Karen Uhlenbeck & Raoul Bott** (1970s-1980s): Extended Morse theory to gauge theory, Yang-Mills equations, and equivariant Morse theory for Lie groups and symmetric spaces.

Application of Morse Theory to PDE's

- Many boundary value problems are equivalent to

$$Au = 0, \tag{4}$$

where $A : X \to Y$ is a mapping between two Banach spaces.

- If a problem is variational, there exists a functional $J : X \to \mathbb{R}$ such that $A = J'$; that is,

$$\langle Au, v\rangle = \lim_{t\to 0} \frac{J(u+tv) - J(u)}{t}, \qquad \text{for all } v \in X.$$

- The space Y corresponds to the topological dual X' of X and the equation (4) is equivalent to $J'(u) = 0$, that is,

$$\langle J'(u), v\rangle = 0, \qquad \text{for all } v \in X. \tag{5}$$

- A *critical point* u_o of J is a solution of (5) and $J(u_o)$ is called a *critical value* of J.
- **Goal:** Find the existence of critical points of J. Are they unique? If not, can we count them?

Abstract setting: Classical and Weak solutions

Let Ω be a bounded domain in $\mathbb{R}^n$ whose boundary $\partial\Omega$ is a smooth manifold. Consider the following linear elliptic problem:

$$\begin{aligned} -\Delta u &= f(x), \quad \text{in } \Omega; \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{6}$$

where $f \in C(\overline{\Omega})$ and Δ denotes the N-dimensional Laplacian.

Definition

A function u is a classical solution of (6) if $u \in C^2(\Omega) \cap C(\overline{\Omega})$.

Abstract setting: Classical and Weak Solutions

If u is a classical solution of (6), we can multiply the equation (6) by $\varphi \in C_0^\infty(\Omega)$ and apply the divergence theorem to obtain

$$\int_\Omega (\nabla u \cdot \nabla \varphi - f\varphi)\, dx = 0. \tag{7}$$

Let $W_0^{1,2}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W_0^{1,2}} = \left(\int_\Omega |\nabla u|^2 \, dx \right)^{1/2}.$$

If $u \in W_0^{1,2}(\Omega)$ and satisfies (7) for all $\varphi \in C_0^\infty(\Omega)$, then u is said to be a **weak solution** of (6).

Abstract setting: Classical and Weak Solutions

- Any classical solution of (6) is a weak solution. The converse is true under stronger hypothesis on f, for example, f being Hölder continuous.
- Let $X = W_0^{1,2}(\Omega)$ and $J : X \to \mathbb{R}$ defined by

$$J(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - fu \right) dx. \tag{8}$$

J is Fréchet differentiable on X and

$$\langle J'(u), \varphi \rangle = \int_\Omega (\nabla u \cdot \nabla \varphi - f\varphi)\, dx, \quad \text{for } \varphi \in X. \tag{9}$$

u is a critical point of $J \Leftrightarrow u$ is a weak solution of (6).

Critical Point Theory Approach to a PDE problem

To analyze the existence and multiplicity of solutions for problems with a variational structure, we proceed with the following steps:

- Associate to a PDE problem an energy functional J defined in an appropriate space X.
- The Sobolev spaces are the appropriate spaces to define such functionals.
- The critical points of the functional J will correspond to *weak* solutions of the PDE Problem.
- Check if J satisfies some a compacteness condition such as the **Palais-Smale condition**, Cerami condition, Brézis-Coron-Nirenberg, etc.
- Verify if the functional J satisfies the hypothesis of some classical existence theorems such as the Mountain Pass, Saddle Point, and Linking theorems of Ambrosetti-Rabinowitz [1].
- Multiplicity Results: Infinite-Dimensional Morse Theory and Algebraic Topology arguments (Long exact sequences).

Presentation of a PDE Problem

Motivated by the results obtained by Castro *et al.* in [5] and Lopera *et al.* in [9], let Ω be an open bounded set in $\mathbb{R}^N$, $N \geqslant 2$, with Lipschitz boundary $\partial\Omega$. In [10], we study the existence and multiplicity of solutions for the problem

$$\begin{cases} -(\Delta)_p^s u(x) + V(x)|u(x)|^{p-2}u(x) = \lambda f(u(x)), & x \in \Omega; \\ u(x) = 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{10}$$

where $V \in L^\infty(\mathbb{R}^N)$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $(-\Delta)_p^s$ denotes the fractional p-Laplacian defined by

$$(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}\, dy, \tag{11}$$

for $x \in \Omega$, with $s \in (0,1)$, $1 < p$, $sp < N$, and $\lambda > 0$.

Presentation of a PDE Problem

In this work, the functions f and V will satisfy the following hypotheses:

(H_1) Assume that $p-1 < q < p_s^* - 1$, where $p_s^* := \frac{Np}{N-sp}$ is the fractional critical Sobolev exponent, and there exist $A, B > 0$ such that

$$A(s^q - 1) \leqslant f(s) \leqslant B(s^q + 1), \qquad \text{for } s > 0, \tag{12}$$

$$f(s) = 0, \qquad \text{for } s \leqslant -1. \tag{13}$$

(H_2) There exist $\theta > p$ and $K \in \mathbb{R}$ such that f satisfies the following Ambrosetti-Rabinowitz type condition

$$sf(s) \geqslant \theta F(s) + K, \qquad \text{for all } s \in \mathbb{R}, \tag{14}$$

where $F(s) = \int_0^s f(\xi)\, d\xi$, for $s \in \mathbb{R}$, is the primitive of f.

(H_3) $V \in L^\infty(\mathbb{R}^N)$ and $V(x) \geqslant -c_V$, for a.e. $x \in \mathbb{R}^N$, where $0 < c_V < \lambda_1$ and λ_1 is the first eigenvalue of $((-\Delta)_p^s, W_0^{s,p}(\Omega))$.

Main Results: $f(0) \neq 0$

Theorem

Assume that Ω is a bounded domain with a Lipschitz boundary $\partial\Omega$ and the hypotheses $(H1)$-$(H3)$ are satisfied with $f(0) \neq 0$. Then, there exists $\lambda_0 > 0$ such that, for all $\lambda \in (0, \lambda_0)$, problem (10) has at least two solutions.

Theorem

In addition to the hypotheses of Theorem 8, assume that $V(x) \geqslant 0$ for a.e $x \in \Omega$, $p \geqslant 2$, Ω is bounded and satisfies the interior ball condition at any $x \in \partial\Omega$, and

$$p - 1 < q < \min\left\{\frac{sp}{N}p_s^*, p_s^* - 1\right\}.$$

Then, there exists $\lambda^* > 0$ such that, for all $0 < \lambda < \lambda^*$, problem (10) has at least two solutions. Moreover:

(a) *If $f(0) > 0$, then both solutions are positive.*

(b) *If $f(0) < 0$, then at least one of the solutions is positive.*

Main Results: $f(0) = 0$

For the case in which the function $u \equiv 0$ is a solution of problem (10), called the trivial solution, to obtain a multiplicity result in this case, we need an additional condition on the primitive of f.

Theorem

Assume that Ω is a bounded domain with a Lipschitz boundary $\partial\Omega$ and the hypotheses $(H1)$-$(H3)$ are satisfied. Moreover, assume that $f(0) = 0$ and

$$\limsup_{s \to 0} \frac{F(s)}{|s|^p} = 0.$$

Then, there exists $\lambda_0 > 0$ such that, for all $\lambda \in (0, \lambda_0)$, problem (10) has at least two nontrivial solutions.

Applications of the Fractional p-Laplacian [4]

- **Nonlinear Partial Differential Equations** in electrostatics, phase transitions, and image processing.
- **Nonlocal Diffusion and Reaction-Diffusion Systems:** Models in biological pattern formation or material science where particles or fields interact over long distances, such as in the diffusion of pollutants or in ecological modeling.
- **Fractional Variational Problems:** Minimizing energy functionals with nonlocal terms, for example, in models of materials with fractured or porous structures.
- **Anomalous Diffusion in Physics:** Model diffusion with memory, useful in materials science and biology, such as in the diffusion of molecules in complex or disordered media.
- **Image Processing and Inpainting:** In image processing, the fractional p-Laplacian operator can be used for image smoothing with better preservation of edges and structures compared to classical methods.
- **Quantum Mechanics and Fractional Schrödinger Equation:** It is used to describe quantum particles in nonlocal or anomalous media.

Strategy to prove the main results

- Define the Variational Framework: Space of admissible functions, Energy Functional, and its Fréchet derivative.
- Verify the functional satisfies the Palais-Smale condition (compactness condition).
- Prove the existence of a critical point u_o of the associated energy functional through a variant of the mountain-pass theorem due to Hofer [8].
- Prove that the solution u_o is positive using a regularity result and a comparison principle.
- **Multiplicity of Results:** Use infinite-dimensional Morse Theory to obtain at least two solutions for the case $f(0) \neq 0$ and three solutions for the case $f(0) = 0$.

Variational Framework

Let $s \in (0,1)$ and $1 \leqslant p < \infty$, and denote by

$$W_0^{s,p}(\Omega) = \{u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ a.e in } \mathbb{R}^N \backslash \Omega\} \tag{15}$$

the subset of the following fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$:

$$W^{s,p}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) : \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}}\, dx\, dy < \infty \right\},$$

endowed with the norm

$$\|u\|_{s,p} := \left(\|u\|_p^p + [u]_{s,p}^p \right)^{1/p}, \tag{16}$$

where $\|\cdot\|_p$ denotes the norm in $L^p(\Omega)$ for $1 \leqslant p < \infty$ and

$$[u]_{s,p} := \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}}\, dx\, dy, \tag{17}$$

is the Gagliardo seminorm. It can be shown that $W^{s,p}(\mathbb{R}^N)$, endowed with the norm $\|\cdot\|_{s,p}$ defined in (16) and (17), is a Banach space, and $W_0^{s,p}(\Omega) \subset W^{s,p}(\mathbb{R}^N)$ is a closed subspace. In the case $1 < p < \infty$, $W^{s,p}(\Omega)$ is a **reflexive** Banach space

Variational Framework

- By virtue of the Sobolev-type inequality (see [7, Theorem 6.7]), it can be shown that the space $W_0^{s,p}(\Omega)$ can also be endowed with the norm

$$\|u\| := [u]_{s,p}, \tag{18}$$

  for $s \in (0,1)$ and $1 \leqslant p < \infty$.

- We shall denote by X the fractional Sobolev space $W_0^{s,p}(\Omega)$.
- Define $J_\lambda : X \to \mathbb{R}$, the energy functional associated with problem (10), by

$$\boxed{J_\lambda(u) = \frac{1}{p}\|u\|^p + \frac{1}{p}\int_\Omega V(x)|u|^p\,dx - \lambda\int_\Omega F(u)\,dx, \quad \text{for } u \in X,} \tag{19}$$

  and $\lambda > 0$ with $\|\cdot\|$ defined in (18).

Variational Framework

- The functional J_λ is well-defined and $J_\lambda \in C^1(X, \mathbb{R})$. It can be shown that the Fréchet derivative of J_λ is given by

$$\begin{aligned}\langle J'_\lambda(u), \varphi\rangle = \int_{\mathbb{R}^{2N}} \frac{\Phi_p(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy \\ + \int_\Omega V(x)|u|^{p-2}u\varphi\,dx - \lambda\int_\Omega f(u)\varphi\,dx,\end{aligned} \tag{20}$$

  for all $\varphi \in X$, where $\Phi_p : \mathbb{R} \to \mathbb{R}$ is given by $\Phi_p(s) = |s|^{p-2}s$, for $s \in \mathbb{R}$.

- We will say that u is a **weak solution** of problem(10) if u is a critical point of J_λ; namely,

$$\langle J'_\lambda(u), \varphi\rangle = 0, \qquad \text{for all } \varphi \in X. \tag{21}$$

The Palais-Smale Condition and Existence of a Solution

Definition

We will say that $(u_n) \subset X$ is a PS–sequence for J if

$$|J(u_n)| \leqslant C \quad \text{for all } n, \quad \text{and} \quad J'(u_n) \to 0, \text{ as } n \to \infty,$$

where C is a positive constant. We say that a functional $J \in C^1(X, \mathbb{R})$ satisfies the Palais–Smale condition (PS–condition) if any PS-sequence $(u_n) \subset X$ possesses a convergent subsequence.

- By using two auxiliary technical lemmas, we prove that the associated energy functional J_λ satisfies the Palais-Smale condition. [10, Lemma 3.3].
- It also allow us to prove the existence of a nontrivial critical point u_o of J_λ by using a variant of a mountain-pass theorem due to Hofer [8].
- By using a comparison principle, we also prove that u_o is positive.
- Next, we introduce the concept of critical groups.

Homology and Critical Groups

- The local behaviour of J near an isolated critical point u is described by the sequence of critical groups:

$$C_q(J,u) = H_q(J^c \cap U, (J^c \cap U)\backslash\{u\}), \quad q \geq 0,$$

where $c = J(u)$ is the corresponding critical value, U is a neighborhood of u, and H_* denotes the relative singular homology of the pair

$$(J^c \cap U, (J^c \cap U)\backslash\{u\})$$

with coefficients in a field $\mathcal{F}$.

- The critical groups are independent of the choice of U by the excision property of homology theory.

Critical Groups of a Mountain-Pass Type Critical Point

Definition

[11, Definition 6.98] Let X be a Banach space, $J \in C^1(X, \mathbb{R})$, and $u_0 \in \mathcal{K}$. We say that u_0 is of *mountain-pass type* if, for any open neighborhood U of u_0, the set $\{w \in U | \, J(w) < J(u_0)\}$ is nonempty and not path-connected.

The critical groups of mountain-pass type can be described by the following proposition found in Montreanu *et al.* [11]:

Proposition

[11, Proposition 6.100] Let X be a **reflexive Banach space**, *$J \in C^1(X, \mathbb{R})$, and $u_0 \in \mathcal{K}$ be isolated in $J(\mathcal{K})$. If u_0 is of mountain-pass type, then $C_1(J, u_0) \ncong 0$.*

Critical Groups at Infinity

Definition

Let $\mathcal{K} = \{u \in E | J'(u) = 0\}$. If $J(\mathcal{K})$ is bounded from below and J satisfies the (PS) condition, the global behavior of J can be described by the critical groups at infinity introduced by Barstch and Li (1997):

$$C_q(J, \infty) = H_q(X, J^a) \quad \text{for } q \in \mathbb{Z},$$

where $a < \inf J(\mathcal{K})$.

- In this work, we prove that the critical groups of J_λ are all trivial except at $q = 0$; namely,

$$\boxed{C_k(J_\lambda, \infty) = H_k(X, J_\lambda^{-M}) \cong \delta_{k,0}\mathbb{F}, \quad \text{for all } k \in \mathbb{Z}.} \tag{22}$$

- The result follows from the fact that there exists $\widetilde{M} > 0$ such that, for all $M \geqslant \widetilde{M}$, J_λ^{-M} is homotopically equivalent to S^∞ which is contractible in X.
- Using an argument with the long exact sequence of the pair (X, J_λ^{-M}) and using the fact X is contractible, we obtain (22).

A Result due to Barstch and Li.

The following result will allow us to obtain the first multiplicity result.

Proposition

[2, Proposition 3.6] Suppose that $J \in C^1(X, \mathbb{R})$ *satisfies the Palais-Smale condition at level* c *for every* $c \in \mathbb{R}$*. If* $\mathcal{K} = \emptyset$*, then*

$$C_k(J, \infty) \cong 0, \qquad \text{for all } k \in \mathbb{Z}.$$

If $\mathcal{K} = \{u_\lambda\}$*, then*

$$C_k(J, \infty) \cong C_k(J, u_\lambda), \qquad \text{for all } k \in \mathbb{Z}.$$

Proof of the First Multiplicity Result: $f(0) \neq 0$

Assume, by a way of contradiction, that $\mathcal{K} = \{u_\lambda\}$ where u_λ is the mountain-pass type solution found in [10, Theorem 3.4]. Then, it follows from Proposition 13 that

$$C_1(J_\lambda, u_\lambda) \ncong 0. \tag{23}$$

Since we are assuming that $\mathcal{K} = \{u_\lambda\}$, we can invoke Proposition 15 to get

$$C_k(J_\lambda, \infty) \cong C_k(J_\lambda, u_\lambda), \qquad \text{for all } k \in \mathbb{Z}. \tag{24}$$

In particular, if $k = 1$ in (24), we obtain from (22) and (23) that

$$0 \cong C_1(J_\lambda, \infty) \cong C_1(J_\lambda, u_\lambda) \ncong 0,$$

which is a contradiction.
Therefore, J_λ must have at least two critical points and this concludes the proof of the theorem.

Critical Groups at the Origin

- In the case where $f(0) = 0$, the trivial solution is a critical point of J_λ.
- To apply the infinite-dimensional Morse Theorey, we need to compute the critical groups of J_λ at the origin.

Lemma

[10, Lemma 5.1] Assume that the nonlinearity f satisfies (H_1) and its primitive F satisfies the condition

$$\limsup_{s \to 0} \frac{F(s)}{|s|^p} = 0. \tag{25}$$

Then, the origin is a local minimizer of the functional J_λ and its critical groups are given by

$$C_k(J_\lambda, 0) \cong \delta_{k,0}\mathbb{F}, \qquad \text{for all } k \in \mathbb{Z}. \tag{26}$$

A Result from Morse Theory

To prove the final result, we apply a result due to Chang [6] as presented in Perera and Schechter [14].

Proposition

[14, Proposition 1.4.1] Let a and b be extended real numbers that satisfy $-\infty < a < b \leqslant +\infty$. Suppose that $J \in C^1(X_0, \mathbb{R})$ has a finite number of critical points at the level $c \in (a, b)$, and that there are no other critical values of J in $[a, b]$. Assume also that J satisfies the $PS_{c'}$ condition for all $c' \in [a, b]$. Then,

$$H_q(J^b, J^a) \cong \bigoplus_{u \in \mathcal{K}^c} C_q(J, u), \qquad \text{for all } q \in \mathbb{Z}, \tag{27}$$

where $\mathcal{K}^c = \{u \in \mathcal{K} | J(u) = c\}$.

Proof of Second Multiplicity Result: $f(0) = 0$

Assume, by a way of contradiction, that $\mathcal{K} = \{0, u_\lambda\}$, where u_λ is the mountain-pass type solution found in [10, Theorem 3.4]. Then, it follows from Proposition 27 that

$$H_k(X, J_\lambda^{-M}) \cong C_k(J_\lambda, 0) \oplus C_k(J_\lambda, u_\lambda), \qquad \text{for all } k \in \mathbb{Z}. \tag{28}$$

In particular, setting $k = 1$ in (28) and using Proposition 13, (22), and (26), we obtain

$$0 \cong C_1(J_\lambda, 0) \oplus C_1(J_\lambda, u_\lambda) \cong 0 \oplus C_1(J_\lambda, u_\lambda) \ncong 0,$$

which is a contradiction. Therefore, the critical set $\mathcal{K}$ must have at least three critical points. This concludes the proof of the theorem.

Thank you!

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