Laws relating runs, long runs, and steps in gambler's ruin, with persistence in two strata

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Abstract

Define a certain gambler's ruin process $\mathbf{X}_j,\ \ j \geq 0$, such that the increments $\varepsilon_j := \mathbf{X}_j - \mathbf{X}_{j-1}$ take values ± 1 and satisfy $P(\varepsilon_{j+1} = 1 | \varepsilon_j = 1, |\mathbf{X}_j| = k) = P(\varepsilon_{j+1} = -1 | \varepsilon_j = -1, |\mathbf{X}_j| = k) = a_k$, all $j \geq 1$, where $a_k = a$ if $0 \leq k \leq f-1$, and $a_k = b$ if $f \leq k < N$. Here $0 < a, b < 1$ denote persistence parameters and $f, N \in \mathbb{N}$ with $f < N$. The process starts at $\mathbf{X}_0 = m \in (-N, N)$ and terminates when $|\mathbf{X}_j| = N$. Denote by $\mathcal{R}'_N, \mathcal{U}'_N$, and $\mathcal{L}'_N$, respectively, the numbers of runs, long runs, and steps in the meander portion of the gambler's ruin process. Define $\mathcal{X}'_N := \left(\mathcal{L}'_N - \frac{1-a-b}{(1-a)(1-b)}\mathcal{R}'_N - \frac{1}{(1-a)(1-b)}\mathcal{U}'_N\right)/N$ and let $f \sim \eta N$ for some $\eta \in (0,1)$. We show $\lim_{N\to\infty} E\{e^{it\mathcal{X}'_N}\} = \hat{\varphi}(t)$ exists in an explicit form. In case $b = 1-a$ and $\eta = a$, $\hat{\varphi}(t) = \sigma^2 t/\{\sinh(\sigma t)[\sigma\cosh(\sigma t) + i(1-2a)^2\sinh(\sigma t)]\}$, for $\sigma^2 := 1 - 3a + 3a^2$. If $b = a$, then $\hat{\varphi}(t) = At/\sinh(At)$, for $A := \sqrt{a/(1-a)}$; see [1] for the case $b = a = \frac{1}{2}$. We obtain a companion scaling limit with order N scaling for the last visit portion of the gambler's ruin process that also extends a result of [1] to the persistence model with two strata. Let $b = a$ and $N = \infty$ and let $\mathbf{R}$, $\mathbf{U}$, and $\mathbf{L}$, denote the numbers of runs, long runs (long inclines), and steps, respectively, in an excursion from the origin (primitive Dyck path). We obtain an explicit formula for the generating function $\overline{K}(r,u,z;a) := E\{r^{\mathbf{R}}u^{\mathbf{U}}z^{\mathbf{L}}\}$. As a consequence, $(1-a)P_a(\mathbf{L} = 2n,\ \mathbf{R} = 2k,\ \mathbf{U} = \ell) = aP_{1-a}(\mathbf{L} = 2n,\ \mathbf{L} - \mathbf{R} = 2k,\ \mathbf{U} = \ell)$, $n \geq 2$. Our proof of this symmetry actually proceeds by utilizing the existence of a one to one correspondence of lattice paths implied by the explicit form of $\overline{K}(r,u,z;\frac{1}{2})$.

References

[1] G.J. Morrow, Laws relating runs and steps in gambler's ruin, *Stoch. Proc. Appl.* **125** (2015) 2010-2025.