AN ELLIPTIC ANALOGUE OF THE ROOK NUMBERS

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The theory of rook numbers was introduced by Kaplinsky and Riordan in 1946, and since then it has been further studied and developed by many other people. In 1986 Garsia and Remmel extended the original rook theory by introducing a q-analogue of rook numbers and Remmel and Wachs generalized the theory even further by introducing (p, q)-analogues in 2004.

In this work, we construct an elliptic analogue of the rook numbers (which contain Garsia and Remmel's q-rook numbers, a.k.a. rook polynomials, as a special case) by using the elliptic weights

$$w_{a,b;q,p}(k) = \frac{\theta(aq^{2k+1}, bq^k, aq^{k-2}/b; p)}{\theta(aq^{2k-1}, bq^{k+2}, aq^k/b; p)} q,$$

where a, b, q are extra independent parameters, and θ is the modified Jacobi theta function with nome p satisfying $\theta(z;p) = \prod_{j\geq 0}(1-p^j z)(1-p^{j+1}/z)$ and $\theta(z_1,\ldots,z_m;p) = \theta(z_1;p)\ldots\theta(z_m;p)$, where $|p|<1$. (An important specialization is $p=0$; then $\theta(z;0) = (1-z)$ reduces to a simple factor.) We define the elliptic number of n (w.r.t. to the parameters a, b, q, p) by

$$[n]_{a,b;q,p} = \frac{\theta(q^n, aq^n, bq^2, a/b; p)}{\theta(q, aq, bq^{n+1}, aq^{n-1}/b; p)}.$$

It is easy to see that for nonnegative integer n the following identity holds

$$[n]_{a,b;q,p} = [n-1]_{a,b;q,p} + W_{a,b;q,p}(n),$$

where $W_{a,b;q,p}(n) = \prod_{k=1}^n w_{a,b;q,p}(j)$, which is crucial for the elliptic theory. We note that our definition of the elliptic number of n originated from the elliptic binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q,p} := \prod_{j=1}^{n-k} \frac{\theta(q^{j+k}, aq^{j+k}, bq^{j+k}, aq^{j-k}/b; p)}{\theta(q^j, aq^j, bq^{j+2k}, aq^j/b; p)}$$

(which happen to satisfy a nice recursion and combinatorial interpretation in terms of lattice paths) defined by the first author, as $[n]_{a,b;q,p} = \begin{bmatrix} n \\ 1 \end{bmatrix}_{a,b;q,p}$.

We use our elliptic weight function to assign weights to the uncancelled cells in the rook cancellation defined by Garsia and Remmel for the q-case and define an elliptic analogue of the rook number $r_k(a, b; q, p; B)$ for any Ferrers board $B = B(b_1, \ldots, b_n)$. We can prove that the elliptic analogue of the rook numbers satisfies the following product formula

$$\sum_{k=0}^n r_{n-k}(a,b;q,p;B) \prod_{j=1}^k [z-j+1]_{aq^{2(n+j-1)},bq^{n+j-1};q,p}$$
$$= \prod_{i=1}^n [z+b_i-i+1]_{aq^{2(n-b_i+i-1)},bq^{n-b_i+i-1};q,p}.$$

This product formula can be used to define an elliptic analogue of Stirling numbers of the second kind and an elliptic analogue of Lah numbers.

Haglund and Remmel proved analogous results in the rook theory for perfect matchings. We are also able to provide an elliptic extension thereof, by utilizing a suitably altered elliptic weight function.

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