Distributed patterns in paths

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Let $\mathcal{L}_n$ denote the set all paths from $(0,0)$ to (n,n) which consists of either unit north steps or unit east steps. $\mathcal{L}_n$ can also be considered as the set of words over the alphabet $\{N,E\}$ with n Ns and n Es and is in bijection with the set of fillings of the $2\times n$ rectangle with the numbers $1,\ldots,2n$. For $L\in\mathcal{L}_n$, let $ps_L(k)$ be the subword of L consisting of the k-th east step and k-th north step. For example, suppose $L = NEEENN \in \mathcal{L}_3$, then $ps_L(1) = NE$, $ps_L(2) = EN$ and $ps_L(3) = EN$. If $Z \subseteq \{1,\ldots,n\}$, we define $ps_L(Z)$ to be the subword consisting of k-th paired steps for all $k\in Z$. For example, if $L = NEEENN$, then $ps_L(1,2) = ps_L(1,3) = NEEN$ and $ps_L(2,3) = EENN$. Given a set of patterns $\Gamma \in \mathcal{L}_k$, we say there is a Γ-match in $L\in\mathcal{L}_n$ starting at the j-th paired step if $ps_L(\{j,j+1,j+2\cdots,j+k-1\})\in\Gamma$. We let Γ-$mch(L)$ denote the number of Γ-matches in L. If $\Gamma=\{P\}$ is a singleton, then we will write P-$mch(L)$ for Γ-$mch(L)$. For example, there six possible patterns of length four, $P_1 = EENN, P_2 = ENEN, P_3 = ENNE, P_4 = NEEN, P_5 = NENE, P_6 = NNEE$.

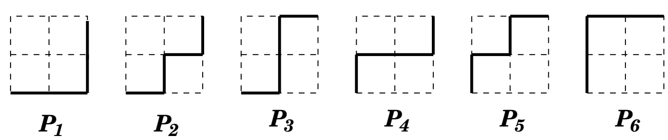

Figure 1: P_1, P_2, P_3, P_4, P_5 and P_6

If $\Delta \subseteq L_2$, we let $L_{n,\Delta}$ denote the set of all paths $Q\in L_n$ such that for $P\in L_2$, if P-$mch(Q)\neq 0$, then $P\in\Delta$. For example, if $\Delta=\{P_1,P_2\}$, then $L_{n,\Delta}$ is the set of all Dyck paths of length $2n$. We study generating functions of the form

$$\begin{aligned} F_\Gamma(x,t) &= 1+\sum_{n\geq 1}\frac{t^n}{(2n)!}\sum_{L\in\mathcal{L}_n} x^{\Gamma\text{-}mch(L)} \text{ and}\\ F_{\Gamma,\Delta}(x,t) &= 1+\sum_{n\geq 1}\frac{t^n}{(2n)!}\sum_{L\in\mathcal{L}_{n,\Delta}} x^{\Gamma\text{-}mch(L)} \end{aligned}$$

where $\Gamma,\Delta\subseteq L_2$.

To find the formulas for $F_\Gamma(x,t)$ and $F_{\Gamma,\Delta}(x,t)$, we generalize the cluster method of Goulden and Jackson. That is, we formulate the original problem as pattern matching problem in row-stricted arrays and then show that the desired generating functions can be expressed in terms of what we call generalized clusters which are certain class of words that can be decomposed into blocks of clusters satisfying certain conditions depending on Γ and/or Δ.