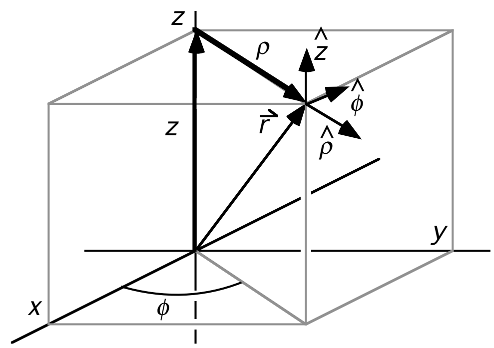
Cylindrical Coordinates



Transforms

The forward and reverse coordinate transformations are

$$\rho = \sqrt{x^2 + y^2} \qquad x = \rho\cos\phi$$
$$\phi = \arctan(y, x) \qquad y = \rho\sin\phi$$
$$z = z \qquad z = z$$

where we *formally* take advantage of the *two argument* arctan function to eliminate quadrant confusion.

Unit Vectors

The unit vectors in the cylindrical coordinate system are functions of position. It is convenient to express them in terms of the *cylindrical* coordinates and the unit vectors of the *rectangular* coordinate system which are *not* themselves functions of position.

$$\hat{\rho} = \frac{\vec{\rho}}{\rho} = \frac{x\hat{x} + y\hat{y}}{\rho} = \hat{x}\cos\phi + \hat{y}\sin\phi$$
$$\hat{\phi} = \hat{z} \times \hat{\rho} = -\hat{x}\sin\phi + \hat{y}\cos\phi$$
$$\hat{z} = \hat{z}$$

Variations of unit vectors with the coordinates

Using the expressions obtained above it is easy to derive the following handy relationships:

$$\frac{\partial\hat{\rho}}{\partial\rho} = 0 \qquad \frac{\partial\hat{\phi}}{\partial\rho} = 0 \qquad \frac{\partial\hat{z}}{\partial\rho} = 0$$
$$\frac{\partial\hat{\rho}}{\partial\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi = \hat{\phi} \qquad \frac{\partial\hat{\phi}}{\partial\phi} = -\hat{x}\cos\phi - \hat{y}\sin\phi = -\hat{\rho} \qquad \frac{\partial\hat{z}}{\partial\phi} = 0$$
$$\frac{\partial\hat{\rho}}{\partial z} = 0 \qquad \frac{\partial\hat{\phi}}{\partial z} = 0 \qquad \frac{\partial\hat{z}}{\partial z} = 0$$

Path increment

We will have many uses for the path increment $d\vec{r}$ expressed in cylindrical coordinates:

$$d\vec{r} = d(\rho\hat{\rho} + z\hat{z}) = \hat{\rho}d\rho + \rho d\hat{\rho} + \hat{z}dz + zd\hat{z}$$
$$= \hat{\rho}d\rho + \rho\left(\frac{\partial\hat{\rho}}{\partial\rho}d\rho + \frac{\partial\hat{\rho}}{\partial\phi}d\phi + \frac{\partial\hat{\rho}}{\partial z}dz\right) + \hat{z}dz + z\left(\frac{\partial\hat{z}}{\partial\rho}d\rho + \frac{\partial\hat{z}}{\partial\phi}d\phi + \frac{\partial\hat{z}}{\partial z}dz\right)$$
$$= \hat{\rho}d\rho + \hat{\phi}\rho d\phi + \hat{z}dz$$

Time derivatives of the unit vectors

We will also have many uses for the time derivatives of the unit vectors expressed in cylindrical coordinates:

$$\dot{\hat{\rho}} = \frac{\partial\hat{\rho}}{\partial\rho}\dot{\rho} + \frac{\partial\hat{\rho}}{\partial\phi}\dot{\phi} + \frac{\partial\hat{\rho}}{\partial z}\dot{z} = \hat{\phi}\dot{\phi}$$

$$\dot{\hat{\phi}} = \frac{\partial\hat{\phi}}{\partial\rho}\dot{\rho} + \frac{\partial\hat{\phi}}{\partial\phi}\dot{\phi} + \frac{\partial\hat{\phi}}{\partial z}\dot{z} = -\hat{\rho}\dot{\phi}$$

$$\dot{\hat{z}} = \frac{\partial\hat{z}}{\partial\rho}\dot{\rho} + \frac{\partial\hat{z}}{\partial\phi}\dot{\phi} + \frac{\partial\hat{z}}{\partial z}\dot{z} = 0$$

Velocity and Acceleration

The velocity and acceleration of a particle may be expressed in cylindrical coordinates by taking into account the associated rates of change in the unit vectors:

$$\vec{v} = \dot{\vec{r}} = \dot{\hat{\rho}}\rho + \hat{\rho}\dot{\rho} + \dot{\hat{z}}z + \hat{z}\dot{z} = \hat{\rho}\dot{\rho} + \hat{\phi}\rho\dot{\phi} + \hat{z}\dot{z}$$

$$\boxed{\vec{v} = \hat{\rho}\dot{\rho} + \hat{\phi}\rho\dot{\phi} + \hat{z}\dot{z}}$$

$$\vec{a} = \dot{\vec{v}} = \dot{\hat{\rho}}\dot{\rho} + \hat{\rho}\ddot{\rho} + \dot{\hat{\phi}}\rho\dot{\phi} + \hat{\phi}\dot{\rho}\dot{\phi} + \hat{\phi}\rho\ddot{\phi} + \dot{\hat{z}}\dot{z} + \hat{z}\ddot{z}$$

$$= \hat{\phi}\dot{\phi}\dot{\rho} + \hat{\rho}\ddot{\rho} - \hat{\rho}\rho\dot{\phi}^2 + \hat{\phi}\dot{\rho}\dot{\phi} + \hat{\phi}\rho\ddot{\phi} + \hat{z}\ddot{z}$$

$$\boxed{\vec{a} = \hat{\rho}\left(\ddot{\rho} - \rho\dot{\phi}^2\right) + \hat{\phi}\left(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}\right) + \hat{z}\ddot{z}}$$

The del operator from the definition of the gradient

Any (static) scalar field u may be considered to be a function of the cylindrical coordinates ρ, ϕ, and z. The value of u changes by an infinitesimal amount du when the point of observation is changed by $d\vec{r}$. That change may be determined from the partial derivatives as

$$du = \frac{\partial u}{\partial\rho}d\rho + \frac{\partial u}{\partial\phi}d\phi + \frac{\partial u}{\partial z}dz .$$

But we also define the gradient in such a way as to obtain the result

$$du = \vec{\nabla}u \cdot d\vec{r}$$

Therefore,

$$\frac{\partial u}{\partial\rho}d\rho + \frac{\partial u}{\partial\phi}d\phi + \frac{\partial u}{\partial z}dz = \vec{\nabla}u \cdot d\vec{r}$$

or, in cylindrical coordinates,

$$\frac{\partial u}{\partial\rho}d\rho + \frac{\partial u}{\partial\phi}d\phi + \frac{\partial u}{\partial z}dz = \left(\vec{\nabla}u\right)_\rho d\rho + \left(\vec{\nabla}u\right)_\phi \rho d\phi + \left(\vec{\nabla}u\right)_z dz$$

and we demand that this hold for any choice of $d\rho$, $d\phi$ and dz. Thus,

$$\left(\vec{\nabla}u\right)_\rho = \frac{\partial u}{\partial\rho}, \quad \left(\vec{\nabla}u\right)_\phi = \frac{1}{\rho}\frac{\partial u}{\partial\phi}, \quad \left(\vec{\nabla}u\right)_z = \frac{\partial u}{\partial z},$$

from which we find

$$\boxed{\vec{\nabla} = \hat{\rho}\frac{\partial}{\partial\rho} + \frac{\hat{\phi}}{\rho}\frac{\partial}{\partial\phi} + \hat{z}\frac{\partial}{\partial z}}$$

Divergence

The divergence $\vec{\nabla}\cdot\vec{A}$ is carried out taking into account, once again, that the unit vectors themselves are functions of the coordinates. Thus, we have

$$\vec{\nabla}\cdot\vec{A} = \left(\hat{\rho}\frac{\partial}{\partial\rho} + \frac{\hat{\phi}}{\rho}\frac{\partial}{\partial\phi} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(A_\rho\hat{\rho} + A_\phi\hat{\phi} + A_z\hat{z}\right)$$

where the derivatives must be taken *before* the dot product so that

$$\begin{aligned}
\vec{\nabla}\cdot\vec{A} &= \left(\hat{\rho}\frac{\partial}{\partial\rho} + \frac{\hat{\phi}}{\rho}\frac{\partial}{\partial\phi} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\vec{A} \\
&= \hat{\rho}\cdot\frac{\partial\vec{A}}{\partial\rho} + \frac{\hat{\phi}}{\rho}\cdot\frac{\partial\vec{A}}{\partial\phi} + \hat{z}\cdot\frac{\partial\vec{A}}{\partial z} \\
&= \hat{\rho}\cdot\left(\frac{\partial A_\rho}{\partial\rho}\hat{\rho} + \frac{\partial A_\phi}{\partial\rho}\hat{\phi} + \frac{\partial A_z}{\partial\rho}\hat{z} + A_\rho\frac{\partial\hat{\rho}}{\partial\rho} + A_\phi\frac{\partial\hat{\phi}}{\partial\rho} + A_z\frac{\partial\hat{z}}{\partial\rho}\right) \\
&\quad + \frac{\hat{\phi}}{\rho}\cdot\left(\frac{\partial A_\rho}{\partial\phi}\hat{\rho} + \frac{\partial A_\phi}{\partial\phi}\hat{\phi} + \frac{\partial A_z}{\partial\phi}\hat{z} + A_\rho\frac{\partial\hat{\rho}}{\partial\phi} + A_\phi\frac{\partial\hat{\phi}}{\partial\phi} + A_z\frac{\partial\hat{z}}{\partial\phi}\right) \\
&\quad + \hat{z}\cdot\left(\frac{\partial A_\rho}{\partial z}\hat{\rho} + \frac{\partial A_\phi}{\partial z}\hat{\phi} + \frac{\partial A_z}{\partial z}\hat{z} + A_\rho\frac{\partial\hat{\rho}}{\partial z} + A_\phi\frac{\partial\hat{\phi}}{\partial z} + A_z\frac{\partial\hat{z}}{\partial z}\right)
\end{aligned}$$

With the help of the partial derivatives previously obtained, we find

$$\begin{aligned}
\vec{\nabla}\cdot\vec{A} &= \hat{\rho}\cdot\left(\frac{\partial A_\rho}{\partial\rho}\hat{\rho} + \frac{\partial A_\phi}{\partial\rho}\hat{\phi} + \frac{\partial A_z}{\partial\rho}\hat{z} + 0 + 0 + 0\right) \\
&\quad + \frac{\hat{\phi}}{\rho}\cdot\left(\frac{\partial A_\rho}{\partial\phi}\hat{\rho} + \frac{\partial A_\phi}{\partial\phi}\hat{\phi} + \frac{\partial A_z}{\partial\phi}\hat{z} + A_\rho\hat{\phi} - A_\phi\hat{\rho} + 0\right) \\
&\quad + \hat{z}\cdot\left(\frac{\partial A_\rho}{\partial z}\hat{\rho} + \frac{\partial A_\phi}{\partial z}\hat{\phi} + \frac{\partial A_z}{\partial z}\hat{z} + 0 + 0 + 0\right) \\
&= \left(\frac{\partial A_\rho}{\partial\rho}\right) + \left(\frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{A_\rho}{\rho}\right) + \left(\frac{\partial A_z}{\partial z}\right) \\
&= \left(\frac{\partial A_\rho}{\partial\rho} + \frac{A_\rho}{\rho}\right) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}
\end{aligned}$$

$$\boxed{\vec{\nabla}\cdot\vec{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(A_\rho\rho\right) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}}$$

Curl

The curl $\vec{\nabla}\times\vec{A}$ is also carried out taking into account that the unit vectors themselves are functions of the coordinates. Thus, we have

$$\vec{\nabla}\times\vec{A}=\left(\hat{\rho}\frac{\partial}{\partial\rho}+\frac{\hat{\phi}}{\rho}\frac{\partial}{\partial\phi}+\hat{z}\frac{\partial}{\partial z}\right)\times\left(A_\rho\hat{\rho}+A_\phi\hat{\phi}+A_z\hat{z}\right)$$

where the derivatives must be taken *before* the cross product so that

$$\begin{aligned}
\vec{\nabla}\times\vec{A}&=\left(\hat{\rho}\frac{\partial}{\partial\rho}+\frac{\hat{\phi}}{\rho}\frac{\partial}{\partial\phi}+\hat{z}\frac{\partial}{\partial z}\right)\times\vec{A}\\
&=\hat{\rho}\times\frac{\partial\vec{A}}{\partial\rho}+\frac{\hat{\phi}}{\rho}\times\frac{\partial\vec{A}}{\partial\phi}+\hat{z}\times\frac{\partial\vec{A}}{\partial z}\\
&=\hat{\rho}\times\left(\frac{\partial A_\rho}{\partial\rho}\hat{\rho}+\frac{\partial A_\phi}{\partial\rho}\hat{\phi}+\frac{\partial A_z}{\partial\rho}\hat{z}+A_\rho\frac{\partial\hat{\rho}}{\partial\rho}+A_\phi\frac{\partial\hat{\phi}}{\partial\rho}+A_z\frac{\partial\hat{z}}{\partial\rho}\right)\\
&\quad+\frac{\hat{\phi}}{\rho}\times\left(\frac{\partial A_\rho}{\partial\phi}\hat{\rho}+\frac{\partial A_\phi}{\partial\phi}\hat{\phi}+\frac{\partial A_z}{\partial\phi}\hat{z}+A_\rho\frac{\partial\hat{\rho}}{\partial\phi}+A_\phi\frac{\partial\hat{\phi}}{\partial\phi}+A_z\frac{\partial\hat{z}}{\partial\phi}\right)\\
&\quad+\hat{z}\times\left(\frac{\partial A_\rho}{\partial z}\hat{\rho}+\frac{\partial A_\phi}{\partial z}\hat{\phi}+\frac{\partial A_z}{\partial z}\hat{z}+A_\rho\frac{\partial\hat{\rho}}{\partial z}+A_\phi\frac{\partial\hat{\phi}}{\partial z}+A_z\frac{\partial\hat{z}}{\partial z}\right)
\end{aligned}$$

With the help of the partial derivatives previously obtained, we find

$$\begin{aligned}
\vec{\nabla}\times\vec{A}&=\hat{\rho}\times\left(\frac{\partial A_\rho}{\partial\rho}\hat{\rho}+\frac{\partial A_\phi}{\partial\rho}\hat{\phi}+\frac{\partial A_z}{\partial\rho}\hat{z}+0+0+0\right)\\
&\quad+\frac{\hat{\phi}}{\rho}\times\left(\frac{\partial A_\rho}{\partial\phi}\hat{\rho}+\frac{\partial A_\phi}{\partial\phi}\hat{\phi}+\frac{\partial A_z}{\partial\phi}\hat{z}+A_\rho\hat{\phi}-A_\phi\hat{\rho}+0\right)\\
&\quad+\hat{z}\times\left(\frac{\partial A_\rho}{\partial z}\hat{\rho}+\frac{\partial A_\phi}{\partial z}\hat{\phi}+\frac{\partial A_z}{\partial z}\hat{z}+0+0+0\right)\\
&=\left(\frac{\partial A_\phi}{\partial\rho}\hat{z}-\frac{\partial A_z}{\partial\rho}\hat{\phi}\right)+\left(-\frac{1}{\rho}\frac{\partial A_\rho}{\partial\phi}\hat{z}+\frac{1}{\rho}\frac{\partial A_z}{\partial\phi}\hat{\rho}+\frac{A_\phi}{\rho}\hat{z}\right)\\
&\quad+\left(\frac{\partial A_\rho}{\partial z}\hat{\phi}-\frac{\partial A_\phi}{\partial z}\hat{\rho}\right)\\
&=\hat{\rho}\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi}-\frac{\partial A_\phi}{\partial z}\right)+\hat{\phi}\left(\frac{\partial A_\rho}{\partial z}-\frac{\partial A_z}{\partial\rho}\right)+\hat{z}\left(\frac{\partial A_\phi}{\partial\rho}+\frac{A_\phi}{\rho}-\frac{1}{\rho}\frac{\partial A_\rho}{\partial\phi}\right)
\end{aligned}$$

$$\boxed{\vec{\nabla}\times\vec{A}=\hat{\rho}\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi}-\frac{\partial A_\phi}{\partial z}\right)+\hat{\phi}\left(\frac{\partial A_\rho}{\partial z}-\frac{\partial A_z}{\partial\rho}\right)+\hat{z}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(A_\phi\rho\right)-\frac{1}{\rho}\frac{\partial A_\rho}{\partial\phi}\right)}$$

Laplacian

The Laplacian is a scalar operator that can be determined from its definition as

$$\nabla^2 u = \vec{\nabla}\cdot\left(\vec{\nabla}u\right) = \left(\hat{\rho}\frac{\partial}{\partial\rho} + \frac{\hat{\phi}}{\rho}\frac{\partial}{\partial\phi} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(\hat{\rho}\frac{\partial u}{\partial\rho} + \frac{\hat{\phi}}{\rho}\frac{\partial u}{\partial\phi} + \hat{z}\frac{\partial u}{\partial z}\right)$$

$$= \hat{\rho}\cdot\frac{\partial}{\partial\rho}\left(\hat{\rho}\frac{\partial u}{\partial\rho} + \frac{\hat{\phi}}{\rho}\frac{\partial u}{\partial\phi} + \hat{z}\frac{\partial u}{\partial z}\right)$$

$$+ \frac{\hat{\phi}}{\rho}\cdot\frac{\partial}{\partial\phi}\left(\hat{\rho}\frac{\partial u}{\partial\rho} + \frac{\hat{\phi}}{\rho}\frac{\partial u}{\partial\phi} + \hat{z}\frac{\partial u}{\partial z}\right)$$

$$+ \hat{z}\cdot\frac{\partial}{\partial z}\left(\hat{\rho}\frac{\partial u}{\partial\rho} + \frac{\hat{\phi}}{\rho}\frac{\partial u}{\partial\phi} + \hat{z}\frac{\partial u}{\partial z}\right)$$

With the help of the partial derivatives previously obtained, we find

$$\nabla^2 u = \hat{\rho}\cdot\left(\hat{\rho}\frac{\partial^2 u}{\partial\rho^2} - \frac{\hat{\phi}}{\rho^2}\frac{\partial u}{\partial\phi} + \frac{\hat{\phi}}{\rho}\frac{\partial^2 u}{\partial\phi\partial\rho} + \hat{z}\frac{\partial^2 u}{\partial z\partial\rho}\right)$$

$$+ \frac{\hat{\phi}}{\rho}\cdot\left(\hat{\phi}\frac{\partial u}{\partial\rho} + \hat{\rho}\frac{\partial^2 u}{\partial\rho\partial\phi} - \frac{\hat{\rho}}{\rho}\frac{\partial u}{\partial\phi} + \frac{\hat{\phi}}{\rho}\frac{\partial^2 u}{\partial\phi^2} + \hat{z}\frac{\partial^2 u}{\partial z\partial\phi}\right)$$

$$+ \hat{z}\cdot\left(\hat{\rho}\frac{\partial^2 u}{\partial\rho\partial z} + \frac{\hat{\phi}}{\rho}\frac{\partial^2 u}{\partial\phi\partial z} + \hat{z}\frac{\partial^2 u}{\partial z^2}\right)$$

$$= \frac{\partial^2 u}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\phi^2} + \frac{\partial^2 u}{\partial z^2}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\phi^2} + \frac{\partial^2 u}{\partial z^2}$$

Thus, the Laplacian operator can be written as

$$\boxed{\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}}$$