All About Work

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Introduction

Q: Exactly what seems to be the problem?

A: Students and often instructors share a deep confusion about work and its relationship to change in energy. (ref: Arons, Bernard, Canagaratna, Erlichson, Kemp, Leff, Penchina, Sherwood.) The confusion generally results from a simple failure to understand that there are a surprising number of distinctly different ways of defining the work done on a system, each useful but with varying applicability to different specific situations. Adding to the confusion, the energy changes to which the various works correspond are varied, occasionally subtle, and often frame-dependent.

Q: So what do you plan to do about it?

A: • Make three simple, very general assumptions.
• Define seven different works—six of which are distinct and useful.
• Derive six “work-energy” relations.
• Determine their frame-dependence.
• Give a couple of examples.
Assumptions

1. Our system consists of a collection of N elements, each of which behaves as a point particle.
   - Point particles obey Newton’s second law.
   - Point particles have no internal energy.

2. Interelement forces are conservative.
   - Allows us to define an internal potential energy function

3. A classical, non-relativistic analysis is adequate.
   - Justifies our use of the simple Galilean transformation
   - Forces and time intervals are frame-invariant.
**Notation**

\[ \mathbf{F}_{ij} \equiv \text{force on element } i \text{ due to element } j. \]

\[ \mathbf{F}_i^{\text{int}} \equiv \sum_{j \neq i} \mathbf{F}_{ij} = \text{net internal force on element } i. \]

\[ \mathbf{F}_i^{\text{ext}} = \text{net external force on element } i. \]

\[ \mathbf{F}_i \equiv \mathbf{F}_i^{\text{int}} + \mathbf{F}_i^{\text{ext}} = \text{net force on element } i. \]

\[ \mathbf{F}_{\text{int}} \equiv \sum_i \mathbf{F}_i^{\text{int}} = \text{net internal force on system.} \]

\[ = 0 \quad \text{(by Newton's third law)} \]

\[ \mathbf{F}_{\text{ext}} \equiv \sum_i \mathbf{F}_i^{\text{ext}} = \text{net external force on system.} \]

\[ \mathbf{F} \equiv \mathbf{F}_{\text{int}} + \mathbf{F}_{\text{ext}} = \text{net force on system.} \]

\[ = \mathbf{F}_{\text{ext}}. \]
Work Definitions and Interrelations

Frame-specific works:

\[ W \equiv \sum_i \int F_i \cdot dr_i \]
\[ W_{\text{ext}} \equiv \sum_i \int F_{i}^{\text{ext}} \cdot dr_i \]
\[ W_{\text{int}} \equiv \sum_i \int F_{i}^{\text{int}} \cdot dr_i \]

(Pseudowork or CM work)
\[ W_{\text{ps}} \equiv \sum_i \int F_i \cdot dr_{\text{cm}} \]

System-specific works:

\[ w \equiv \sum_i \int F_i \cdot d\vec{r}_i \]
\[ w_{\text{ext}} \equiv \sum_i \int F_{i}^{\text{ext}} \cdot d\vec{r}_i \]
\[ w_{\text{int}} \equiv \sum_i \int F_{i}^{\text{int}} \cdot d\vec{r}_i \]

Interrelations:

\[ W = W_{\text{ext}} + W_{\text{int}} \]
\[ W_{\text{ext}} = W_{\text{ps}} + w_{\text{ext}} \]
\[ W_{\text{int}} = w_{\text{int}} \]
\[ w = w_{\text{ext}} + w_{\text{int}} \]
Work-Energy Relations

(Derived with the aid of Newton’s Second law, \( F_i = m_i \frac{dv_i}{dt} \).)

\[
W = \Delta \left[ \sum_i \frac{1}{2} m_i v_i^2 \right] \equiv \Delta K
\]

\[
W_{ps} = \Delta \left[ \frac{1}{2} M v_{cm}^2 \right] \equiv \Delta K_{tr}
\]

\[
w = \Delta \left[ \sum_i \frac{1}{2} m_i \vec{v}_i^2 \right] \equiv \Delta K_{int} = \Delta K - \Delta K_{tr}
\]

\[
W_{int} = \Delta \left[ \sum_{all\ pairs} \int_{\infty}^{\vec{r}_{ij}} \vec{F}_{ij} \cdot d\vec{r}_{ij} \right] \equiv -\Delta \Phi
\]

\[
w_{ext} = \Delta \Phi + \Delta K_{int} \equiv \Delta U
\]

\[
W_{ext} = \Delta K_{tr} + \Delta U \equiv \Delta E
\]

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## Summary Table

<table>
<thead>
<tr>
<th>Work</th>
<th>Associated ΔEnergy</th>
<th>Frame-dependence</th>
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</thead>
<tbody>
<tr>
<td>$W \equiv \sum_i \int \mathbf{F}<em>i \cdot d\mathbf{r}<em>i = W</em>{\text{ext}} + W</em>{\text{int}}$</td>
<td>$\Delta K \equiv \Delta \left[ \sum_i \frac{1}{2} m_i \mathbf{v}<em>i^2 \right] = \Delta K</em>{\text{tr}} + \Delta K_{\text{int}}$</td>
<td>Frame-dependent*</td>
</tr>
<tr>
<td>$W_{\text{ext}} \equiv \sum_i \int \mathbf{F}<em>i^{\text{ext}} \cdot d\mathbf{r}<em>i = W</em>{\text{ps}} + w</em>{\text{ext}}$</td>
<td>$\Delta E \equiv \Delta K_{\text{tr}} + \Delta U$</td>
<td>Frame-dependent*</td>
</tr>
<tr>
<td>$W_{\text{int}} \equiv \sum_i \int \mathbf{F}_i^{\text{int}} \cdot d\mathbf{r}<em>i = w</em>{\text{int}}$</td>
<td>$-\Delta \Phi \equiv \Delta \left[ \sum \text{all pairs} \int_{\infty} \mathbf{F}<em>{ij} \cdot d\mathbf{r}</em>{ij} \right]$</td>
<td>Invariant</td>
</tr>
<tr>
<td>$W_{\text{ps}} \equiv \sum_i \int \mathbf{F}<em>i \cdot d\mathbf{r}</em>{\text{cm}}$</td>
<td>$\Delta K_{\text{tr}} \equiv \Delta \left[ \frac{1}{2} M \mathbf{v}_{\text{cm}}^2 \right]$</td>
<td>Frame-dependent*</td>
</tr>
<tr>
<td>$w \equiv \sum_i \int \mathbf{F}<em>i \cdot d\mathbf{\bar{r}}<em>i = w</em>{\text{ext}} + w</em>{\text{int}}$</td>
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<td>$w_{\text{ext}} \equiv \sum_i \int \mathbf{F}_i^{\text{ext}} \cdot d\mathbf{\bar{r}}_i$</td>
<td>$\Delta U \equiv \Delta K_{\text{int}} + \Delta \Phi$</td>
<td>Invariant</td>
</tr>
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<td>$w_{\text{int}} \equiv \sum_i \int \mathbf{F}_i^{\text{int}} \cdot d\mathbf{\bar{r}}_i$</td>
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</tr>
</tbody>
</table>

* $\Delta K_{\text{tr}}$ (frame traveling at $\mathbf{u}$ wrt “lab frame”) = $\Delta K_{\text{tr}}$ ("lab frame") − $M \mathbf{u} \cdot \Delta \mathbf{v}_{\text{cm}}$

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Mirror-Symmetric Collision

(Consider mass pair on the right.)

Before the collision:

- \( W_{ps} = W_{ext} = w_{ext} = 0 \)
  \( \Rightarrow K_{tr}, E, U \) are constant
- \( W = w = -w_{int} \) (oscillatory)
  \( \Rightarrow \) Energy shuttles between \( K_{int} \) and \( \Delta \Phi \)

During the collision:

- \( W_{ps} = \Delta K_{tr} > 0 \) (By assumption.)
- \( W_{ext} = \Delta E = 0 \) (Because the external force acts through zero displacement.)
- \( w_{ext} = \Delta U = W_{ext} - W_{ps} < 0 \) (As expected.)

Note: \( w_{ext} \) involves the force exerted by the mass pair on the left and the motion of its point of application (the interface) relative to the system CM.
Free Expansion of an Ideal Gas
(“Ideal” ⇒ Φ = 0, so that \( w_{\text{int}} = 0 \) for any process.)

Before wall collapses:
\( K_{\text{tr}} = 0 \) and \( E = U = K_{\text{int}} \)

During the first few moments after the wall collapses:
\( W_{\text{ps}} = \Delta K_{\text{tr}} > 0 \) and \( w_{\text{ext}} = w = -W_{\text{ps}} = \Delta K_{\text{int}} < 0 \)
- Internal kinetic energy becomes translational kinetic energy as a net leftward force from the left wall pushes on the system.
- \( w \) and \( w_{\text{ext}} \) are negative because the left wall pushes left while moving right relative to the center of mass.

As gas is decelerated at the opposite end:
\( W_{\text{ps}} = \Delta K_{\text{tr}} < 0 \) and \( w_{\text{ext}} = w = -W_{\text{ps}} = \Delta K_{\text{int}} > 0 \)
- Translational kinetic energy becomes internal kinetic energy again as the rightward force from the right wall brings the system CM to rest.
- \( w \) and \( w_{\text{ext}} \) are positive because the right wall exerts a dominant rightward force while moving right relative to the center of mass.

And so on until equilibrium is reestablished ...

Note: \( W = W_{\text{ext}} = \Delta K = \Delta E = 0 \) throughout the process!
Conclusions

• There are (at least) seven different and useful ways to define the work done on a system. (Although two of them turn out to be identical.)

• For each of the six distinguishable works, there exists a “work-energy” relationship which relates its value in any process to the change in some form of system energy.

• Careful analysis of processes involving multiparticle systems or extended bodies is not possible without an understanding of the subtle differences between the various definitions of work and the forms of energy whose changes they mediate and of their frame-transformation properties.
What Lies Behind the Symbols?

\[ \sum_i \int \mathbf{F}_i \cdot d\mathbf{r}_i = \Delta \left[ \sum_i \frac{1}{2} m_i v_i^2 \right] \]

\[ \sum_i \int \mathbf{F}_i^{\text{ext.}} \cdot d\mathbf{r}_i = \Delta \left[ \sum_i \frac{1}{2} m_i v_i^2 - \sum_{\text{all pairs}} \int_{\infty} \mathbf{F}_{ij} \cdot d\mathbf{r}_{ij} \right] \]

\[ W = \Delta K \]

\[ W_{\text{ext}} = \Delta E \]

\[ W_{\text{int}} = -\Delta \Phi \]

\[ W_{ps} = \Delta K_{tr} \]

\[ w = \Delta K_{int} \]

\[ \sum_i \int \mathbf{F}_i \cdot d\mathbf{r}_{cm} = \Delta \left[ \frac{1}{2} M v_{cm}^2 \right] \]

\[ \sum_i \int \mathbf{F}_i^{\text{int.}} \cdot d\mathbf{r}_i = \Delta \left[ \sum_i \frac{1}{2} m_i \mathbf{v}_i^2 \right] \]

\[ \sum_i \int \mathbf{F}_i^{\text{ext.}} \cdot d\mathbf{r}_i = \Delta \left[ \sum_i \frac{1}{2} m_i \mathbf{v}_i^2 - \sum_{\text{all pairs}} \int_{\infty} \mathbf{F}_{ij} \cdot d\mathbf{r}_{ij} \right] \]

\[ W_{\text{ext}} = \Delta U \]

\[ W_{\text{int}} = -\Delta \Phi \]