Transport II

Lecture Notes (2014)

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Chapter 1

Introduction to Heat Transfer

1.1 Definition of heat transfer

Heat transfer is the science which seeks to predict the rate of energy transfer between material bodies as a result of a temperature difference.

- **Definition of 'heat'**
  Heat is energy in transit solely as a result of a temperature difference.

- **Definition of 'temperature'**
  Temperature is a measure of the mean kinetic energy of molecules. Absolute zero (0°K) is a state of complete motionless of molecules.

- **'Rate'**
  'Rate' implies an element of speed, how fast an event happens, and time.

- **'System'**
  A system is any designated region of a continuum of fixed mass. The boundaries of a system may be deformable but they always enclose the same mass. In thermodynamics, the universe can be divided into two parts. One part is the system, the other part is the rest of the universe called the surroundings.

![Figure 1.1 Schematic diagram of the "universe", showing a system and the surroundings.](image)

- **'Control volume'**
  A 'control volume' is also any designated region of a continuum except that it may permit matter to cross its boundaries. If the boundaries of a control volume are such that matter may not enter or leave the control volume, the control volume is identical to a system. In these respects, a 'system' is a subset of a 'control volume'.

- **'Equilibrium'**
  'Equilibrium' means that there are no spatial differences in the variables that describe the condition of the system, also called the 'state' of a system, such as its pressure, temperature, volume, and mass (P, T, V, m), and that any changes which occur do so infinitesimally slowly.
1.2 Importance of Thermodynamics in the Study of Heat Transfer

Thermodynamics is the science which seeks to predict the amount of energy needed to bring about a change of state of a system from one equilibrium state to another. While thermodynamics tells us nothing about the mechanisms of energy transfer, rates of change, and time associated with a system changing from one equilibrium state to another, it is still the linchpin that allow us to answer these questions.

Heat transfer analysis is based on the laws of thermodynamics. Because heat is transferred only when a temperature difference exists, and thus when a system or control volume is not in equilibrium, we have to acknowledge that there is an inherent dilemma in the application of the laws of thermodynamics to systems or control volumes that are not in equilibrium. Heat transfer is an essentially non-equilibrium science since it involves the rate processes.

1.3 Inherent Dilemma in the Study of Heat Transfer

The laws of thermodynamics are applicable only to equilibrium states which means that the state does not really change significantly with time, differences in variables between the state of a system and its surroundings are of infinitesimal magnitude and that within the system itself there are no spatial variations of the variables that determine its state. Heat transfer concerns processes associated with differences, time and rate - aspects which are excluded from thermodynamics analysis. So how can we study a system which is experiencing a change of state when we can rigorously define nothing about the system during the change?

This problem is resolved by assuming that states can still be specified for a system which is not in equilibrium provided that the rate of change of the state is not too fast, or that differences in state variables within a system are not too large. Just what constitute 'too fast' and 'too large' is a matter of experience. For the vast majority of engineering processes of practical importance, this assumption is an excellent one.

The standard practice is to assume that the state of a system can be assigned specific values of such quantities like T, P, V at any time and location even when it is known that these quantities are changing or not uniform over the whole of the system, then to proceed with applying the laws of thermodynamics to the system under these conditions of inherent nonequilibrium, and to compare the results of the analyses with observed behavior. Fortunately, differences are small enough for most engineering problems as to make worthwhile our study of heat transfer. Heat transfer relies explicitly on the validity of the laws of thermodynamics, makes intimate use of these laws, and assumes that they are applicable to a high degree of approximation for systems that are not in equilibrium.

The following example illustrates the information that thermodynamics can provide about the change of state of system when a constraint is removed. The example is of a hot bearing being dropped into a cold oil bath. The constraint that is being removed to allow the change of state of the bearing and oil to occur is the physical separation of the bearing from the oil bath. Removal of the constraint occurs when the bearing is dropped into the oil. The problem is to compute the final temperature of the bearing and oil given information about the initial temperature of the oil and bearing, their heat capacities, and masses.
Example 1.3-1

A bearing with mass \( m_b \), heat capacity \( C_b \), at an initial temperature \( T_{bi} \) is dropped into the oil bath with mass \( m_o \), heat capacity \( C_o \), at an initial temperature \( T_{oi} \). Compute the heat exchanged between the bearing and oil and the final equilibrium temperature \( T_{bf} \) in terms of \( m_b, C_b, T_{bi}, m_o, C_o, \) and \( T_{oi} \).

![Diagram showing \( m_b, C_b, T_{bi} \) and \( m_o, C_o, T_{oi} \), with \( m_b, C_b, T_{bf} \) and \( m_o, C_o, T_{of} \).]

Solution

1. Compute the heat exchanged between the bearing and oil

Apply the first law of thermodynamics to bearing

\[
\delta Q_b = \delta W_b + dE_b
\]

where

- \( \delta Q_b \) = heat transfer between the bearing and the oil
- \( \delta W_b \) = work exchanged between the bearing and the oil
- \( dE_b \) = accumulated energy of the bearing

Work is defined as any other transfer of energy except the energy transfer due to a difference in temperature between the objects. Work is done on a system whenever a piston is pushed, a liquid within a container is stirred, or a current is run through a resistor. In each case, the system's energy will increase, and usually its temperature too. However the system is not being heated since the flow of energy is not a spontaneous one caused by a difference in temperature. Notice that both heat and work refer to energy in transit. The total energy inside a system can be defined but not heat or work. It is only meaningful to specify how much heat entered a system, or how much work was done on a system.

The usual sign convention for the first law of thermodynamics is as follows:

**The flow of heat into a system is a positive flow, while a flow of work into a system is a negative flow.**

Thus, if 10 J of heat \( \delta Q \) flow into a system, it is regarded as \( \delta Q = +10 \text{ J} \), while if 10 J of work \( \delta W \) flow into a system, it is regarded as \( \delta W = -10 \text{ J} \).

The sign convention for heat flow stated above has been universally used; unfortunately, the sign convention for work flow has not been universally accepted. In the U. S. the convention...
for work as stated above has been in widespread use while in Europe the opposite convention for work has been more commonly used [1].

The symbol '\( d \)' means an exact differential quantity where \( \int dE = E_f - E_i \). Energy is a state function. Heat and work are path functions and the differentials of heat and work, \( \delta Q \) and \( \delta W \), respectively, are nonexact differentials so that \( \int \delta Q \neq Q_f - Q_i \).

There are various forms of energy that matter may possess, in particular, kinetic energy \( KE \), potential energy \( PE \), internal energy \( U \), electrical energy \( EE \), and magnetic energy \( ME \). Only kinetic, potential, and internal energies will mostly be considered in this text. A system possesses kinetic energy by virtue of its velocity, a system possesses potential energy by virtue of its height above a reference plane, and a system possesses internal energy by virtue of the random thermal motion of the atoms and molecules of which it is composed.

For this example, the kinetic and potential energies will be considered to be zero

\[
dE_b = dU_b
\]

Since there is no work done on the rigid bearing

\[
\delta Q_b = dU_b
\]

From the thermodynamic postulate, the state of a pure, homogeneous system is known if any two of the independent variables are known. The internal energy \( U \) can then be expressed in terms of the independent variables \( T \) and \( V \) as,

\[
U_b = U_b(T_b, V_b)
\]

From the chain rule of calculus

\[
dU_b = \left. \frac{\partial U_b}{\partial T_b} \right|_V dT_b + \left. \frac{\partial U_b}{\partial V_b} \right|_T dV_b
\]

The second term is zero since \( dV_b = 0 \) for a rigid bearing. From the definition of the heat capacity at constant volume

\[
C_{vb} = \left. \frac{\partial U_b}{\partial T_b} \right|_V
\]

and the fact that heat capacity at constant volume is almost the same as heat capacity at constant pressure for a solid \( C_{vb} \approx C_{pb} \).

\[
dU_b = C_{pb}dT_b
\]

Return to the first law on the bearing.
\[ \delta Q_b = C_{pb}dT_b \]

\[ \int_i^f \delta Q_b = \int_i^f C_{pb}dT_b \]

Since the differential \( \delta Q_b \) is nonexact, the integral from state \( i \) to state \( f \), denoted \( \int_i^f \delta Q_b \), depends on the path connecting state \( i \) to state \( f \). Defining \( Q_b = \int_i^f \delta Q_b \) and assuming that \( C_{pb} \) is a constant, it follows that

\[ Q_b = C_{pb}(T_{bf} - T_{bi}) \]

Let \( C_{pb} = m_b c_{pb} \)

where \( c_{pb} \) is the specific heat or heat capacity per unit mass of the bearing

then

\[ Q_b = m_b c_{pb}(T_{bf} - T_{bi}) \]

Similarly for a system consisting of the oil only

\[ Q_o = m_o c_{po}(T_{of} - T_{oi}) \]

2. Compute the final equilibrium temperature \( T_{bf} \)

Apply the first law to the composite system of bearing and oil, assuming no heat loss to the surroundings

\[ \delta Q = \delta W + dU \]

Since \( \delta Q = \delta W = 0 \)

\[ dU = dU_o + dU_b = 0 \]

\[ m_o c_{po}dT_o + m_b c_{pb}dT_b = 0 \]

\[ \int_{T_o}^{T_{of}} m_o c_{po}dT_o + \int_{T_b}^{T_{bf}} m_b c_{pb}dT_b = 0 \]

Let \( \zeta = \frac{m_b c_{pb}}{m_o c_{po}} \)
Since the composite system of bearing and oil is in equilibrium $T_{bf} = T_{of} = T_f$

Solve for $T_f$ gives

$$T_f = \frac{T_{of} + \xi T_{bi}}{1 + \xi}$$

If one is interested in knowing the time it takes for the temperature of the bearing to reach the final equilibrium temperature $T_f$ or the temperature of the bearing $T_b$ at any intervening time, the answers can be obtained from the study of heat transfer not from thermodynamics.

**Example 1.3-2**

A slab of ice in a thin-walled container 20 mm thick and 400 mm on each side is placed on a well-insulated pad. At its top surface, the ice is exposed to ambient air for which $T_\infty = 25^\circ$C and the convection coefficient is 25 W/m$^2$-°K. The density and latent heat of fusion of ice are 920 kg/m$^3$ and 334 kJ/kg, respectively. Neglecting heat transfer from the sides and assuming the ice-water mixture remains at 0°C, how long (in sec) will it take to completely melt the ice?

**Solution**

The rate of heat transfer, $q$, to the slab of ice is given by

$$q = hA(T_\infty - T)$$

In this equation $h$ is the heat transfer coefficient and $A$ is the top surface of the ice. The energy required to melt the ice is $\rho AL\Delta h_{\text{fusion}}$ where $L$ is the thickness of the ice slab. The time to melt the ice is then:

$$t = \frac{\rho AL\Delta h_{\text{fusion}}}{hA(T_\infty - T)} = \frac{\rho L\Delta h_{\text{fusion}}}{25h} = \frac{(920)(0.02)(334000)}{(25)(25)} = 9833 \text{ s}$$
Chapter 1

1.4 Reason for Studying Heat Transfer

To precisely describe the way in which the dissimilarity between two temperatures governs the flow of heat between them

Applications:
- Generation of electrical power
- Cooling of engines and electronic equipment
- Refrigeration
- Control of pollution generation from combustion of fossil fuels
- Biological systems

Efficient heat transfer from the human body to its ambient maintains the average body temperature at 37°C. The human body constantly generates heat from the conversion of nutrients chemical bond to thermal energy. The heat is removed from the body by the three modes of heat transfer: conduction, convection, and radiation. Hyperthermia is the process when the body temperature increases above normal due to insufficient heat removal. On the other hand, hypothermia is the process when the body temperature decreases below normal because heat loss from the body is higher than the heat generated.

1.5 Modes of Heat Transfer

Conduction

Conduction refers to energy transfer by molecular interactions. Energy carriers on the molecular level are 'electrons' and 'phonons' where the latter is a quantized lattice vibration. The interaction is a nearest-neighbor process that extends only a few molecular dimensions. Energy transport over a distance is by a staged transfer through molecular distances.

Convection

Convection refers to energy transport over macroscopic distances by bulk movement of matter. Once matter reaches its destination, energy dissipated by conduction. In general, the total heat transfer is a superposition of energy transport by molecular interactions and by the bulk motion of the fluid.

Radiation

Radiation refers to energy transfer by propagation of electromagnetic waves. Energy is absorbed or emitted by electrons changing their energy levels as a result of the temperature of the body. A packet of energy emitted this way is called a 'photon' which has an energy \( E \) given by Planck's Law,

\[
E = h \nu
\]

where \( h = 6.625 \times 10^{-34} \text{ J-s/(molecule)} \) is 'Planck's constant' and \( \nu \) is the frequency of the electromagnetic wave.
Unlike conduction and convection, radiation heat transfer does not require any matter in the region over which the temperature difference exists to promote the transport of heat. The following figure gives an analogy to show the differences between the three modes of heat transfer.

**Figure 1.5-1** Three modes of heat transfer.

Each mode of heat transfer has a different constitutive equation that relates the energy flux $q^n$ to temperature ($T$) as expressed in general form

$$q^n \propto f(T)$$

where the function $f(T)$ is different for the three modes of heat transfer. For conduction,

$$q^n_n \propto \frac{\partial T}{\partial n}$$

In this expression, $n$ is direction (e.g., $x$, $y$, or $z$) in which energy flux is transported. The symbol $q^n_n$ means the heat flux in the $n$ direction. Written as an equality, the above equation is

$$q^n_n = -k_n \frac{\partial T}{\partial n}$$

This equation is called Fourier’s Law of conduction which serves as the defining equation for the thermal conductivity, $k_n$, in the $n$ direction. The units of $k_n$ in SI system are W/m·°K. The explicit assumption is that each direction (e.g., for a Cartesian coordinate system ($n = x$, $y$, or $z$)) can potentially have its own unique thermal conductivity (i.e., $k_x$, $k_y$, $k_z$). Such a material is termed orthotropic as a special case of an anisotropic material. Certain selected woods and semiconductor materials (e.g., silicon in very thin film form for material thickness on the order of micrometers) are anisotropic. When all components of the thermal conductivity are the same,
(k_x = k_y = k_z = k) the material is termed isotropic. In this text, most of the materials we consider will be isotropic.

For convection \( f(T) = T \) and \( q'' \propto T \), writing the above equation as an equality,

\[
q'' = h(T - T_x)
\]

This equation is known as Newton’s law of cooling where the proportionality constant is the heat transfer coefficient \( h \). The units of \( h \) are \( \text{W/m}^2\cdot\text{K} \). This equation is technically only useful when the heat transfer coefficient is independent of temperature. In reality, this situation is rarely realized. Still, computing the heat transfer coefficient continues to be the most important parameter in convection heat transfer analysis.

Finally, for radiation \( q'' \propto T^4 \). This equation is known as Planck radiation equation and

\[
q'' = \sigma T^4
\]

where \( \sigma = 5.67 \times 10^{-8} \text{ W/m}^2\cdot\text{K}^4 \) is Stefan-Boltzmann constant. An additional discussion of these three laws follows in the next few pages.

1.6 Nanoscale Origin of Heat Conduction

\[
\lambda_p = \frac{\hbar}{(2\pi mk_B T)^{1/2}}
\]
where $\hbar = 6.625 \times 10^{-34}$ J·s/(molecule) is Planck's constant, $k_B = 1.38025 \times 10^{-23}$ J/(molecule·°K) is Boltzmann constant and $m$ is the mass of the molecule.

### 1.7 Thermal Conductivity

In this section, a scale analysis for molecular heat flux will be presented so that the thermal conductivity of solids can be derived in terms of the carriers’ mean free paths and other properties. The one-dimensional heat flux due to electrons as the energy carries is considered first.

![Figure 1.7-1](image)

**Figure 1.7-1** Electron energy as a function of distance.

At location $x_0$, the energy of electrons are the energy they have after their previous collisions at locations at $(x_0 - \lambda_e)$ and $(x_0 + \lambda_e)$. The flux of energy transported, $q''_e$, is proportional to the electron energy and the electron flux:

$$q''_e \sim \frac{\text{Energy}}{\text{Electron} \times \text{Area} \times \text{Time}}$$

Let $n_e$ be the electron number density (electrons/volume) and $u_e$ the mean electron speed, the electron flux at any location is given by

$$n_e u_e = n_e u_e \bigg|_{x_0-\lambda_e} = n_e u_e \bigg|_{x_0+\lambda_e}$$

The above relations come from the fact that electron flux is a constant at any location therefore there must be no net flux of electrons at $x_0$. Let $e_e$ be the energy of the electron, the energy
transported from locations \((x_0 - \lambda_e)\) and \((x_0 + \lambda_e)\) are \(n_e u_e e_{e|_{x_0 - \lambda_e}}\) and \(n_e u_e e_{e|_{x_0 + \lambda_e}}\), respectively. The net energy transported across \(x = x_0\) is then

\[
q'' e = n_e u_e \left[ e_{e|_{x_0 - \lambda_e}} - e_{e|_{x_0 + \lambda_e}} \right] \tag{1.7-3}
\]

The negative sign for the second term on the RHS is due the transport in the negative direction. Expanding the electron energy \(e_e(x)\) using Taylor series and retaining only the first order terms gives

\[
e_e(x) = e_e(x_0) + \frac{d e_e}{d x} \bigg|_{x=x_0} (x - x_0) + \ldots \tag{1.7-4}
\]

Let \(x = (x_0 - \lambda_e)\) and \(x = (x_0 + \lambda_e)\) we have

\[
e_{e|_{x_0 - \lambda_e}} = e_e(x_0) + \frac{d e_e}{d x} \bigg|_{x=x_0} (x_0 - \lambda_e - x_0) \approx e_e(x_0) + \frac{d e_e}{d x} (-\lambda_e) \tag{1.7-5a}
\]

\[
e_{e|_{x_0 + \lambda_e}} = e_e(x_0) + \frac{d e_e}{d x} \bigg|_{x=x_0} (x_0 + \lambda_e - x_0) \approx e_e(x_0) + \frac{d e_e}{d x} (\lambda_e) \tag{1.7-5b}
\]

Substituting Eqs. (1.7-5a,b) into Eq. (1.7-3) we have

\[
q'' e = 2 n_e u_e \lambda_e \frac{d e_e}{d x} \tag{1.7-6}
\]

The electron energy gradient can be rewritten as

\[
\frac{d e_e}{d x} = \frac{d e_e}{d T} \frac{d T}{d x} \tag{1.7-7}
\]

Since the solid is incompressible we have

\[
\frac{d e_e}{d T} = \frac{\partial e_e}{\partial x} \bigg|_v \equiv C_{ve} \tag{1.7-8}
\]

The above equation defines the heat capacity per electron which is a constant independent of temperature

\[
C_{ve} = \frac{3}{2} k_B \neq f(T) \tag{1.7-9}
\]
The heat capacity can also be defined in mass unit so that

$$C_{ve} = C_{ve}m_e$$ \hspace{1cm} (1.7-10)

In this equation, $C_{ve}$ is the heat capacity per unit mass and $m_e$ mass of the electron. The energy flux is then

$$q_e'' = -2n_eu_e\lambda_e\frac{de_e}{dx} = q_e'' = -2n_eu_e\lambda_eC_{ve}m_e\frac{dT}{dx}$$ \hspace{1cm} (1.7-11)

In terms of the electron density defined as $\rho_e = n_em_e$, Eq. (1.7-11) may be written as

$$q_e'' = -2\rho_e u_e\lambda_eC_{ve}\frac{dT}{dx}$$

This equation may be expressed in terms of an equality using a proportional constant $C$.

$$q_e'' = -C\rho_e u_e\lambda_eC_{ve}\frac{dT}{dx}$$ \hspace{1cm} (1.7-12)

Taking into account of the distribution of molecular speeds about the mean, the constant C is given$^3$ as $C = \frac{1}{3}$. Therefore

$$q_e'' = -\frac{1}{3}\rho_e u_e\lambda_eC_{ve}\frac{dT}{dx}$$ \hspace{1cm} (1.7-13)

The Fourier’s law for 1 dimension is written as

$$q_e'' = -k_e\frac{dT}{dx}$$ \hspace{1cm} (1.7-14)

In this equation, $k_e$ is the thermal conductivity contribution due to the flow of free electrons. Comparing Eq. (1.7-3) and Eq. (1.7-4) we have

$$k_e = \frac{1}{3}\rho_e u_e\lambda_eC_{ve}$$ \hspace{1cm} (1.7-15)

Similar arguments for phonons as the heat carrier give a similar formula for thermal conductivity, $k_p$, contribution due to lattice vibrational waves and molecular collisions.

$$k_p = \frac{1}{3}\rho_p u_p\lambda_pC_{vp}$$ \hspace{1cm} (1.7-16)

$^3$ Kruger, p. 19
In this equation \( u_p \) is the speed of sound in the solid state and \( C_{vp} \) the lattice heat capacity which is a function of temperature. In general

\[
q'' = q''_e + q''_p, \text{ or } k = k_e + k_p.
\]

The thermal conductivity \( k \) has two contributions \( k_e \) and \( k_p \). The contribution \( k_e \) is due to the flow of free electrons and the contribution \( k_p \) is due to lattice vibrational waves (energy quanta = phonon) and molecular collisions. For pure metals, the contribution due to electron flow is dominant \( (k_e >> k_p) \), for alloys the two contributions are comparable \( (k_e \approx k_p) \), and for non metals the phonon contribution is more important \( (k_e << k_p) \). If \( \lambda_e \) and \( \lambda_p \) are much smaller than the characteristic dimension \( L \) of the system, \( k = (k_e + k_p) \) is called the bulk thermal conductivity.

Diamond has the highest known thermal conductivity for a solid with \( k > 1500 \text{ W/m-K} \). In some device like the heat pipe the thermal conductivity can approach infinity. Figure 1.7-2 shows a schematic of a heat pipe for a horizontal position\(^4\). In this configuration, the heat pipe is a hollow cylinder with a layer of wicking material covering the inside surface with a hollow core in the center. Heat transfers along the pipe by the movement of a condensable fluid contained in the pipe where the liquid permeates the wicking materials by capillary action. Liquid is vaporized in the evaporator end of the pipe where heat is added. The vapor then moves to the condenser end where heat is removed. The condensed liquid flows back to the evaporator section by capillary action.

![Figure 1.7-2 Basic heat pipe configuration in horizontal position.](image)

Materials at the nanoscale, 1 to 100 nanometer (1 nanometer = \( 10^{-9} \) m) length scale, have properties (i.e. chemical, electrical, magnetic, mechanical, and optical) very different from the bulk materials. In fact, nanoparticles possess enhanced performance properties compared to bulk materials when they are used in similar applications\(^5\).

![Figure 1.7-3 The four allotropes of carbon\(^6\).](image)

\(^4\) To be determined


Carbon nanotube shown in Figure 1.7-3 is an allotrope of carbon beside graphite, diamond, and fullerene. Fullerene or Buckminsterfullerene is nanostructure of 60 carbon atoms (C_{60}). Other fullerenes with larger number of carbon atoms (C_{76}, C_{80}, C_{240}, etc.) also exist. Fullerenes were discovered by Kroto and collaborators using laser to evaporate graphite.

Carbon nanotubes were discovered by Iijima during the synthesis of fullerenes using an electric arc-evaporation reactor to vaporize carbon graphite under an inert atmosphere. The nanotubes produced by Iijima appeared to made up of a perfect network of hexagonal graphite rolled up to form a hollow tube. The nanotube diameter range is from one to several nanometers. The diameter range is much smaller than the nanotube length range, which is from one to a few micrometers.

Carbon nanotubes and fullerenes have unusual photochemical, electronic, thermal and mechanical properties. Single-walled carbon nanotubes (SWCNTs) could behave metallic, semi-metallic, or semi-conductive one-dimensional objects, and their longitudinal thermal conductivity could exceed the in-plane thermal conductivity of graphite. The thermal conductivity of nanotube could exceed 2000 W/m-K.

The temperature dependent of bulk thermal conductivity is shown in Figure 1.7-4 for various metallic and nonmetallic solids. When \( \lambda_e \) and \( \lambda_p \) are comparable to the characteristic dimension \( L \) of the system, we must account for the distribution of heat carriers by applying the Boltzmann Tranport Equation. This approach to heat transfer is beyond the scope of this text. In the following we will assume bulk properties.

Figure 1.7-4 The temperature dependence of \( k \) for various solids.

---

Chapter 2

Constitutive relation between $q$ and $T$

2.1 Conduction

Fourier's law (1822), developed from observed phenomena, states that the rate of heat transfer in the 'n' direction is proportional to the temperature gradient $\frac{\partial T}{\partial n}$

$$q_n \propto \frac{\partial T}{\partial n}$$

where $n$ is the direction of heat transfer and $\partial n$ is the rate of change of distance in the direction $n$.

$n$ is the unit normal vector, and $t$ is the unit tangential vector with the following properties,

$$|n| = 1; \ |t| = 1; \ n \cdot t = 0; \ n \cdot n = 1; \ t \cdot t = 1$$

$$q_n = C \frac{\partial T}{\partial n}, \text{ where } C = -A \cdot k_n$$

$$q_n = -k_wA \frac{\partial T}{\partial n}$$

where
\[ k_n = \text{thermal conductivity in 'n' direction, W/m·K} \]
\[ A = \text{area of surface perpendicular to } n \text{ through which } q_n \text{ flows} \]

The minus is a sign convention so that \( q_n \) is positive in the direction it transfers. In this text, we will usually consider the isotropic materials where the thermal conductivity \( k \) is independent of direction. For one dimensional heat transfer in the \( x \)-direction only, the heat transfer rate is then

\[
q_x = -kA \frac{dT}{dx}
\]

or in terms of the heat flux \( q_x^* \)

\[
q_x^* = -k \frac{dT}{dx}
\]

The thermal conductivity \( k \) has two contributions \( k_e \) and \( k_L \).

\[ k = k_e + k_L \]

The contribution \( k_e \) is due to the flow of free electrons and the contribution \( k_L \) is due to lattice vibrational waves (energy quanta = phonon) and molecular collisions.

- Pure metals \( k_e \gg k_L \)
- Alloys \( k_e \approx k_L \)
- Non metals \( k_e \ll k_L \)

Diamond has the highest known thermal conductivity for a solid with \( k > 1500 \text{ W/m·K} \). In some device like the heat pipe the thermal conductivity can approach infinity. Figure 2.1 shows a schematic of a heat pipe for a horizontal position [1]. In this configuration, the heat pipe is a hollow cylinder with a layer of wicking material covering the inside surface with a hollow core in the center. Heat transfers along the pipe by the movement of a condensable fluid contained in the pipe where the liquid permeates the wicking materials by capillary action. Liquid is vaporized in the evaporator end of the pipe where heat is added. The vapor then moves to the condenser end where heat is removed. The condensed liquid flows back to the evaporator section by capillary action.
Figure 2.1-1 Basic heat pipe configuration in horizontal position.

The temperature dependent of thermal conductivity is shown in Figure 2.2 for various metallic and nonmetallic solids [2].

Figure 2.1-2 The temperature dependence of $k$ for various solids.
2.2 Convection

Convection refers to energy transport over macroscopic distances by bulk movement of matter. Once matter reaches its destination, energy dissipated by conduction. In general, the total heat transfer is a superposition of energy transport by molecular interactions and by the bulk motion of the fluid. Convection occurs between a solid surface and a fluid when the two are at different temperature. Convection heat transfer is usually classified as forced convection or free convection. Forced convection occurs when the flow is caused by an external means, such as a fan or blower, a compressor, a pump, or atmospheric winds. Free convection occurs when the flow is caused only by the density differences due to temperature variation in the fluid. It should be note that free convection also exists in forced convection, however the contribution of free convection in this situation is negligible. Convection can occur with or without a phase change. When there is no change in phase, the energy that is being transfer is the sensible energy of the fluid. When a phase change occurs such as boiling or condensation there is an additional heat exchange due to the latent heat of the fluid from the change in physical molecular bonds.

Boundary Layer Concept

Consider a flow over a flat plate where the free stream velocity is \( u_x \). The fluid flow can be divided into two regions: a velocity boundary layer region next to the solid surface in which momentum transfer exists and a region outside the boundary layer in which momentum transfer is negligible.

![Figure 2.2.-1 Velocity boundary layer on a flat plate [2].](image)

A thermal boundary layer also exists when the fluid flows over a surface if the fluid free stream temperature \( T_x \) is not the same as the surface temperatures \( T_s \). Heat transfer is significant within the thermal boundary layer region.

![Figure 2.2-2 Thermal boundary layer on an isothermal flat plate [2].](image)

Let \( \delta(x) \) be the thickness of the velocity boundary layer and \( \delta_t(x) \) be the thickness of the thermal boundary layer. \( \delta(x) \) is typically defined as the normal distance from the surface to the
location where the velocity in the $x$-direction $u = 0.99u_\infty$. Similarly $\delta(x)$ is defined as the normal distance from the surface to the location where $[(T_s - T)/(T_s - T_\infty)] = 0.99$. $T$ is the temperature of the fluid that varies between $T_\infty$ and $T_s$. In general the velocity boundary layer thickness will not be the same as the thermal boundary layer thickness.

Constitutive Equation

Newton (1701)

$$\frac{dT}{dt} \propto (T_s - T_\infty)$$

The symbol $\propto$ means 'proportional to'.

From the first law of thermodynamics

$$\delta Q = \delta W + dU$$

Consider the energy flow due to the temperature difference only: $\delta W = 0$, then

$$\delta Q = dU$$

Divide the above expression by $dt$

$$\frac{\delta Q}{dt} = \frac{dU}{dt}$$

Since $dU = C_p dT$; so

$$\frac{dU}{dt} = C_p \frac{dT}{dt}$$

or

$$\frac{\delta Q}{dt} = C_p \frac{dT}{dt}$$

$\frac{\delta Q}{dt}$ is the rate of heat transfer $q$, therefore

$$q = C_p \frac{dT}{dt} \propto (T_s - T_\infty)$$

The proportional sign can be removed by the use of $C$, proportionality constant

$$q = C(T_s - T_\infty)$$

where $C$ is defined as

$$C = \overline{h} \cdot A_s$$
where is $A_s$ the surface area exposed to fluid and $\bar{h}$ is the average heat transfer coefficient with unit of W/m$^2$-K.

$$q = \bar{h} \cdot A_s (T_s - T_\infty)$$

In terms of the heat flux $q''$ defined as the heat transfer rate per unit area $q'' = q/A_s$

$$q'' = \bar{h} (T_s - T_\infty)$$

This formula is really useful only if the heat transfer coefficient is $\bar{h}$ constant. Table 2.1 shows representative values of the heat transfer coefficient $\bar{h}$.

<table>
<thead>
<tr>
<th>Type of convection</th>
<th>$\bar{h}$ (W/m$^2$-K.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Free convection in gas</td>
<td>2 – 25</td>
</tr>
<tr>
<td>Free convection in liquid</td>
<td>50 – 1,000</td>
</tr>
<tr>
<td>Forced convection in gas</td>
<td>25 – 250</td>
</tr>
<tr>
<td>Forced convection in liquid</td>
<td>50 – 20,000</td>
</tr>
<tr>
<td>Boiling and condensation</td>
<td>2,500 – 100,000</td>
</tr>
</tbody>
</table>

It should be noted that the heat transfer coefficient $\bar{h}$ is not a fluid property whereas the thermal conductivity $k$ is a fluid property. $\bar{h}$ depends on parameters external to the fluid,

$$\bar{h} = f(u_\infty, \rho_{\text{fluid}}, \mu_{\text{fluid}}, T_s, C_p_{\text{fluid}} + \text{others})$$
Chapter 2

2.3 Thermal Radiation

Thermal radiation is energy emitted by matter entirely because of its temperature. The mechanism of emission is electrons changing energy states with the frequency $\nu$ determined by a material's temperature. Thermal radiation exists in a vacuum. It may be viewed as the propagation of electromagnetic wave with the wavelength $\lambda$ confined from 0.1 $\mu$m to 100 $\mu$m. The wavelength is related to the frequency by

$$\lambda = \frac{c}{\nu}$$

where $c$ is the speed of light in the medium. In the visible part of the spectrum, which extends from 0.4 $\mu$m (violet) to 0.7 $\mu$m (red), the various wavelengths are associated with the color of the light. There is no thermal radiation if the matter is at 0 K. Figure 2.5 shows the emissive power $E_{\lambda,b}$ of a blackbody as a function of temperature and wavelength [2]. The surface emissive power is the rate at which energy is released per unit area. A blackbody is an ideal surface that absorbs all incident radiation, regardless of wavelength and direction. It is also an ideal emitter with radiation-emitted independent of direction.

![Figure 2.3-1 Spectral blackbody emissive power](image)

---

The total emissive power of a blackbody is the rate of thermal radiation energy emitted over the entire spectrum at a given temperature,

\[ E_b = \int_0^\infty E_{\lambda,b} d\lambda \]

From experiments, the blackbody (total) emissive power is given by

\[ E_b = \sigma T^4 \]

where \( \sigma = 5.67 \times 10^{-8} \text{ W/m}^2\text{-K}^4 \) is Stefan-Boltzmann constant. The emissive power of a real surface is less than that of a blackbody or

\[ E = \varepsilon E_b \]

where \( \varepsilon \) is the "emissivity" of the real surface with value between 0 and 1: \( 0 \leq \varepsilon \leq 1 \). In general the emissivity depends on the temperature, the wavelength, and the surface finish of the materials,

\[ \varepsilon = f(T, \lambda, \text{surface finish, …}) \]

**Radiation exchange between two surfaces**

Consider two infinite flat plates facing each other with the left surface 1 at temperature \( T_1 \) and the right surface 2 at temperature \( T_2 \).

\[
\begin{array}{c|c|c|c}
| & T_1 & & T_2 |
\end{array}
\]

a) If surfaces 1 and 2 are ideal emitters (\( \varepsilon_1 = \varepsilon_2 = 1 \))

\[ q_{1-2} = A \cdot E_{b_1} - A \cdot E_{b_2} \]

where

- \( E_{b_1} \) = total radiation flux leaving surface 1
- \( E_{b_2} \) = total radiation flux leaving surface 2

\[ q_{1-2} = A(E_{b_1} - E_{b_2}) = A \cdot \sigma(T_1^4 - T_2^4) \]

b) If surfaces 1 and 2 are not ideal emitters

\[ q_{1-2} = \Xi A \cdot \sigma(T_1^4 - T_2^4) \]
The rate at which radiant energy is absorbed per unit area may be determined from the surface absorptivity \( \alpha \). A surface is called a gray surface if the absorptivity is equal to the emissivity. In a special case where a small gray surface at \( T_s \) is completely enclosed within another surface at \( T_{\text{sur}} \), the net rate of radiation heat transfer from the surface is

\[
q = A \varepsilon \sigma (T_s^4 - T_{\text{sur}}^4)
\]

Figure 2.3-2 Radiation exchange a surface enclosed within the surroundings\(^2\).

2.4 Energy Balance

Energy balance is the cornerstone of heat transfer analysis. The first law of thermodynamics is the conservation of energy, which states that energy is neither created nor destroyed. The first law can be written for a system as

\[
\delta Q = \delta W + dE
\]

where

\( \delta Q \) = heat transfer between the system and the surroundings

\( \delta W \) = work exchanged between the system and the surroundings

\( dE \) = accumulated energy of the system

The first law postulates the existence of a "function of state" called the accumulated energy such that for an adiabatic system (\( \delta Q = 0 \)) the work output is balanced by a reduction in the accumulated energy:

\[
dE = -\delta W
\]

While \( \delta Q \) and \( \delta W \) are not themselves a "function of state", the difference \( \delta Q - \delta W \) is a function of state.

A quantity is a \textit{function of state} when the difference in its values between two states only depends on the initial and final states and not on the paths connecting these two states. The accumulated energy \( E \) is a state function so that,

\[
\left[ E(T_2, P_2, \ldots) - E(T_1, P_1, \ldots) \right]_a = \left[ E(T_2, P_2, \ldots) - E(T_1, P_1, \ldots) \right]_b
\]

The differential of \( E \) is an exact differential for which the integral from state 1 to state 2 is simply the difference \( E(T_2, P_2, \ldots) - E(T_1, P_1, \ldots) \).

\[
\int_{1}^{2} dE = E|^{2}_{1} = E(T_2, P_2, \ldots) - E(T_1, P_1, \ldots)
\]

Heat and work are path functions and the differentials of heat and work, \( \delta Q \) and \( \delta W \), respectively, are nonexact differentials so that \( \int_{1}^{2} \delta Q \neq Q_2 - Q_1 \) and \( \int_{1}^{2} \delta W \neq W_2 - W_1 \). The following example will show that work is path dependent.

\textbf{Example 2.4-1}

A gas is contained within a cylinder and piston system shown. Assuming a 'simple' system (expansion and compression work only), calculate the work done by the system in transforming from state \( P_1, V_1 \) to state \( P_2, V_2 \).

\begin{figure}
\centering
\includegraphics[width=\textwidth]{example2.4-1.png}
\caption{Example 2.4-1}
\end{figure}

\textbf{Solution}

1. Compute the work using path \( a \) with constant volume followed by constant pressure

\[
\int_{1}^{2} \delta W = \int_{P_1, V_1}^{P_2, V_2} PdV = \int_{P_1, V_1}^{P_2, V_1} PdV + \int_{P_2, V_1}^{P_2, V_2} PdV = 0 + P_2(V_2 - V_1) = W_a
\]

2. Compute the work using path \( b \) with constant pressure followed by constant volume

\[
\int_{1}^{2} \delta W = \int_{P_1, V_1}^{P_2, V_2} PdV = \int_{P_1, V_1}^{P_2, V_2} PdV = 0 + P_2(V_2 - V_1) = W_b
\]
\[ \int_{V_1}^{V_2} \delta W = \int_{P_1}^{P_2} PdV = \int_{P_1}^{P_2} PdV + \int_{P_2}^{P_1} PdV = P_1(V_2 - V_1) + 0 = W_b \]

Clearly \( W_a \neq W_b \)

It should be noted that a constant pressure process makes \( \delta W \) a function of state.

\[ PdV = d(PV) - VdP \]

\[ \int_{V_1}^{V_2} \delta W = \int_{V_1}^{V_2} d(PV) - \int_{V_1}^{V_2} VdP \]

\( \int_{V_1}^{V_2} d(PV) \) is a function of state while \( \int_{V_1}^{V_2} VdP \) is not a function of state. For constant \( P \)

\[ \int_{V_1}^{V_2} \delta W = \int_{V_1}^{V_2} d(PV) = PV_2^1 = P_2V_2 - P_1V_1 = P_{1 \text{ or } 2} (V_2 - V_1) \]

A function of state is one whose integral of a differential of itself recovers the original function, for example

\[ \int dU = U; \int dP = P; \int d(PV) = PV \]

**First law as a rate equation**

Apply the first law to the system shown over time interval \( \Delta t \)

\[ \Delta Q = \Delta W + \Delta E \quad (2.4-5) \]

Divide the above equation by \( \Delta t \)

\[ \frac{\Delta Q}{\Delta t} = \frac{\Delta W}{\Delta t} + \frac{\Delta E}{\Delta t} \quad (2.4-6) \]
We are departing from the classical thermodynamic view that deals with equilibrium because time is not a relevant parameter for equilibrium systems. Take the limit of Eq. (2.4-6) as \( \Delta t \to 0 \)

\[
\frac{\Delta Q}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta E}{\Delta t}
\]

\[
\frac{\delta Q}{\delta t} = \frac{\delta W}{\delta t} + \frac{dE}{dt} \tag{2.4-7}
\]

\[
q = \dot{W} + \frac{dE}{dt} \tag{2.4-8}
\]

where \( q = \) net heat input plus heat generated (W)

\[
q = q_{in} - q_{out} + q_{gen} \tag{2.4-9}
\]

\( \dot{W} = \) net work output (W)

\( \frac{dE}{dt} = \) accumulated energy change (W)

A control volume must be defined to apply Eqs. (2.4-8) and (2.4-9).

**Example 2.4-2**

A metallic wire of electrical resistance \( \Omega \), diameter \( D \), and length \( L \) is heated by passing an electrical current \( i \) through the wire to induce a uniform heat generation \( q_{gen} \). The ambient air around the wire is at a temperature \( T_\infty \) with an average heat transfer coefficient \( h \). Assuming the wire temperature is uniform, obtain an expression for the unsteady-state wire temperature \( T_w(t) \).

For \( \Omega = 5 \times 10^{-3} \, \Omega/m, D = 0.020 \, m, L = 0.50 \, m, i = 100 \, A, T_\infty = 25^\circ C, h = 25 \, W/m^2 \cdot K, \) wire density \( \rho = 8900 \, kg/m^3, \) and wire specific heat \( c_p = 380 \, J/kg \cdot K, \) plot \( T_w(t) \).

**Solution**

The control volume is the wire with diameter \( D \) and length \( L \). The internal electrical energy generation within the wire is

\[
q_{gen} = i^2 \Omega L
\]

Apply the first law with no work to the wire,
\[ \frac{dE}{dt} = q_{in} - q_{out} + q_{\text{gen}} \]

\[ q_{in} = 0, \quad q_{out} = \bar{h} \pi DL(T_w - T_{\infty}) \]

Neglecting kinetic and potential energies

\[ dE = dU = mc_p dT_w = \frac{\pi D^2}{4} L \rho c_p dT_w \]

\[ \frac{dE}{dt} = \frac{\pi D^2}{4} L \rho c_p \frac{dT_w}{dt} \]

The first law becomes

\[ \frac{\pi D^2}{4} L \rho c_p \frac{dT_w}{dt} = - \bar{h} \pi DL(T_w - T_{\infty}) + i^2 \Omega L \]

This equation can be integrated using the initial condition at \( t = 0, T_w = T_{\infty} \) to obtain

\[ T_w = T_{\infty} + \frac{i^2 \Omega}{\bar{h} \pi D} \left( 1 - \exp \left( - \frac{4 \bar{h}}{\rho c_p D} t \right) \right) \]

\[ \frac{i^2 \Omega}{\bar{h} \pi D} = \frac{100^2 \times 5 \times 10^{-3}}{25 \times \pi \times 0.02} = 31.83^\circ C \]

\[ \frac{4 \bar{h}}{\rho c_p D} = \frac{4 \times 25}{8900 \times 380 \times 0.02} = 1.478 \times 10^3 \text{s}^{-1} = 5.32 \text{ hr}^{-1} \]

\[ T_w = 25^\circ C + 31.83^\circ C \left( 1 - \exp(-5.32t) \right) \]
**Example 2.4-3**

A large slab with thermal conductivity $k$ and thickness $L$ is maintained at temperatures $T_1$ and $T_2$ at the two surfaces. Determine the heat flux through this material at steady-state condition.

![Diagram of a slab with temperatures T1 and T2]

**Solution**

The $x$-coordinate is assigned in the direction normal to the slab with $x = 0$ at the left surface where the temperature is $T_1$. Since the temperature varies across the slab or the $x$-direction, a differential control volume with the same cross-sectional area $A$ as that of the slab and a thickness $dx$ will be considered. An energy balance (first law) is then applied to this differential control volume

$$\frac{dE}{dt} = q_{in} - q_{out} + q_{gen}$$

For steady state with no heat generation

$$q_{in} = q_{out} \Rightarrow A q_x^+ = A q_x^- \Rightarrow \frac{q_x^+|_{x+dx} - q_x^-|_x}{dx} = 0$$

In the limit when $dx$ approaches zero

$$\frac{q_x^+|_{x+dx} - q_x^-|_x}{dx} = \frac{dq_x^+}{dx} = 0 \Rightarrow q_x^- = -k \frac{dT}{dx} = \text{constant}$$

If the thermal conductivity $k$ is a constant,

$$\frac{dT}{dx} = \frac{T_2 - T_1}{L} = \text{constant}$$

Therefore the heat flux $q_x^-$ through the slab is simply: $q_x^- = k \frac{T_1 - T_2}{L} = \text{constant}$
Chapter 2
Example 2.4-4

A horizontal copper plate is coated on the top surface such that it absorbs 90% of a solar radiation flux of 1000 W/m². The bottom surface of the plate is well insulated and the plate is thin (0.005 m thick) so that the temperature of the plate might be assumed to be uniform at any time. The plate is initially at 300°K and is suddenly exposed to ambient air at 295°K with an average heat transfer coefficient $h = 25$ W/m²·K. The emissivity of the top surface is 0.30. For copper: density $\rho = 8900$ kg/m³, specific heat $c_p = 380$ J/kg·K, plot $T_w(t)$.

a) Determine the temperature of the plate when steady-state conditions are reached.

b) Neglecting radiation loss, determine the time for the plate temperature to reach 320°K.

Solution

The control volume is the plate with area $A$ and thickness $L$. Apply the first law with no work to the plate,

$$\frac{dE}{dt} = q_{in} - q_{out} + q_{gen}$$

$q_{in} = 0.9 \times 1000$ W/m²×$A = 900A$; $q_{out} = \bar{h} \: A(T - T_{\infty}) + \varepsilon \sigma A T^4$; $q_{gen} = 0$

a) Steady-state temperature of the plate

$T$ is the uniform plate temperature and the surrounding temperature $T_{\text{sur}}$ for radiation energy exchange with the plate is assumed to be at 0°K. Neglecting kinetic and potential energies

$$dE = dU = mc_p dT = LA \rho c_p dT = 0 \text{ at steady-state}$$

The energy equation becomes

$$q_{in} = q_{out} \Rightarrow 900A = \bar{h} \: A(T - T_{\infty}) + \varepsilon \sigma A T^4$$

or

$$900 + \bar{h} \: T_{\infty} = \bar{h} \: T + \varepsilon \sigma T^4 = T(\bar{h} + \varepsilon \sigma T^3)$$
Since $\sigma = 5.67 \times 10^{-8}$ W/m$^2$-K$^4$ the above equation is rearranged to

$$T = \frac{8275}{25 + 1.701 \times 10^{-8} T^3}$$

The steady-state temperature can be solved iteratively by direct substitution with the first guess neglecting heat loss by radiation

$$T = \frac{8275}{25} = 331^\circ K$$

$$T = \frac{8275}{25 + 1.701 \times 10^{-8} \times 331^3} = 323.03^\circ K$$

$$T = \frac{8275}{25 + 1.701 \times 10^{-8} \times 323.03^3} = 323.58^\circ K$$

$$T = \frac{8275}{25 + 1.701 \times 10^{-8} \times 323.58^3} = 323.54^\circ K$$

b) Time for the plate temperature to reach $320^\circ$K

The unsteady-state energy balance is

$$\frac{dE}{dt} = q_{in} - q_{out} + q_{gen}$$

Neglecting heat loss by radiation

$$LA\rho c_p \frac{dT}{dt} = 900A - hA(T - T_\infty)$$

$$0.005 \times 8900 \times 380 \frac{dT}{dt} = 900 - 25(T - 295) = 8275 - 25T$$

$$t = \int_0^T dt = 16910 \int_{300}^T \frac{dT}{8275 - 25T} = 676.4 \int_{300}^T \frac{dT}{331 - T} = 676.4 \times \ln \left( \frac{331 - 300}{331 - T} \right)$$

when $T = 320^\circ$K, $t = 701$ s
2.5 Engineering Application: Temperature of a Focal Plane Array

2.5 A) Background Information

Today, many of the everyday conveniences we are accustomed to depend on satellites. These satellites are responsible for communication, weather prediction, research and defense. In order to function properly, electronic components on satellites must operate within a strictly defined temperature range. This is especially true of infrared focal plane arrays (IRFPA). IRFPA’S are an arrangement of Infrared sensors arranged in rows on a plane, hence the name. The following diagram (Fig. 2.5-1) depicts an actual IRFPA.

![Figure 2.5-1: Focal plane array](image)

Infrared focal plane arrays are used to obtain IR signatures given off by differences in temperature observed on or above the earth’s surface. If the operating temperature is exceeded, the infrared sensors cannot effectively differentiate between the source of the temperature difference and its surroundings. Assuming the proper operational temperature is maintained, the IR signature is converted to an electronic signal that is then amplified and processed, resulting in useful data. This is depicted in the following schematic (Fig. 2.5-2).

![Figure 2.5-2 Operational principle of focal plane array.](image)

IRFPA’s are used for weather prediction and defense purposes primarily. Northrop Grumman, the exclusive contractor for the IR components on defense satellites is involved in developing a passive cooling system for satellites that will be cost effective, light weight and have all the necessary properties to operate effectively in outer-space.

2.5 B) Temperature of a Focal plane Array

Since the temperature of the focal plane array is a critical factor in its operation, we want to predict its temperature during various conditions. For a surface in the high orbit above the earth,
the solar energy flux arrives at the surface is about 1350 W/m² if the sun ray is normal to surface as shown in Figure 2.5-3(A).

![Figure 2.5-3 Orientation of the surface with respect to sunlight.](image)

During normal operation, the focal plane array is oriented so that its surface makes an angle $\theta$ of 3 degree with the light ray as shown in Figure 2.5-3(B). In general a plane can be oriented with two different angles with respect to the $xy$-surface as shown in Figure 2.5-3(C). If the focal plane is only rotated about the $x$ or $y$-axis with an angle $\theta$, the energy flux received by the focal plane array is $(1350 \times \sin \theta)$. However not all the energy arrived at the surface will be absorbed by the plane as shown in Figure 2.5-4.

![Figure 2.5-4 Fate of radiation incident upon a surface.](image)

The amount of energy absorbed ($q_{\text{absorbed}}$) is given by

$$q_{\text{absorbed}} = A\alpha(1350 \times \sin \theta) \quad (2.5-1)$$

In this equation $A$ is the surface area of the plane array and $\alpha$ is the absorptivity for solar radiation. In general the absorptivity depends on the temperature, the wavelength, and the surface finish of the materials. The plane itself radiates energy to the surrounding, which is mostly dark empty space, by the following equation

$$q = A\varepsilon\sigma(T_{s}^{4} - T_{\text{sur}}^{4}) \quad (2.5-2)$$

In this equation $\sigma = 5.67 \times 10^{-8}$ W/m²-K⁴ is Stefan-Boltzmann constant, $T_{\text{sur}}$ is the surrounding background temperature ($2.73^\circ$K), and $\varepsilon$ is the "emissivity" of the real surface with value between 0 and 1: $0 \leq \varepsilon \leq 1$. Similarly to the absorptivity, the emissivity depends on the
temperature, the wavelength, and the surface finish of the materials. At steady state operation, the energy absorbed by the focal plane array is equal to the energy emitted

\[ q_{\text{absorbed}} = q \Rightarrow A \alpha (1350 \sin \theta) = A \varepsilon \sigma (T_s^4 - T_{\text{sur}}^4) \]

The temperature of the focal plane array \( (T_s) \) is then given by

\[ T_s = \left( \frac{1350 \alpha \sin \theta}{\varepsilon \sigma} + T_{\text{sur}}^4 \right)^{1/4} \]

Figure 2.5-5 shows the effects of the absorptivity and emissivity on the temperature of the focal plane array. The temperature increases with absorptivity and decreases with emissivity where the lowest curve has the highest emissivity.

![Dependence of surface temperature on absorptivity and emissivity.](image)

For normal operation of the focal plane array, its surface temperature should be less than 150\(^o\)K. The solar radiation arrived at the surface of the plane array is in the visible range that has different wavelength than the infrared radiation leaving the surface (Figure 2.5-6). Therefore it is possible to manufacture coating materials with low solar absorptivity and high infrared emissivity. The coating material can be applied to the surface of the plane array to keep its temperature below 150\(^o\)K. A typical material developed by Northrop Grumman possesses a solar
absorptivity of about 0.15 and an infrared emissivity of about 0.9. This coating may be used as a passive cooling system if it can last in the high earth orbit for the duration of the satellites’ life.

We now want to determine the temperature of the focal plane array when it at a random orientation toward the sun. This happens during the initial period when the satellite is first in orbit and twice per year at the equinox when the sun crosses the equator. We need to determine randomly the angles the plane array makes with the $y$-axis ($y$-angle) and with the $x$-axis ($x$-angle) as show in Figure 2.5-7. We arbitrarily assign a value of less than $\pi$ for both the $x$-angle and $y$-angle in order for the plane array to face the sun.

When the plane array faces the sun it will receive solar energy that will cause a rise in temperature. The following Matlab program may be used to predict the temperature during the period the plane array is at a random orientation toward the sun. An emissivity of 0.9 is assumed

---

Footnote:

for the calculation. The temperature for the normal operation is also calculated for comparison. Figure 2.5-8 shows the results from the calculation.

% Temperature at a random orientation and at normal operational orientation
theta=3*pi/180;Tsur=2.73;
flux=1350;con=5.67e-8;
ntry=1000;ein=0;nhit=0;
absor=.1:.02:1;
for i=1:ntry;
    xangle=rand;
    yangle=rand;
    if yangle<.5 & xangle<.5,
        ein=flux*sin(xangle*2*pi)*sin(yangle*2*pi) + ein;
        nhit=nhit+1;
    end
end
ein=ein/ntry;
emiss=.9;
Ts=(ein*absor/(con*emiss)+Tsur^4).^.25;
Ts3=(flux*sin(theta)*absor/(con*emiss)+Tsur^4).^.25;
plot(absor,Ts,absor,Ts3,'--');grid on
xlabel('Absorptivity');ylabel('Temperature (K)')
legend('Random position','3 degree')
fprintf('nhit = %g
',nhit)
>> arrayr
nhit = 215

--------------------------------------------------------------------------------------------------
Figure 2.5-8 Temperature of the plane array at random and 3 degree orientations.
Chapter 2

2.6 Examples in Conduction, Convection, and Radiations

Example 2.6-1

A vacuum system, as used in sputtering electrically conducting thin films on microcircuits, is comprised of a baseplate maintained by an electrical heater at 300°K and a shroud within the enclosure maintained at 77°K by a liquid nitrogen coolant loop. The baseplate, insulated on the lower side, is 0.3 m in diameter and has an emissivity of 0.25.

(a) How much electrical power must be provided to the baseplate heater?

(b) At what rate must liquid nitrogen be supplied to the shroud if its heat of vaporization is 125 kJ/kg?

(c) To reduce the liquid nitrogen consumption, it is proposed to bond a thin sheet of aluminum foil (ε = 0.09) to the baseplate. Will this have the desired effect?

Solution

(a) How much electrical power must be provided to the baseplate heater?

At steady state, the power supplied to the baseplate heater, $\dot{E}_{in}$, is equal to rate of radiation energy transferred to the shroud, $\dot{E}_{out}$:

$$\dot{E}_{in} = \dot{E}_{out} = \varepsilon\sigma A_p(T_p^4 - T_{sh}^4) = (0.25)(5.67\times10^{-8})(\pi \times 0.3^2/4)(300^4 - 77^4) = 8.12 \text{ W}$$

(b) At what rate must liquid nitrogen be supplied to the shroud if its heat of vaporization is 125 kJ/kg?

The heat transfer to the shroud is removed by liquid nitrogen through the latent heat, $h_{fg}$, therefore the rate of liquid nitrogen, $\dot{m}$, supplied to the shroud is given by

$$\dot{m} = \frac{\dot{E}_{in}}{h_{fg}} = \frac{8.12}{125} = 0.0649 \text{ g/s} = 0.234 \text{ kg/hr}$$

---

5 Fundamentals of Heat Transfer by Incropera and DeWitt.
(c) To reduce the liquid nitrogen consumption, it is proposed to bond a thin sheet of aluminum foil ($\varepsilon = 0.09$) to the baseplate. Will this have the desired effect?

The rate of heat transfer by radiation to the shroud is then

$$\dot{E}_{\text{out}} = 0.09 \sigma A_p (T_p^4 - T_{\text{sh}}^4) = (0.09 / 0.25)(8.12) = 2.92 \text{ W}$$

This will reduce liquid nitrogen consumption.

**Example 2.6-2**

In the thermal processing of semiconductor materials, annealing is accomplished by heating a silicon wafer according to a temperature-time recipe and then maintaining a fixed elevated temperature for a prescribed period of time. For the process tool arrangement shown as follows, the wafer is in an evacuated chamber whose walls are maintained at 27°C and within which heating lamps maintain a radiant flux $q_r$ at its upper surface. The wafer is 0.78 mm thick, has a thermal conductivity of 30 W/m-K, and an emissivity that equals its absorptivity to the radiant flux ($\varepsilon = \alpha_l = 0.65$). For $q_r = 3.0 \times 10^5 \text{ W/m}^2$, the temperature on its lower surface is measured by a radiation thermometer and found to have a value of $T_{w,l} = 997°C$. To avoid warping the wafer and inducing slip planes in the crystal structure, the temperature difference across the thickness of the wafer must be less than 2°C. Is this condition being met?

**SCHEMATIC:**

Perform an energy balance on the upper surface of the wafer we have

$$q_{\text{in}} = q_{\text{out}}$$

$$0.65(3.0 \times 10^5) = \varepsilon \sigma (T_{w,u}^4 - 300^4) + \frac{k}{L} \left[ T_{w,u} - (997 + 273) \right]$$

$$0.65(3.0 \times 10^5) = (0.65)(5.67 \times 10^{-8})(T_{w,u}^4 - 300^4) + (30 / 7.8 \times 10^{-4})(T_{w,u} - 1270)$$

$$T_{w,u}^4 - 300^4 + 1.0436 \times 10^{12}(T_{w,u} - 1270) = 5.291 \times 10^{12} = 0$$

---

6 Fundamentals of Heat Transfer by Incropera and DeWitt.
The nonlinear algebraic equation can be solved by the following Matlab statements:

```matlab
>> fu=('x^4-300^4+1.0436e12*(x-1270)-5.2910e12');
>> Twu = fsolve(fu,1000,optimset('Display','off'))
```

\[ \text{Twu} = 1.2726e+003 \]

Therefore \( T_{w,u} = 1272.6^\circ\text{K} = 999.6^\circ\text{C} \)

The difference in temperature is \( \Delta T = 999.6^\circ\text{C} - 997^\circ\text{C} = 2.6^\circ\text{C} \)

Since \( \Delta T > 2^\circ\text{C} \), the condition is not met.

**Example 2.6-3**

A spherical stainless steel (AISI 302) canister is used to store reacting chemicals that provide for a uniform heat flux \( q_i^* \) to its inner surface. The canister is suddenly submerged in a liquid bath of temperature \( T_\infty < T_i \), where \( T_i \) is the initial temperature of the canister wall.

(a) Assuming negligible temperature gradient in the canister wall and a constant heat flux \( q_i^* \), develop an equation that governs the variation of the wall temperature with time during the transient process. What is the initial rate of change of the wall temperature if \( q_i^* = 10^5 \text{ W/m}^2 \)?

(b) What is the steady-state temperature of the wall?

![Diagram of a spherical canister with heat fluxes and temperatures labeled](image)

**Solution**

(a) Performing an energy balance on the spherical stainless steel canister we have

\[
\frac{dE}{dt} = q_{in} - q_{out}
\]

\[
\frac{4}{3} \pi (r_o^3 - r_i^3) \rho C_p \frac{dT}{dt} = 4 \pi r_i^2 q_i^* - 4 \pi r_o^2 h(T - T_\infty)
\]

---

7 *Fundamentals of Heat Transfer* by Incropera and DeWitt.
Solving for \( \frac{dT}{dt} \) gives

\[
\frac{dT}{dt} = \frac{3}{\rho C_p \left( r_i^3 - r_o^3 \right)} \left[ r_i^2 q_i'' - r_o^2 h(T - T_\infty) \right]
\]

For stainless steel (AISI 302), \( \rho = 8055 \text{ kg/m}^3 \), \( C_p = 510 \text{ J/kg} \cdot ^\circ\text{C} \), the initial rate of change of the wall temperature is

\[
\left. \frac{dT}{dt} \right|_{t=0} = \frac{3 \left[ 0.5^2 \times 10^5 - 0.6^2 \times 500 \times (500 - 300) \right]}{(8055)(510)(0.6^3 - 0.5^3)} = -0.088 \text{ } ^\circ\text{C/s}
\]

(b) What is the steady-state temperature of the wall?

At steady state \( q_{\text{in}} = q_{\text{out}} \)

\[
4\pi r_i^2 q_i'' - 4\pi r_o^2 h(T - T_\infty)
\]

\[
T = T_\infty + \frac{q_i''}{h \left( r_i^2 - r_o^2 \right)} = 300 + \frac{10^5 \left( \frac{0.5}{0.6} \right)^2}{500} = 439 \text{ } ^\circ\text{C}
\]

Example 2.6-4

Electronic power devices are mounted to a heat sink having an exposed surface area of 0.045 m\(^2\) and an emissivity of 0.80. When the devices dissipate a total power of 20 W and the air and surroundings are at 27°C, the average sink temperature is 42°C. What average temperature will the heat sink reach when the devices dissipate 30 W for the same environmental condition?

\[\text{Solution}\]

Perform an energy balance on the heat sink we have

\[
q_{\text{in}} = q_{\text{out}}
\]

\[
20 = A \varepsilon \sigma (T_s^4 - T_{\text{sur}}^4) + Ah(T_s - T_\infty)
\]

\[
20 = (0.045)(0.8)(5.67 \times 10^{-8})(315^4 - 300^4) + 0.045h(42 - 27)
\]

\[\text{8 Fundamentals of Heat Transfer by Incropera and DeWitt.}\]
20 = 0.675h + 3.563 \Rightarrow h = 24.4 \text{ W/m}^2\cdot \text{K}

When the devices dissipate 30 W for the same environmental condition we have

\[ 30 = A\varepsilon\sigma(T_s^4 - T_{\text{sur}}^4) + Ah(T_s - T_{\infty}) \]

\[ 30 = (0.045)(0.8)(5.67 \times 10^{-8})(T_s^4 - 300^4) + (0.045)(24.4)(T_s - 300) \]

\[ 2.041 \times 10^{-9}(T_s^4 - 300^4) + 1.098(T_s - 300) - 30 = 0 \]

The nonlinear algebraic equation can be solved by the following Matlab statements:

\[
\begin{align*}
>> & \text{fu=}'2.041e-9*(x^4-300^4)+1.098*(x-300)-30';
>> & \text{Ts} = \text{fsolve(fu,300,optimset('Display','off'))}
\end{align*}
\]

\[ \text{Ts} = 322.3170 \]

The average temperature of the heat sink is \( 322.3^\circ\text{K} = 49^\circ\text{C} \)

**Example 2.6-5**

A computer consists of an array of five printed circuit boards (PCBs), each dissipating \( P_b = 20 \text{ W} \) power. Cooling of the electronic components on a board is provided by the forced flow of air, equally distributed in passages formed by adjoining boards, and the convection coefficient associated with heat transfer from the components to the air is approximately \( h = 200 \text{ W/m}^2\cdot\text{K} \). Air enters the computer console at a temperature of \( T_i = 20^\circ\text{C} \), and flow is driven by a fan whose power consumption is \( P_f = 25 \text{ W} \).

(a) If the temperature rise of the air flow, \( (T_o - T_i) \), is not to exceed \( 15^\circ\text{C} \), what is the minimum allowable volumetric flow rate of the air? The density and specific heat of the air may be approximate as \( \rho = 1.161 \text{ kg/m}^3 \) and \( C_p = 1007 \text{ J/kg}\cdot\text{K} \), respectively.

(b) The component that is most susceptible to thermal failure dissipates 1 W/cm\(^2\) of surface area. To minimize the potential for thermal failure, where should the component be installed on a PCB? What is its surface temperature at this location?
Solution

(a) If the temperature rise of the air flow, \((T_o - T_i)\), is not to exceed 15°C, what is the minimum allowable volumetric flow rate of the air?

\[
\dot{m} \ C_p (T_o - T_i) = 5P_b + P_f \Rightarrow \dot{m} = \frac{5P_b + P_f}{C_p (T_o - T_i)}
\]

\[
\dot{m} = \frac{5 \times 20 + 25}{(1007)(15)} = 8.28 \times 10^{-3} \text{ kg/s}
\]

The minimum allowable volumetric flow rate, \(\dot{\nu}\), of the air is then

\[
\dot{\nu} = \frac{\dot{m}}{\rho} = 8.28 \times 10^{-3}/1.161 = 7.13 \times 10^{-3} \text{ m}^3/\text{s}
\]

(b) The component that is most susceptible to thermal failure dissipates 1 W/cm² of surface area. To minimize the potential for thermal failure, where should the component be installed on a PCB? What is its surface temperature at this location?

The component should be mounted at the bottom of one of the PCBs where the air is coolest.

\[
q'' = h(T_s - T_i)
\]

\[
T_s = T_i + \frac{q''}{h} = 20 + \frac{10^4 \text{ W/m}^2 \cdot \text{s}}{200 \text{ W/m}^2 \cdot \text{K}} = 70°C
\]
Chapter 3

Conduction Differential Equation

3.1 Derivation

This section presents the derivation of the differential equation that must be solved to determine the temperature distribution in a solid. Then, the heat transfer rate can be determined. The differential equation can be derived by applying an energy balance on a "tiny" or "differential" element.

Apply the first law to a 3-D control volume in Cartesian coordinates with the following assumptions:

- No movable surfaces ($\dot{W} = 0$)
- Negligible kinetic and potential energies
- Isotropic solid ($k_x = k_y = k_z = k$)
- $C_p = C_v$ (specific heat at constant $P$ and $V$ equal)
- Properties independent of temperature

$\frac{dE}{dt} = V\rho c_p \frac{dT}{dt} = q_{in} - q_{out} + q_{gen}$

where $V =$ volume of the control volume

Since $T(x, y, x, t)$ is a function of more than one independent variable, the derivative $\frac{dT}{dt}$ is a total derivative with respect to time which can be obtained from the chain rule of calculus as follow
\[dT = \left(\frac{\partial T}{\partial x}\right)_{x,y,z} \, dx + \left(\frac{\partial T}{\partial y}\right)_{x,y,z} \, dy + \left(\frac{\partial T}{\partial z}\right)_{x,y,z} \, dz + \frac{\partial T}{\partial t}\bigg|_{x,y,z} \, dt\]

Divide the above equation by \(dt\)

\[\frac{dT}{dt} = \left(\frac{\partial T}{\partial x}\right)_{x,y,z} \frac{dx}{dt} + \left(\frac{\partial T}{\partial y}\right)_{x,y,z} \frac{dy}{dt} + \left(\frac{\partial T}{\partial z}\right)_{x,y,z} \frac{dz}{dt} + \frac{\partial T}{\partial t}\bigg|_{x,y,z}\]

where

\[
\frac{dx}{dt} = v_x = \text{velocity in the x-direction}
\]

\[
\frac{dy}{dt} = v_y = \text{velocity in the y-direction}
\]

\[
\frac{dz}{dt} = v_z = \text{velocity in the z-direction}
\]

\[
\frac{dT}{dt} = v_x \left(\frac{\partial T}{\partial x}\right)_{x,y,z} + v_y \left(\frac{\partial T}{\partial y}\right)_{x,y,z} + v_z \left(\frac{\partial T}{\partial z}\right)_{x,y,z} + \frac{\partial T}{\partial t}\bigg|_{x,y,z} = \mathbf{v} \cdot \nabla T + \frac{\partial T}{\partial t}
\]

For a solid \(v_x = v_y = v_z = 0; \nabla \mathbf{v} = 0\)

\[
\frac{dT}{dt} = \frac{\partial T}{\partial t}
\]

The energy balance becomes

\[V \rho c_p \frac{\partial T}{\partial t} = q = q_{in} - q_{out} + q_{gen}\]

The control volume is defined as \(V = \Delta x \cdot \Delta y \cdot \Delta z\), so

\[
\rho c_p \Delta x \cdot \Delta y \cdot \Delta z \frac{\partial T}{\partial t} = q_{in} - q_{out} + q_{gen}
\]

\[
q_{gen} = q'' \cdot V = q'' \cdot \Delta x \cdot \Delta y \cdot \Delta z
\]

\[
q_{in} - q_{out} + q_{gen} = \left[q_x + q_y + q_z\right] - \left[q_{x+\Delta x} + q_{y+\Delta y} + q_{z+\Delta z}\right] + q'' \cdot \Delta x \cdot \Delta y \cdot \Delta z
\]

The energy equation becomes
\[ \rho c_p \Delta t \Delta x \Delta y \Delta z \frac{\partial T}{\partial t} = [q_x - q_{x+\Delta x}] + [q_y - q_{y+\Delta y}] + [q_z - q_{z+\Delta z}] + q'' \cdot \Delta x \Delta y \Delta z \]

Divide the equation by \( \Delta x \Delta y \Delta z \) and take the limit as \( \Delta x \Delta y \Delta z \to 0 \)

\[
\begin{align*}
\lim_{\Delta x \to 0} & \lim_{\Delta y \to 0} \lim_{\Delta z \to 0} \\
& = \text{Point (Volume is } dxdydz) \\
\end{align*}
\]

In the limiting process of making \( \Delta x, \Delta y, \Delta z \to 0 \)

\( \Delta x \to dx \)
\( \Delta y \to dy \)
\( \Delta z \to dz \)

\[
\begin{align*}
\lim_{\Delta x \to 0} & \lim_{\Delta y \to 0} \lim_{\Delta z \to 0} \\
\rho c_p & \frac{\partial T}{\partial t} = \frac{q_x - q_{x+\Delta x}}{\Delta x} \frac{1}{\Delta y \Delta z} + \frac{q_y - q_{y+\Delta y}}{\Delta y \Delta z} \frac{1}{\Delta x \Delta z} + \frac{q_z - q_{z+\Delta z}}{\Delta z \Delta x} \frac{1}{\Delta x \Delta y} + q'' \\
\end{align*}
\]

This limit process produces partial derivatives

\[
\begin{align*}
\lim_{\Delta x \to 0} \left[ \frac{q_x - q_{x+\Delta x}}{\Delta x} \right] & = -\frac{\partial q_x}{\partial x} \\
\lim_{\Delta y \to 0} \left[ \frac{q_y - q_{y+\Delta y}}{\Delta y} \right] & = -\frac{\partial q_y}{\partial y} \\
\lim_{\Delta z \to 0} \left[ \frac{q_z - q_{z+\Delta z}}{\Delta z} \right] & = -\frac{\partial q_z}{\partial z} \\
\end{align*}
\]

From Fourier's law

\[ q_x = -k dydz \frac{\partial T}{\partial x} \]
Similarly
\[
\lim_{\Delta y \to 0} \left[ \frac{q_y - q_{y+\Delta y}}{\Delta y} \frac{1}{\Delta x \Delta z} \right] = \frac{\partial}{\partial y} \left( k \frac{\partial T}{\partial y} \right)
\]

\[
\lim_{\Delta z \to 0} \left[ \frac{q_z - q_{z+\Delta z}}{\Delta z} \frac{1}{\Delta x \Delta y} \right] = \frac{\partial}{\partial z} \left( k \frac{\partial T}{\partial z} \right)
\]

Therefore the energy equation for heat conduction in a solid becomes

\[
\rho c_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left( k \frac{\partial T}{\partial z} \right) + q''
\]

The gradient of a scalar \( s \) is defined as a vector in the direction in which \( s \) increases most rapidly with distance. The gradient operator in the rectangular Cartesian coordinate system is given as

\[
\nabla = \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k}
\]

where \( \mathbf{i}, \mathbf{j}, \) and \( \mathbf{k} \) are the unit vectors in the x, y, and z-direction respectively. In terms of the gradient operator, the conduction differential equation becomes,

\[
\rho c_p \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) + q''
\]

The gradient operator can be derived for any coordinate system: rectangular, cylindrical, spherical, bispherical, or other [1].
The temperature gradient in cylindrical coordinates is

\[ \nabla T = \mathbf{i} \frac{\partial T}{\partial r} + \mathbf{j} \frac{1}{r} \frac{\partial T}{\partial \phi} + \mathbf{k} \frac{\partial T}{\partial z} \]

The gradient denotes a change with respect to distance. The length increment in the \( \phi \) direction is \( r \partial \phi \) not \( \partial \phi \). Similarly, the temperature gradient in spherical coordinates is

\[ \nabla T = \mathbf{i} \frac{\partial T}{\partial r} + \mathbf{j} \frac{1}{r} \frac{\partial T}{\partial \theta} + \mathbf{k} \frac{1}{r \sin(\theta)} \frac{\partial T}{\partial \phi} \]

It should be noted that the length increment in the \( \phi \) direction for spherical coordinates is \( r \sin(\theta) \partial \phi \).
Special cases of the heat conduction equation

\[ \rho c_p \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) + q'' \]

- \( k \) is independent of \( T \)

\[ \nabla \cdot (k \nabla T) = k \nabla \cdot \nabla T = k \nabla^2 T \]

\[ \rho c_p \frac{\partial T}{\partial t} = k \nabla^2 T + q'' \]

- steady state and \( k \) is independent of \( T \)

\[ \nabla^2 T + \frac{q''}{k} = 0 \]

- steady state, no heat generation, and \( k \) is independent of \( T \)

\[ \nabla^2 T = 0 \]

The general heat conduction equation \( \rho c_p \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) + q'' \) can be simplified for each particular case as shown in the following example.

**Example 3.1-1**

Derive a differential equation for temperature for the one dimensional heat transfer in rectangular coordinate system. The system is at steady state with no heat generation, however the thermal conductivity is dependent on temperature.

**Solution**

The general heat conduction equation is

\[ \rho c_p \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) + q'' \]

Since \( \rho c_p \frac{\partial T}{\partial t} = 0 \) for steady state and \( q'' = 0 \) for no heat generation, the heat conduction equation becomes

\[ \nabla \cdot (k \nabla T) = 0 \]

For 1-dimensional heat transfer in Cartesian coordinate
\[
\mathbf{i} \left( \frac{d}{dx} \right) \left( k \frac{dT}{dx} \right) \mathbf{i} = \frac{d}{dx} \left( k \frac{dT}{dx} \right) \mathbf{i} \mathbf{i} = \frac{d}{dx} \left( k \frac{dT}{dx} \right) = 0
\]

Apply the product rule of differentiation

\[
\frac{d}{dx} \left( k \frac{dT}{dx} \right) = k \frac{d^2T}{dx^2} + \frac{dk}{dx} \frac{dT}{dx} = 0
\]

From the chain rule \( dk = \frac{dk}{dT} dT \)

\[
\frac{dk}{dx} = \frac{dk}{dT} \frac{dT}{dx}
\]

The 1-dimensional heat transfer equation becomes

\[
k \frac{d^2T}{dx^2} + \frac{dk}{dT} \left( \frac{dT}{dx} \right)^2 = 0
\]

The above equation can be solved for \( T(x) \) once \( k(x) \) and the boundary conditions are given.

The calculation of 1-dimensional heat transfer rate \( q_x \) depends on whether \( T(x) \) is known or unknown once the thermal conductivity \( k(T) \) is given as a function of temperature.

- If \( T(x) \) is known, \( q_x \) can be determined from Fourier's law directly,

  \[
  q_x = -kA \frac{dT}{dx}
  \]

- If \( T(x) \) is unknown or difficult to evaluate, \( q_x \) can be determined from the direct integration of Fourier's law,

  \[
  q_x = -kA \frac{dT}{dx}
  \]

  \[
  \int_{r_1}^{r_2} \frac{q_x}{A} \, dx = -\int_{r_1}^{r_2} k \, dT
  \]

For the special case of steady state with no heat generation, \( q_x \) is a constant and can be moved out of the integral sign.
\[ q_x \int_{x_1}^{x_2} \frac{1}{A} \, dx = - \int_{T_1}^{T_2} k(T) \, dT \]

\[ q_x = - \frac{\int_{T_1}^{T_2} k(T) \, dT}{\int_{x_1}^{x_2} \frac{1}{A} \, dx} \]

The fact that \( q_x \) is a constant can be derived from an energy balance around a differential control volume \( A \, dx \),

\[ \frac{dE}{dt} = q_{in} - q_{out} + q_{gen} \]

For steady state with no heat generation

\[ q_{in} = q_{out} \]

\[ q_x \bigg|_x = q_x \bigg|_{x+dx} \]

\[ \frac{q_x \bigg|_{x+dx} - q_x \bigg|_x}{dx} = 0 \]

In the limit when \( dx \) approaches zero

\[ \frac{q_x \bigg|_{x+dx} - q_x \bigg|_x}{dx} = \frac{dq_x}{dx} = 0 \Rightarrow q_x = -kA \frac{dT}{dx} = \text{constant} \]
Chapter 3

Example 3.1-2

The temperature distribution across a wall 0.40 m thick at a certain instant of time is given as

\[ T(x) = 300 + 100x + 150x^2 \]

where \( T \) is in degree Celsius and \( x \) is the distance in meter from the left side of the wall. Heat is generated uniformly in the wall at 2000 W/m\(^3\). The thermal conductivity of the wall is a function of temperature,

\[ k(T) = 20 + 0.02T \]

where \( T \) is in degree Celsius and \( k \) is in W/m·K. Properties of the wall are known: density \( \rho = 1900 \text{ kg/m}^3 \), specific heat \( c_p = 8 \text{ J/kg·K} \).

1) Determine the rate of heat transfer at \( x = 0 \) and \( x = 0.40 \text{ m} \).

2) Determine the rate of change of energy accumulated in the wall.

3) Determine the rate of temperature change at \( x = 0 \) and \( x = 0.40 \text{ m} \).

Solution

1) Since \( T(x) \) is known, the heat transfer rate can be determined directly from Fourier's law,

\[ q_x = - kA \frac{dT}{dx} = - kA(100 + 300x) \]

At \( x = 0 \), \( T = 300^\circ\text{C} \), \( k = 26 \text{ W/m·K} \)

\[ q_x(x = 0) = -26 \text{ W/m·K} \times 1 \text{ m}^2 \times 100^\circ\text{C/m} = -2600 \text{ W} \]

Heat is leaving the wall at \( x = 0 \).

At \( x = 0.40 \text{ m} \), \( T = 364^\circ\text{C} \), \( k = 27.28 \text{ W/m·K} \)

\[ q_x(x = 0.40) = -27.28 \text{ W/m·K} \times 1 \text{ m}^2 \times 100^\circ\text{C/m} = -2728 \text{ W} \]
\[ q_s(x = 0.40 \text{ m}) = -27.28 \text{ W/m-K} \times 1 \text{ m}^2 \times 220 \text{ °C/m} = -6002 \text{ W} \]

Heat is entering the wall at \( x = 0.40 \text{ m} \)

2) The rate of change of energy accumulated in the wall may be determined from the energy with the system as the wall,

\[
\frac{dE}{dt} = q_{in} - q_{out} + q_{gen}
\]

where

\[ q_{in} = 6002 \text{ W}; q_{out} = 2600 \text{ W}; \text{ and } \]

\[ q_{gen} = AL q'' = 1 \text{ m}^2 \times 0.40 \text{ m} \times 2000 \text{ W/m}^3 = 800 \text{ W} \]

The rate of change of accumulated energy is

\[
\frac{dE}{dt} = 6002 \text{ W} - 2600 \text{ W} + 800 \text{ W} = 4202 \text{ W}
\]

3) The rate of temperature change at any position in the wall can be determined from the heat conduction equation

\[
\rho c_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right) + q''
\]

\[
\frac{\partial}{\partial t} \left( k \frac{\partial T}{\partial x} \right) = \frac{\partial}{\partial x} \left( k \frac{\partial^2 T}{\partial x^2} \right) + \frac{\partial k}{\partial T} \left( \frac{\partial T}{\partial x} \right)^2
\]

\[
T(x) = 300 + 100x + 150x^2 \Rightarrow \frac{\partial T}{\partial x} = 100 + 300x \Rightarrow \frac{\partial^2 T}{\partial x^2} = 300 \text{ °C/m}^2
\]

\[ k(T) = 20 + 0.02T \Rightarrow \frac{\partial k}{\partial T} = 0.02 \text{ W/m} \]

At \( x = 0, T = 300\text{ °C}, k = 26 \text{ W/m-K}, \rho c_p = 15,200 \text{ J/m}^3\cdot\text{K} \]

\[
\frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right) = 26 \text{ W/m-K} \times 300 \text{ °C/m}^2 + 0.02 \text{ W/m-K}^2 \times (100 \text{ °C/m})^2 = 8,000 \text{ W/m}^3
\]

\[
\frac{\partial T}{\partial t} = \frac{8,000 \text{ W/m}^3}{15,200 \text{ J/m}^3\cdot\text{K}} + \frac{2,000 \text{ W/m}^3}{15,200 \text{ J/m}^3\cdot\text{K}} = 0.658 \text{ °C/s}
\]
At \( x = 0.40 \) m, \( T = 364^\circ C \), \( k = 27.28 \) W/m·K \( \rho c_p = 15,200 \) J/m\(^3\)·K

\[
\frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right) = 27.28 \text{ W/m·K} \times 300 \text{ °C/m}^2 + 0.02 \text{ W/m·K}^2 \times (220 \text{ °C/m})^2 = 9,152 \text{ W/m}^3
\]

\[
\frac{\partial T}{\partial t} = \frac{9,152 \text{ W/m}^3}{15,200 \text{ J/m}^3\cdot K} + \frac{2,000 \text{ W/m}^3}{15,200 \text{ J/m}^3\cdot K} = 0.734 \text{ °C/s}
\]

**Example 3.1-3**

The quarter cylindrical system shown has negligible temperature variation in the \( r \) and \( z \) directions. \( r_2 - r_1 \) is small compare to \( r_1 \) and the length in the \( z \) direction, normal to the page, is \( L \). The cylindrical surfaces at \( r_2 \) and \( r_1 \) are insulated and \( T_2 \) is greater than \( T_1 \). For steady-state conditions with no heat generation and constant properties, determine the temperature distribution \( T(\phi) \) and the heat transfer rate \( q_\phi \).

![Diagram of the quarter cylindrical system](image)

**Solution**

1) \( T(\phi) \) may be determined from the energy equation

\[
\rho c_p \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) + q''
\]

For steady state with no heat generation and constant \( k \)

\[
\nabla^2 T = 0
\]

The Laplacian \( \nabla^2 T \) is given in cylindrical coordinate as

\[
\nabla^2 T = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} = 0
\]

Since temperature is a function of \( \phi \) only
\[ \frac{\partial^2 T}{\partial \phi^2} = \frac{d^2 T}{d\phi^2} = 0 \]

Integrate the above equation twice to obtain

\[ T = C_1 \phi + C_2 \]

where the two constants of integration \( C_1 \) and \( C_2 \) may be determined from the boundary conditions: \( \phi = 0, T = T_1; \phi = \frac{\pi}{2}, T = T_2 \)

\[ C_1 = (T_2 - T_1) \frac{2}{\pi} \]
\[ C_2 = T_1 \]

The temperature distribution is then

\[ T = (T_2 - T_1) \frac{2}{\pi} \phi + T_1 \]

2) The heat transfer rate \( q_\phi \) may be determined from Fourier's law in the \( \phi \) direction

\[ q_\phi = -kA \frac{dT}{ds} \]

where \( A = \Delta rL, ds = \bar{r} \ d\phi \), and \( \bar{r} = (r_1 + r_2)/2 \)

\[ \frac{dT}{ds} = \frac{dT}{d\phi} \frac{d\phi}{ds} \]

since \( \frac{d\phi}{ds} = \frac{1}{\bar{r}} \Rightarrow \frac{dT}{ds} = \frac{1}{\bar{r}} \frac{dT}{d\phi} \)
\[ q_\phi = -k \frac{L}{r} \frac{dT}{d\phi} \Delta r \]

From the temperature profile \( T = (T_2 - T_1) \frac{2}{\pi} \phi + T_1 \Rightarrow \frac{dT}{d\phi} = (T_2 - T_1) \frac{2}{\pi} \)

Therefore

\[ q_\phi = -k \frac{L}{r} (T_2 - T_1) \frac{2}{\pi} \Delta r = -\frac{4Lk}{\pi} \frac{r_2 - r_1}{r_2 + r_1} (T_2 - T_1) \]

**Example 3.1-4**

The quarter cylindrical system shown has negligible temperature variation in the \( r \) and \( z \) directions. \( r_2 - r_1 \) is small compared to \( r_1 \) and the length in the \( z \) direction, normal to the page, is \( L \). The cylindrical surfaces at \( r_2 \) and \( r_1 \) are insulated and \( T_2 \) is greater than \( T_1 \). For steady-state conditions with no heat generation and constant properties, derive the energy equation by applying the first law to a differential element.

\[ \Delta s = \bar{r} \Delta \phi \]

**Solution**

The control volume has an area \( \Delta r \bar{r} \Delta \phi \) with a unit distance in the direction normal to the page. Apply the first law to the control volume at steady state

\[ \frac{dE}{dt} = q_{in} - q_{out} + q_{gen} = 0 \]

where

\[ q_{in} = q_\phi ; q_{out} = q_{\phi + \Delta \phi} ; \text{ and } q_{gen} = 0 \]

\[ q_\phi - q_{\phi + \Delta \phi} = 0 \]

Divide by \( \Delta \phi \) and take the limit as \( \Delta \phi \to 0 \)
\[
\lim_{\Delta \phi \to 0} \left[ \frac{q_\phi - q_{\phi + \Delta \phi}}{\Delta \phi} \right] = - \frac{dq_\phi}{d\phi} = 0
\]

From the Fourier's law

\[
q_\phi = -kA \frac{dT}{ds}
\]

where \( A = \Delta rL \), \( ds = \bar{r} \, d\phi \), and \( \bar{r} = (r_1 + r_2)/2 \)

\[
\frac{dT}{ds} = \frac{dT}{d\phi} \frac{d\phi}{ds}
\]

since \( \frac{d\phi}{ds} = \frac{1}{\bar{r}} \Rightarrow \frac{dT}{ds} = \frac{1}{\bar{r}} \frac{dT}{d\phi} \)

\[
q_\phi = -k \frac{L}{\bar{r}} \Delta r \frac{dT}{d\phi}
\]

From the energy balance

\[
\frac{dq_\phi}{d\phi} = 0
\]

or

\[
\frac{d}{d\phi} \left[ -k \frac{L}{\bar{r}} \Delta r \frac{dT}{d\phi} \right] = 0
\]

Finally

\[
\frac{d}{d\phi} \left( \frac{dT}{d\phi} \right) = \frac{d^2T}{d\phi^2} = 0
\]


Chapter 3
3.2 Boundary and Initial Conditions

The temperature distribution for heat conduction in a medium may be obtained from the differential energy equation and the boundary and/or initial conditions. Since the differential energy equation is second order in the spatial coordinate, two boundary conditions are required for each coordinate. However only one initial condition is required since the energy equation is first order in time.

![Control volume diagram](image)

**Figure 3.2** Illustration of a control volume at boundary.

Apply the first law of thermodynamics to the control volume at boundary shown

\[
\frac{dE}{dt} = V \rho c_p \frac{dT}{dt} = q_{in} - q_{out} + q_{gen}
\]

where

\[ V = A_s \delta = 0 \text{ since } \delta = 0 \]

\[ q_{gen} = q'' V = 0 \]

Therefore

\[ q_{in} - q_{out} = \int_{A_s} q \cdot n dA = 0 \]

\[
\begin{align*}
q & \rightarrow q \cdot n = + q \\
\rightarrow n & \\
q & \rightarrow q \cdot n = - q
\end{align*}
\]
A boundary cannot store heat because its volume is zero.

**Typical Boundary Conditions Encountered**

- **Constant temperature**

  
  at $x = 0$, $T = T_w$ or $T(0, y, z, t) = T_w$

- **Imposed heat flux**

  Apply $q_{in} - q_{out} = 0$ to the boundary

  
  $q_{in}'' = q_o'' = \text{input flux to the surface (a known value)}$

  
  $q_{out}'' = q_o'' = -k \frac{\partial T}{\partial x}_{x=0}$

  
  $\frac{\partial T}{\partial x}_{x=0} = -\frac{q_o''}{k}$
Gradient or ‘slope’ at $x = 0$

If $q_{in}^* = 0$ then $q_{out}^* = q^* + q_{o}^* = -k \frac{\partial T}{\partial x}_{x=0} + q_{o}^* = 0$, or

$$\frac{\partial T}{\partial x}_{x=0} = \frac{q_{o}^*}{k}$$

Be careful of signs when writing boundary conditions

- Establish directions of fluxes from information given
- Write balance statement for boundary
  $$q_{in}^* - q_{out}^* = 0$$
  - Choose sign in Fourier's law to make flux direction consistent with assumed direction
Example 3.2-1

A 1-m thick slab is maintained at constant temperatures $T_1$ and $T_2$ at its two surfaces. If $T_1 = 400^\circ K$, $T_2 = 600^\circ K$, and the thermal conductivity $k$ of the slab is 100 W/m-K, determine the temperature gradient $\frac{dT}{dx}$ and the heat flux $q_x$ for the three cases (a), (b), and (c) shown.

Solution

a) $\frac{dT}{dx} = \frac{T_2 - T_1}{L} = \frac{600 - 400}{1.0} = 200^\circ K/m$

$q_x^\circ = -k \frac{dT}{dx} = -100 \text{ W/m-K} \times 200^\circ K/m = -20,000 \text{ W/m}^2$

Heat is transferred in the negative $x$ direction

b) $\frac{dT}{dx} = \frac{T_1 - T_2}{L} = \frac{400 - 600}{1.0} = -200^\circ K/m$

$q_x^\circ = -k \frac{dT}{dx} = -100 \text{ W/m-K} \times (-200^\circ K/m) = +20,000 \text{ W/m}^2$

Heat is transferred in the positive $x$ direction

c) $\frac{dT}{dx} = \frac{T_2 - T_1}{L} = \frac{600 - 400}{1.0} = 200^\circ K/m$

$q_x^\circ = -k \frac{dT}{dx} = -100 \text{ W/m-K} \times 200^\circ K/m = -20,000 \text{ W/m}^2$

Heat is transferred in the negative $x$ direction
- Two surface rubbing against each other

- Convection at a surface, $T_\infty$ is known

\[ q'' = -k \frac{\partial T}{\partial x} \bigg|_{x=0} \]

\[ q_c'' = h(T_\infty - T_w) \]

From the energy balance: $q''_{in} - q''_{out} = 0$

\[ h(T_\infty - T_w) + k \frac{\partial T}{\partial x} \bigg|_{x=0} = 0 \]

\[ \frac{\partial T}{\partial x} \bigg|_{x=0} = -\frac{h}{k} (T_\infty - T_w) \]
Convection and Radiation

Heat transfers from the fluid to the solid surface by convection from the bulk fluid temperature \( T_\infty \) and by radiation from the surroundings at \( T_{\text{sur}} \).

\[
T(x,y,x,t) = q''(x) + q_c''(x) - q''(x) = 0
\]

From the energy balance: \( q'' + q_c'' - q'' = 0 \)

\[
q'' = -k \frac{\partial T}{\partial x} \bigg|_{x=0}
\]

\[
q_c'' = h(T_\infty - T_w)
\]

\[
q_{\text{rad}}'' = \Im \cdot \sigma (T_\infty^4 - T_{\text{sur}}^4)
\]

\( \Im \) is a view factor that depends on emissivity only. This factor will be discussed in later chapter. Substitute the above terms into the energy equation and solve for the temperature gradient in the solid,

\[
\frac{\partial T}{\partial x} \bigg|_{x=0} = -\frac{h}{k} (T_\infty - T_w) - \frac{\Im \sigma}{k} (T_\infty^4 - T_{\text{sur}}^4)
\]

References:


Example 3.2-2

A thin flat plate of length $L = 0.8 \text{ m}$, thickness $t = 5 \text{ mm}$, and width $w >> L$ is thermally joined to two large heat sinks that are maintained at a temperature of $60^\circ \text{C}$ and $0^\circ \text{C}$, respectively.

The bottom of the plate is well insulated, while the net heat flux $q''$ to the top surface is known to have a uniform value of $4000 \text{ W/m}^2$. The thermal conductivity of the plate is $100 \text{ W/m}^\circ \text{K}$. Determine the temperature distribution along the plate.

Solution

Making an energy balance on the control volume $wtdx$, we have

$$-kwt \frac{dT}{dx} \bigg|_x + kwt \frac{dT}{dx} \bigg|_{x+dx} + q''wdx = 0$$

In the limit as $dx \rightarrow 0$, we have

$$\frac{d^2T}{dx^2} + \frac{q''}{kt} = 0$$

Integrating this equation we obtain

$$\frac{dT}{dx} = -\frac{q''}{kt}x + C_1$$

$$T = -\frac{q''}{2kt}x^2 + C_1x + C_2 = -\frac{4000}{2(0.005)(100)}x^2 + C_1x + C_2 = -4000x^2 + C_1x + C_2$$

At $x = 0$, $T = 60^\circ \text{C} \Rightarrow C_2 = 60^\circ \text{C}$

At $x = 0.8 \text{ m}$, $T = 0^\circ \text{C} \Rightarrow 0 = -4000(0.8)^2 + C_1(0.8) + 60 \Rightarrow C_1 = 3,125^\circ \text{C/m}$

---

The temperature distribution along the plate is then

\[ T = -4,000x^2 + 3,125x + 60 \]

In this equation \( T \) is in \( ^\circ C \) and \( x \) is in m.

**Example 3.2-3**

In the two-dimensional body illustrated, the gradient at surface \( A \) is found to be \( \partial T / \partial y = 40 \, \text{K/m} \). What are \( \partial T / \partial y \) and \( \partial T / \partial x \) at surface \( B \)?

**Solution**

The temperature along the \( y \)-direction of surface \( B \) is constant, therefore \( \partial T / \partial y = 0 \) at surface \( B \).

Since \( \partial T / \partial y = 40 \, \text{K/m} > 0 \), the rate of heat leaving the body at surface \( A \) per m depth is given by

\[ q_{\text{out}} = -k(1)(1) \frac{\partial T}{\partial y}|_A \]

For steady state, the rate of heat entering the body at surface \( B \) must be equal to the rate of heat leaving the body at surface \( A \).

\[ q_{\text{in}} = -k(1)(0.6)\frac{\partial T}{\partial x}|_B = -k(1)(1) \frac{\partial T}{\partial y}|_A \]

Hence

\[ \frac{\partial T}{\partial y}|_A = (0.6) \frac{\partial T}{\partial x}|_B \Rightarrow \frac{\partial T}{\partial x}|_B = 40/0.6 = 66.7 \, \text{K/m} \]

---

Chapter 4

Analysis of 1-Dimensional Conduction

4.1 Steady state, constant properties, and no heat generation

From the general heat conduction equation

$$\rho c_p \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) + q''$$

For steady state ($\frac{\partial T}{\partial t} = 0$) with no heat generation ($q'' = 0$) and constant $k$

$$\nabla^2 T = 0$$

4.2 Cartesian system, isothermal surfaces

For 1-dimensional conduction in Cartesian coordinate system $\nabla^2 T = 0$ becomes

$$\frac{d^2 T}{dx^2} = 0$$

The two boundary conditions required to determine the temperature profile $T(x)$ are

$$x = 0, \ T = T_1$$

$$x = L, \ T = T_2$$

Integrate the 1-dimensional conduction equation twice to obtain

$$T = C_1 x + C_2$$
Apply the boundary conditions to obtain

\[ C_1 = (T_2 - T_1) \frac{1}{L} \]

\[ C_2 = T_1 \]

The temperature distribution is then

\[ T = (T_2 - T_1) \frac{x}{L} + T_1 \]

The heat transfer rate may be determined from Fourier's law

\[ q_x = -kA \frac{dT}{dx} \]

where \( \frac{dT}{dx} = (T_2 - T_1) \frac{1}{L} \), therefore

\[ q_x = \frac{kA}{L} (T_1 - T_2) = \frac{T_1 - T_2}{L} \frac{L}{kA} \]

The heat transfer rate is proportional to the thermal conductivity \( k \), the area of heat transfer \( A \), the temperature difference \( (T_1 - T_2) \), and inversely proportional to the thickness of the system \( L \).

### 4.3 Radial system, isothermal surfaces

The Laplacian \( \nabla^2 T \) is given in cylindrical coordinate as
\[ \nabla^2 T = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} = 0 \]

For heat transfer in the radial direction only \( \nabla^2 T = 0 \) becomes

\[ \frac{1}{r} \frac{d}{dr} \left( r \frac{dT}{dr} \right) = 0 \]

Integrate the 1-dimensional conduction equation twice to obtain

\[ T = C_1 \ln r + C_2 \]

Apply the boundary conditions

\[ r = r_1, \quad T = T_1 \]
\[ r = r_2, \quad T = T_2 \]

to obtain

\[ C_1 = (T_1 - T_2) \frac{1}{\ln(r_1 / r_2)} \]
\[ C_2 = - (T_1 - T_2) \frac{\ln(r_1)}{\ln(r_1 / r_2)} + T_2 \]

Therefore

\[ T = (T_1 - T_2) \frac{\ln(r)}{\ln(r_1 / r_2)} + T_2 \]

The heat transfer rate in the radial direction is given as

\[ q_r = -kA \frac{dT}{dr} \]

where \( A = 2\pi rL \), and \( \frac{dT}{dr} = \frac{T_1 - T_2}{\ln(r_1 / r_2)} \frac{1}{r} \), therefore

\[ q_r = -k2\pi rL \frac{T_1 - T_2}{\ln(r_1 / r_2)} \frac{1}{r} = k \cdot 2\pi \cdot \frac{r_2 - r_1}{\ln(r_2 / r_1)} \cdot \frac{T_1 - T_2}{r_2 - r_1} \]
The heat transfer rate \( q_t \) is proportional to the thermal conductivity \( k \), the area of heat transfer \( A = 2\pi L \frac{r_2 - r_1}{\ln(r_2/r_1)} \), the temperature difference \( T_1 - T_2 \), and inversely proportional to the thickness of the system \( r_2 - r_1 \). Since the area for heat transfer depends on the radial distance, the area based on the log mean average distance is used for the calculation of the heat transfer rate. The log mean average distance \( r_{ln} \) is defined as,

\[
r_{ln} = \frac{r_2 - r_1}{\ln(r_2/r_1)} = \frac{r_1 - r_2}{\ln(r_1/r_2)}
\]

4.4 Spherical system, isothermal surfaces

For heat transfer in the radial direction only

\[
\nabla^2 T = \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dT}{dr} \right) = 0
\]

Integrate the 1-dimensional conduction equation twice to obtain

\[
T = \frac{C_1}{r} + C_2
\]

Apply the boundary conditions
\[ r = r_1, \quad T = T_1 \]
\[ r = r_2, \quad T = T_2 \]

to obtain
\[ C_1 = (T_1 - T_2) \frac{r_1 r_2}{r_2 - r_1} \]
\[ C_2 = T_2 - \frac{T_1 - T_2}{r_2 - r_1} r_1 \]

Therefore
\[ T = \frac{T_1 - T_2}{r_2 - r_1} r_1 \left[ \frac{r_2}{r} - 1 \right] + T_2 \]

The heat transfer rate in the radial direction is given as

\[ q_r = -kA \frac{dT}{dr} \]

where \( A = 4\pi r^2 \), and \( \frac{dT}{dr} = -\frac{T_1 - T_2}{r_2 - r_1} \frac{r_1 r_2}{r^2} \), therefore

\[ q_r = -k[4\pi r^2] \left[ -\frac{T_1 - T_2}{r_2 - r_1} \frac{r_2}{r^2} \right] = k[4\pi r_1 r_2] \left[ \frac{T_1 - T_2}{r_2 - r_1} \right] \]
\[ q_r = \frac{T_1 - T_2}{\frac{r_2 - r_1}{4\pi kr_2}} \]

The heat transfer rate \( q_r \) is proportional to the thermal conductivity \( k \), the area of heat transfer \( A = 4\pi r_1 r_2 \), the temperature difference \( (T_1 - T_2) \), and inversely proportional to the thickness of the system \( r_2 - r_1 \). Since the area for heat transfer depends on the radial distance, the area based on the geometric average distance is used for the calculation of the heat transfer rate. The geometric average distance \( r_{ave} \) is defined as,

\[ r_{ave} = [r_1 r_2]^{1/2} \]
4.5 Electrical and Thermal Analogy

There is an analogy between the heat conduction and the electric current flow. For 1-dimensional conduction in Cartesian coordinate system the flow of heat is given by

\[ q_x = \frac{kA}{L} (T_1 - T_2) = \frac{T_1 - T_2}{L} \frac{L}{kA}. \]

The thermal resistance is defined as the ratio of a driving potential for heat transfer \( T_1 - T_2 \) to the rate of heat transfer \( q_x \)

\[ R = \frac{T_1 - T_2}{q_x} = \frac{L}{kA}. \]

When a potential \( V_1 - V_2 \) is applied across a conductor of thickness \( L \) with an electrical conductivity \( \sigma_e \) and a cross section \( A \), an electric current flow \( i_e \) is given as

\[ i_e = \frac{V_1 - V_2}{R_e} = \frac{V_1 - V_2}{L} \frac{L}{\sigma_e A}. \]

where the electrical resistance is defined as

\[ R_e = \frac{V_1 - V_2}{i_e} = \frac{L}{\sigma_e A}. \]

The thermal resistance is then the distance of heat transfer divided by the thermal conductivity and the area of heat transfer.
<table>
<thead>
<tr>
<th>Coordinate</th>
<th>Thermal resistance</th>
<th>Distance of heat transfer</th>
<th>Area of heat transfer</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cartesian</td>
<td>( \frac{L}{kA} )</td>
<td>( L )</td>
<td>( A )</td>
</tr>
<tr>
<td>Cylindrical</td>
<td>( \frac{r_2 - r_1}{k2\pi(r_2 - r_1)L} \frac{L}{\ln(r_2 / r_1)} )</td>
<td>( r_2 - r_1 )</td>
<td>( \frac{2\pi(r_2 - r_1) L}{\ln(r_2 / r_1)} )</td>
</tr>
<tr>
<td>Spherical</td>
<td>( \frac{r_2 - r_1}{k4\pi r_2^2 r_1} )</td>
<td>( r_2 - r_1 )</td>
<td>( 4\pi r_1 r_2 )</td>
</tr>
</tbody>
</table>

A thermal resistance for heat transfer by convection can also be defined. From Newton's law of cooling

\[
q_c = hA(T_s - T_{\infty}) = \frac{T_s - T_{\infty}}{1/hA}
\]

The thermal resistance for convection is then

\[
R_{\text{con}} = \frac{1}{hA}
\]

For a system with more than one thermal resistance, they can be combined into an equivalent resistance according to the rules of series and parallel resistances.

\[
R_{\text{eq, series}} = R_1 + R_2 + R_3 + \\
R_{\text{eq, parallel}} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + ...}
\]
Example 4.5-1

Determine the equivalent thermal circuit for the following multi-layered wall.

Calculate (a) the heat loss, \( q \), given \( T_{\infty i}, T_{\infty o}, h_{i}, h_{o}, k_{a}, k_{b} \); (b) the temperatures \( T_{1}, T_{2}, \) and \( T_{3} \).

Solution

Use electrical analogy

a)

\[
q = \frac{T_{\infty i} - T_{\infty o}}{R_{i} + R_{a} + R_{b} + R_{o}}
\]

where

\[
R_{i} = \frac{1}{h_{i}A}, \quad R_{a} = \frac{L_{a}}{k_{a}A}, \quad R_{b} = \frac{L_{b}}{k_{b}A}, \quad R_{o} = \frac{1}{h_{o}A}
\]

b) With \( q \) known, the temperature \( T_{1} \) is determined from

\[
q = \frac{T_{\infty i} - T_{1}}{R_{i}}
\]

\[
T_{1} = T_{\infty i} - qR_{i}
\]

Similarly

\[
T_{2} = T_{1} - qR_{a}
\]

\[
T_{3} = T_{2} - qR_{b}
\]

In general, an interface temperature \( T_{j} \) can be determined from
Example 4.5-2

Determine the equivalent thermal circuit for the following composite cylinder that is long.

Calculate the heat loss, $q$, given $T_i$, $T_o$, $h_i$, $h_o$, $k_a$, $k_b$ ($T_i > T_o$).

Solution

Use electrical analogy

$$q = \frac{T_i - T_o}{R_i + R_a + R_b + R_o}$$

where

$$R_i = \frac{1}{h_i A_i}, \quad R_a = \frac{\ln(r_i / r_1)}{2\pi L k_a}, \quad R_b = \frac{\ln(r_i / r_2)}{2\pi L k_b}, \quad R_o = \frac{1}{h_o A_o}$$

$A_i = 2\pi r_1 L, \quad A_o = 2\pi r_3 L$
Determine the equivalent thermal circuit for the following multi-layered wall. There is radiation heat exchange from the outside surface to the surrounding at temperature $T_o$.

Calculate the heat loss, $q$, given $T_i$, $T_o$, $h_i$, $h_o$, $k_a$, $k_b$.

**Solution**

Use electrical analogy

The radiation heat exchange between the outer surface and the surroundings is given by

$$q_r = h_r A (T_3 - T_o)$$

where the heat transfer coefficient associated with radiation is defined as

$$h_r \equiv \varepsilon \sigma (T_3 + T_o) \left( \frac{T_3^2 + T_o^2}{T_o} \right)$$

The heat transfer rate is then

$$q = \frac{T_i - T_o}{R_i + R_a + R_b + R_{eq}}$$

where

$$R_{eq} = \frac{1}{1/R_i + 1/R_r + 1/R_c}$$, $R_r = \frac{1}{h_r A}$, and $R_c = \frac{1}{h_o A}$
Chapter 4
4.6 Overall Heat Transfer Coefficient

The heat transfer rate in general may be obtained from the driving force over the resistance for heat transfer

\[ q = \frac{\Delta T}{R_{eq}} \]

where \( R_{eq} \) is a combination (series/parallel) of resistances. The heat transfer rate may also be expressed in terms of an overall heat transfer coefficient

\[ q = UA \Delta T \]

where \( U \) is the overall heat transfer coefficient (W/m\(^2\)) defined as

\[ UA = U_i A_i = U_o A_o = \frac{1}{R_{eq}} \]

The overall heat transfer coefficient \( U \) is match with any area \( A \) for heat transfer. For a composite cylinder the heat transfer rate is given as

\[
q = \frac{1}{A_i h_i} + \frac{\ln(r_2/r_i)}{2\pi L_{a}} + \frac{\ln(r_3/r_2)}{2\pi L_{b}} + \frac{1}{A_o h_o} (T_i - T_o)
\]
where
\[ A_i = 2\pi r_1 L, \quad A_o = 2\pi r_3 L \]

The inside area \( A_i \) can be factored out from the expression for the heat transfer rate

\[
q = A_i \left( \frac{1}{h_i} + \frac{A_i \ln(r_2 / r_1)}{2\pi L k_a} + \frac{A_i \ln(r_5 / r_2)}{2\pi L k_b} + \frac{1}{A_o h_o} \right) (T_i - T_o)
\]

In terms of the overall heat transfer coefficient based on the inside area

\[
q = A_i U_i (T_i - T_o)
\]

Therefore

\[
U_i = \frac{1}{\frac{1}{h_i} + \frac{A_i \ln(r_2 / r_1)}{2\pi L k_a} + \frac{A_i \ln(r_5 / r_2)}{2\pi L k_b} + \frac{1}{A_o h_o}}
\]

The outside area \( A_o \) can also be factored out from the expression for the heat transfer rate

\[
q = A_o \left( \frac{1}{A_i h_i} + \frac{A_o \ln(r_2 / r_1)}{2\pi L k_a} + \frac{A_o \ln(r_5 / r_2)}{2\pi L k_b} + \frac{1}{h_o} \right) (T_i - T_o)
\]

In terms of the overall heat transfer coefficient based on the outside area

\[
q = A_o U_o (T_i - T_o)
\]

Therefore

\[
U_o = \frac{1}{\frac{1}{A_i h_i} + \frac{A_o \ln(r_2 / r_1)}{2\pi L k_a} + \frac{A_o \ln(r_5 / r_2)}{2\pi L k_b} + \frac{1}{h_o}}
\]

Since

\[
q = A_i U_i (T_i - T_o) = A_o U_o (T_i - T_o)
\]

\[
A_i U_i = A_o U_o
\]

However \( U_i \neq U_o \)
Example 4.6.1

A spherical aluminum tank, inside radius \( R_1 = 3 \) m, and wall thickness \( l_1 = 4 \) mm, contains liquid-vapor oxygen at 1 atm pressure and 90.18 K. Heat of evaporation of oxygen is \( 2.123 \times 10^5 \) J/kg. Under steady state, at the liquid gas surface, the heat flowing (leak) into the tank causes boil off at a rate \( \dot{M}_g \). In order to prevent the pressure of the tank from rising, the gas resulting from boil off is vented through a safety valve as shown in Figure 4.6-1. An evacuated air gap, extending to location \( r = R_2 = 3.1 \) m, is placed where the combined conduction-radiation effect for this gap is represented by a conductivity \( k_a = 0.004 \) W/m-K. A layer of insulation with \( k_i = 0.033 \) W/m-K and thickness \( l_2 = 10 \) cm is added. The outside surface temperature is kept constant at \( T_2 = 283.15 \) K. Neglect the heat resistance through the aluminum.

Figure E 4.6-1. Liquid oxygen in a spherical container

(a) Determine the rate of heat leak \( Q_{k,2-1} \) in W.

(b) Determine the amount of boil off \( \dot{M}_g \) in kg/s.

(c) Determine the temperature at the inner surface of the insulation.

Solution

(a) Determine the rate of heat leak \( Q_{k,2-1} \) in W

The inside radius of the air layer is \( R_{i,\text{air}} = R_1 + l_1 = 3.000 + 0.004 = 3.004 \) m. The thermal resistance through the air layer, \( R_{t,\text{air}} \), is given by

\[
R_{t,\text{air}} = \frac{R_2 - R_{i,\text{air}}}{(4\pi k_a)(R_{i,\text{air}} R_2)} = \frac{0.096}{(0.004)(4\pi)(3.004 \times 3.1)} = 0.2051 \text{ K/W}
\]

The outside radius of the insulation layer is \( R_{o,\text{ins}} = R_2 + l_2 = 3.1 + 0.1 = 3.2 \) m. The thermal resistance through the insulation layer, \( R_{t,\text{insulation}} \) is given by

---

The rate of heat leak is the heat transfer to the liquid oxygen

\[ Q_{k,2-1} = \frac{T_2 - T_1}{R_{t,air} + R_{t,insulation}} = \frac{283.15 - 90.18}{0.2051 + 0.0243} = 841 \text{ W} \]

(b) Determine the amount of boil off \( \dot{M}_g \) in kg/s.

For steady state, the heat transfer to the inside wall will boil the liquid oxygen off at a rate \( \dot{M}_g \).

The vapor oxygen is then vented off the container.

\[ \dot{M}_g = \frac{841}{2.123 \times 10^5} = 3.96 \times 10^{-3} \text{ kg/s} \]

(c) Determine the temperature at the inner surface \((r = R_2)\) of the insulation using the thermal resistance concept through the insulation layer.

The thermal resistance through the insulation layer, \( R_{t,insulation} \), is given by

\[ R_{t,insulation} = \frac{R_{o,ins} - R_2}{(4\pi k_i)(R_2 R_{o,ins})} = \frac{.10}{(0.033)(4\pi)(3.1 \times 3.2)} = 0.0243 \text{ K/W} \]

The rate of heat transfer through the insulation layer can also be evaluated using the thermal resistance concept through the insulation layer.

\[ Q_{k,2-1} = \frac{T_2 - T}{R_{t,insulation}} \]

\[ 841 = \frac{T_2 - T}{R_{t,insulation}} \]

\[ T = 283.15 - (841)(0.0243) = 262.71 \text{ K} \]

\[ R_{t,insulation} = \frac{R_{o,ins} - R_2}{(4\pi k_i)(R_2 R_{o,ins})} = \frac{.10}{(0.033)(4\pi)(3.1 \times 3.2)} = 0.0243 \text{ K/W} \]
4.7 Systems that contain heat sources.

We now want to consider the situations for which thermal energy is being generated within the medium. The temperature and the heat flux within this medium cannot be obtained using the concept of thermal resistance. The differential energy equation must first be solved for the temperature distribution within the medium and then the heat flux can be obtained from Fourier’s law.

![Composite cylinder system with heat generation](image)

**Figure 4.7-1** A composite cylinder system with heat generation within material $a$.

Figure 4.7-1 shows a composite cylinder system with heat generation within material $a$. The temperature within this material must be obtained by integrating the differential energy equation. However, the temperature and heat flux within material $b$ can still be obtained by using the thermal resistance since there is no heat generated within this material.

**Example 4.7-1**

Determine the steady state temperature distribution of a rectangular plate with uniform heat generation and surface temperature maintained at $T_w$ on both sides. The thickness of the plate is $2L$.

![Rectangular plate](image)

**Figure 4.7-2** A rectangular plate with uniform heat generation $q''$.

**Solution**
The $x$-coordinate is assigned in the direction normal to the plate with $x = 0$ at the left surface where the temperature is $T_w$. The steady state heat conduction equation with constant thermal conductivity is given as

$$\nabla^2 T + \frac{q''}{k} = 0$$

For one-dimensional heat transfer in the $x$ direction

$$\frac{d^2 T}{dx^2} + \frac{q''}{k} = 0$$

The above equation is integrated twice to obtain

$$T = -\frac{q''}{2k} x^2 + C_1x + C_2$$

From the boundary conditions

a) $x = 0$, $T = T_w$; b) $x = 2L$, $T = T_w$

the two constants of integration are obtained as

$$C_1 = \frac{q''L}{k}; \quad C_2 = T_w$$

The temperature distribution is then

$$T = -\frac{q''}{2k} x^2 + \frac{q''L}{k} x + T_w = \frac{q''}{k} \left( Lx - \frac{x^2}{2} \right) + T_w$$

The heat flow at $x = 0$ is determined as

$$q_o = -kA \left. \frac{dT}{dx} \right|_{x=0}$$

$$\frac{dT}{dx} = -\frac{q''}{k} x + \frac{q''L}{k}$$

$$\left. \frac{dT}{dx} \right|_{x=0} = \frac{q''L}{k} \Rightarrow q_o = -kA \frac{q''L}{k} = -LA q''$$

The system is at steady state and is symmetrical with respect to the plane at the center of the plate. Therefore half of the energy generated within the plate must leave the surface at $x = 0$. The
heat flow is in the negative $x$ direction. We can verify the heat flow at $x = 2L$ to be $LAq''$ by using Fourier’s law again

$$q_{2L} = -kA \frac{dT}{dx} \bigg|_{x=2L}$$

$$\frac{dT}{dx} = -\frac{q''}{k} x + \frac{q''L}{k} \Rightarrow \frac{dT}{dx} \bigg|_{x=2L} = -\frac{q''L}{k}$$

$$q_{2L} = -kA \left( -\frac{q''L}{k} \right) = LAq''$$

In solving the differential equation for the temperature distribution, if you can find a symmetry plane, the integration will generally simpler. For this problem, the temperature is symmetric with respect to the plane at the center of the plate where $\frac{dT}{dx} = 0$. The system is equivalent to a plate of half the thickness with one side (left side) insulated.

![Figure 4.7-3 The plate with thickness 2L is symmetric with respect to a plane at the center.](image)

The $x$-coordinate is assigned in the direction normal to the plate with $x = 0$ at the left surface where $\frac{dT}{dx} = 0$. For one-dimensional heat transfer in the $x$ direction

$$\frac{d^2T}{dx^2} + \frac{q''}{k} = 0$$

The above equation is first integrated to obtain

$$\frac{dT}{dx} = -\frac{q''}{k} x + C_1$$
The boundary condition at \( x = 0 \), \( \frac{dT}{dx} = 0 \) can be applied to yield \( C_1 = 0 \)

The resulting equation is then integrated again

\[
T = -\frac{q^m}{2k} x^2 + C_2
\]

From the boundary condition \( x = L \), \( T = T_w \)

\[
C_2 = T_w + \frac{q^m}{2k} L^2
\]

The temperature distribution is then

\[
T = \frac{q^m}{2k} \left( L^2 - x^2 \right) + T_w
\]

If the system does not have a symmetry plane as shown in Figure 4.7-4, there will be no simplification of the integration process.

\[
\text{Figure 4.7-4 A plate with thickness } 2L \text{ without symmetry at the center.}
\]
Chapter 4

Example 4.7-2

Determine the steady state temperature distribution of a solid cylinder of length \( L \) with uniform heat generation. The cylinder is in a convective environment with heat transfer coefficient \( h \) and fluid temperature \( T_\infty \).

![Figure 4.7-5](image.png)

**Figure 4.7-5** A solid cylinder with uniform heat generation \( q'' \).

**Solution**

The \( r \)-coordinate is at the center of the cylinder where the temperature is a maximum. The steady state heat conduction equation with constant thermal conductivity in cylindrical coordinate is given as

\[
\nabla^2 T + \frac{q''}{k} = 0
\]

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} + \frac{q''}{k} = 0
\]

For one-dimensional heat transfer in the \( r \) direction

\[
\frac{1}{r} \frac{d}{dr} \left( r \frac{dT}{dr} \right) + \frac{q''}{k} = 0
\]

The above equation is integrated to obtain
\[ r \frac{dT}{dr} = -\frac{q^m}{2k} r^2 + C_1 \]

The boundary condition at \( r = 0 \), \( \frac{dT}{dr} = 0 \) can be applied to yield \( C_1 = 0 \)

The resulting equation is then integrated again

\[ T = -\frac{q^m}{4k} r^2 + C_2 \]

From the condition at the surface \( r = R, T = T_s \).

\[ C_2 = T_s + \frac{q^m}{4k} R^2 \]

It should be noted that the condition at the surface \( r = R \) is not an actual boundary condition since \( T_s \) is unknown. We will need to solve for \( T_s \) in terms of the known parameters latter. The temperature distribution is then

\[ T = \frac{q^m R^2}{4k} \left( 1 - \frac{r^2}{R^2} \right) + T_s \]

\( T_s \) can be solved from the energy balance for steady state system that requires all the energy generated within the cylinder must leave the system by convection (neglecting radiation)

\[ q^m \pi R^2 L = h(2\pi RL)(T_s - T_\infty) \Rightarrow T_s = T_\infty + \frac{q^m R}{2h} \]

The final temperature distribution is then

\[ T - T_\infty = \frac{q^m R^2}{4k} \left( 1 - \frac{r^2}{R^2} \right) + \frac{q^m R}{2h} \]  \hspace{1cm} (E-1)

We now want to put the equations in dimensionless form so that the independent and dependent variables are of order one (scaling). We first rearrange equation (E-1) as

\[ T - T_\infty = \frac{q^m R}{4h} \left[ \frac{hR}{k} \left( 1 - \frac{r^2}{R^2} \right) + 2 \right] \]  \hspace{1cm} (E-2)

Using the dimensionless quantities defined as \( \eta = \frac{r}{R} \) and \( Bi = \frac{hR}{k} \), equation (E-2) becomes
\[ T - T_\infty = \frac{q''R}{4h} [Bi(1 - \eta^2) + 2] \]  

(E-3)

At the center of the cylinder \((r = 0 \text{ or } \eta = 0)\), we have

\[ T_c - T_\infty = \frac{q''R}{4h} (Bi + 2) \]  

(E-4)

In this equation, \(T_c\) is the temperature at the center of the cylinder. Dividing equation (E-3) by equation (E-4) we obtain

\[ \Theta = \frac{T - T_\infty}{T_c - T_\infty} = \frac{Bi(1 - \eta^2) + 2}{Bi + 2} \]  

(E-5)

\(\Theta\) is the dimensionless temperature that ranges from unity at the center of the wire to zero in the bulk air. In this equation the maximum values of the independent variable, \(\eta\), and of dependent variable, \(\Theta\), are 1. Therefore they are of order 1. The temperature profile in the cylinder can be plotted at various values of the Biot number, \(Bi\), by the following Matlab statements

```matlab
ena=0:.02:1;enas=ena.*ena;
Bi=.01; Theta1=(Bi*(1-enas)+2)./(Bi+2);
Bi=.1; Theta2=(Bi*(1-enas)+2)./(Bi+2);
Bi=1; Theta3=(Bi*(1-enas)+2)./(Bi+2);
Bi=10; Theta4=(Bi*(1-enas)+2)./(Bi+2);
Bi=100;Theta5=(Bi*(1-enas)+2)./(Bi+2);
Theta=[Theta1; Theta2; Theta3; Theta4; Theta5];
plot(ena,Theta)
grid on; xlabel('r/R'); ylabel('Theta')
```

4-21
Figure 1.5-2 shows the dimensionless temperature profile for several values of Bi. For Bi << 1, the wire is almost isothermal and the main temperature drop is in the air. The resistance by heat conduction in the cylinder is so much smaller than the resistance by heat convection in the air. For Bi >> 1, the external temperature drop is negligible and the temperature at the cylinder surface is very close to the ambient value. The resistance by conduction in the cylinder is so much larger than the resistance by heat convection in the air. Thus, the Bi number represents the ratio of the heat transfer resistance within the cylinder to that within the surrounding air.

\[
\Theta = \frac{T - T_\infty}{T_c - T_\infty} = \frac{Bi(1 - \eta^2) + 2}{Bi + 2} \tag{E-5}
\]

From equation (E-5), for Bi << 1

\[
\frac{T - T_\infty}{T_c - T_\infty} \approx 1 \Rightarrow T \approx T_c
\]

At \( \eta = 1 \) and for Bi >> 1

\[
\frac{T - T_\infty}{T_c - T_\infty} \approx 0 \Rightarrow T \approx T_\infty
\]
**Example 4.7-3**

A bare copper wire, 0.4 cm in diameter, has its outer surface maintained at 25°C. It has an electrical resistance of $1.73 \times 10^{-4}$ ohm/m (per meter of wire length). If the centerline temperature is not to exceed 110°C, what is the maximum current it will carry?

![Temperature distribution T(r)](image)

**Figure 4.7-6** A copper wire with uniform heat generation $q''$ by electrical current.

**Solution**

The $r$-coordinate is at the center of the cylinder where the temperature is a maximum. The steady state heat conduction equation with constant thermal conductivity in cylindrical coordinate is given as

$$
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} + \frac{q''}{k} = 0
$$

For one-dimensional heat transfer in the $r$ direction

$$
\frac{1}{r} \frac{d}{dr} \left( r \frac{dT}{dr} \right) + \frac{q''}{k} = 0
$$

The above equation is integrated to obtain

$$
r \frac{dT}{dr} = - \frac{q''}{2k} r^2 + C_1
$$

The boundary condition at $r = 0$, $\frac{dT}{dr} = 0$ can be applied to yield $C_1 = 0$
The resulting equation is then integrated again

\[ T = -\frac{q''}{4k} r^2 + C_2 \]

From the boundary condition at the surface \( r = R \), \( T = T_s \).

\[ C_2 = T_s + \frac{q''}{4k} R^2 \]

The temperature distribution is then

\[ T = \frac{q'' R^2}{4k} \left( 1 - \frac{r^2}{R^2} \right) + T_s = \frac{q'' R^2}{4k} \left( 1 - \frac{r^2}{R^2} \right) + 25^\circ C \]

The heat generation per unit volume \( q'' \) can be solved from the requirement that the centerline temperature of the wire is not to exceed 110\(^\circ\)C.

\[ 110 = \frac{q'' R^2}{4k} + 25^\circ C \]

The thermal conductivity of copper at an average temperature \( 0.5(110 + 25) = 67.5^\circ\)C \( k = 398 \) W/m\(^\circ\)K.

\[ q'' = \frac{4 \times 85 \times k}{R^2} = \frac{4 \times 85 \times 398}{0.002^2} = 3.383 \times 10^{10} \text{ W/m}^3 \]

The heat generated per meter of wire is then

\[ P R_e = q'' \pi R^2 \]

where \( R_e \) is the electrical resistance per meter of wire length, \( R_e = 1.73 \times 10^{-4} \) ohm/m

\[ P^2 = \frac{q'' \pi R^2}{R_e} = \frac{3.383 \times 10^{10} \times \pi \times 0.002}{1.73 \times 10^{-4}} = 2.457 \times 10^9 \text{ A}^2 \]

The maximum current the wire can carry is then

\[ I = 4.96 \times 10^4 \text{ A} \]
Example 4.7-4

The air inside a chamber at $T_{x,i} = 50^\circ C$ is heated convectively with $h_i = 25 \text{ W/m}^2\text{K}$ by a 0.25-m-thick wall having a thermal conductivity of 5 W/m-K and a uniform heat generation of 1500 W/m$^3$. To prevent any heat generated within the wall from being lost to the outside of the chamber at $T_{x,o} = 15^\circ C$ with $h_o = 10 \text{ W/m}^2\text{K}$, a very thin electrical strip heater is placed on the outer wall to provide a uniform heat flux, $q_o$.

![Diagram of a chamber with a wall and strip heater](image)

(a) If no heat generated within the wall is lost to the outside of the chamber, determine the temperature at the wall boundary $T(L)$.

(b) If the heat generation in the wall were switched off while the heat flux to the strip heater remained constant at 600 W/m$^2$, what would be the steady-state temperature, $T(0)$, of the outer wall surface?

Solution

(a) If no heat generated within the wall is lost to the outside of the chamber, determine the temperature at the wall boundary $T(L)$.

For steady state, the rate of heat transfer generated within the wall is equal to the rate of heat transfer to the air

$$1500LA = h_iA(T_L - T_{x,i}) \Rightarrow T_L = \frac{(1500)(0.25)}{25} + 50 = 65^\circ C$$

(b) If the heat generation in the wall were switched off while the heat flux to the strip heater remained constant at 600 W/m$^2$, what would be the steady-state temperature, $T(0)$, of the outer wall surface?

The heat generated by the strip heater is transferred to both the inside and outside chamber air

$$600 = \frac{T_o - T_{x,i}}{L} + \frac{T_o - T_{x,o}}{k_w h_i} = \frac{T_o - 50}{0.25} + \frac{T_o - 15}{1} = \frac{1}{0.09}(T_o - 50) + 10T_o - 150$$

$$(0.09)(600) = 1.9 \ T_o - 50 - (0.09)(150) \Rightarrow T_o = \frac{50 + 0.09(600 + 150)}{1.9} = 61.8^\circ C$$
Example 4.7-5

A teacup is filled with water having temperature $T_w = 90^\circ C$. The cup is made of porcelain with $k = 1.5 \text{ W/m-K}$. The cup wall inside radius is $R$ and its thickness is $L = 3$ mm as shown in Figure 4. The water is assumed to be well mixed and at a uniform temperature. The ambient air is otherwise quiescent with a temperature $T_{\infty} = 20^\circ C$, and adjacent to the cup the air undergoes a thermobuoyant motion resulting a surface-convection resistance $A_{ku}R_{ku} = 10^{-3} \text{ K/(W/m}^2\text{)}$ where $A_{ku}$ is the outside surface area of the cup (contacting with the air). Since $L << R$, you can assume the outside surface area is the same as the inside surface area (you can approximate the wall as a slab). Determine the cup outside surface temperature.\(^2\)

![Diagram of a teacup with thermobuoyant air motion](image)

**Solution**

The cup outside surface temperature, $T$, can be obtained by equating the heat transfer from the liquid to the outside surface and from the outside surface to the air. The thermal resistance for heat transfer through the wall is given by

$$R_{\text{cond}} = \frac{L}{kA} = \frac{3 \times 10^{-3}}{1.5 A}$$

The thermal resistance for heat transfer from the cup outside surface to the air is given by

$$R_{\text{conv}} = \frac{A_{ku}R_{ku}}{A} = \frac{10^{-3}}{A}$$

Equating the heat transfer from the liquid to the outside surface and from the outside surface to the air, we obtain

$$\frac{90 - T}{R_{\text{cond}}} = \frac{T - 20}{R_{\text{conv}}} \Rightarrow \frac{1.5(90 - T)}{3 \times 10^{-3}} = \frac{T - 20}{10^{-3}}$$

$$135 - 1.5T = 3T - 30 \Rightarrow T = \frac{135 + 60}{4.5} = 43.3^\circ C$$

---

Example 4.7-6

In IC (internal combustion) engines, during injection of liquid fuel into the cylinder, it is possible for the injected fuel droplets to form a thin liquid film over the piston as shown in Figure 1. The heat transferred from the gas above the film and from the piston beneath the film causes surface evaporation. The liquid-gas interface is at the boiling \( T_{lg} \), corresponding to the vapor pressure. The heat transfer from the piston side is by conduction through the piston and then by conduction through the thin liquid film. The surface-convection heat transfer from the gas side to the surface of the thin liquid film is 13,500 W.

Data:
Heat of evaporation of fuel = 3.027 \( \times 10^5 \) J/kg, thermal conductivity of fuel, \( k_f = 0.083 \) W/m·K, \( T_{lg} = 398.9 \) K, liquid fuel density \( \rho_l = 900 \) kg/m\(^3\), thermal conductivity of piston \( k_s = 236 \) W/m·K, temperature of piston at distance \( L = 3 \) mm from the surface is \( T_1 = 500 \) K. Piston diameter \( D = 12 \) cm, thickness of liquid film \( L_f = 0.05 \) mm.

![Diagram of an IC engine showing liquid film formation on top of the piston](image)

**Figure 1.** An IC engine, showing liquid film formation on top of the piston.

(a) Estimate the time it will take for the liquid film to evaporate completely assuming the thermal resistance to the liquid film remain constant at the initial value.
(b) Estimate the time it will take for the liquid film to evaporate completely if the thermal resistance to the liquid film is not a constant.

**Solution**

(a) Since the liquid at the gas interface is at the boiling point, it will evaporate with heat input. Let \( \dot{m} \) be the rate of evaporation of the liquid film in kg/s, we have

\[
13,500 + \frac{500 - 398.9}{L} \frac{L}{k_s A} + \frac{L_f}{k_f A} = 3.027 \times 10^5 \dot{m}
\]

In this equation \( A \) is the area for heat transfer given by

---

\[A = \frac{\pi D^2}{4} = \frac{\pi(0.12)^2}{4} = 1.131 \times 10^{-2} \text{ m}^2\]

Hence

\[\dot{m} = \frac{13,500 + \frac{(1.131 \times 10^{-2})(101.1)}{3 \times 10^{-3} + 5 \times 10^{-5}}}{236 \frac{0.083}{3.027 \times 10^5}} = 5.07 \times 10^{-2} \text{ kg/s}\]

The time it will take for the liquid film to evaporate completely is

\[t = \frac{\rho_f L_f A}{\dot{m}} = \frac{(900)(5 \times 10^{-5})(1.131 \times 10^{-2})}{5.07 \times 10^{-2}} = 0.010 \text{ s}\]

(b) Since \(L_f\) is changing with time, the thermal resistance to the liquid film will decrease resulting in a shorter time for the liquid film to evaporate completely. The following equation may be solved for the evaporation time

\[-3.027 \times 10^5 \frac{d(\rho_f L_f A)}{dt} = 13,500 + \frac{500 - 398.9}{L_k A} + \frac{L_f}{k_f A}\]

\[-(3.027 \times 10^5)(900) \frac{dL_f}{dt} = \frac{13,500}{A} + \frac{(101.1)}{3 \times 10^{-3} + \frac{L_f}{0.083}}\]

\[- \frac{dL_f}{dt} = 4.381 \times 10^{-3} + \frac{3.711 \times 10^{-7}}{1.271 \times 10^{-5} + 12.048 L_f}\]

\[t = \int_{0}^{5 \times 10^{-5}} \frac{dL_f}{4.381 \times 10^{-3} + \frac{3.711 \times 10^{-7}}{1.271 \times 10^{-5} + 12.048 L_f}} = \int f(L_f) dL_f\]

<table>
<thead>
<tr>
<th>(L_f)</th>
<th>0</th>
<th>2.5 \times 10^{-5}</th>
<th>5.0 \times 10^{-5}</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(L_f))</td>
<td>2.9785 \times 10^1</td>
<td>1.7974 \times 10^2</td>
<td>2.0061 \times 10^2</td>
</tr>
</tbody>
</table>

Integrating the equation with three points Simpson’s rule yields

\[t = \frac{2.5 \times 10^{-5}}{3} (2.9785 \times 10^1 + 4 \times 1.7974 \times 10^2 + 2.0061 \times 10^2) = 7.9113 \times 10^3 \text{ s}\]
Chapter 5

Analysis of Fins and "Extended Surfaces"

5.1 Introduction

Consider the area $A$ on the surface shown in Figure 5.1 where heat is being transfer from the surface at a fixed temperature $T_s$ to the surrounding fluid at a temperature $T_\infty$ with a heat transfer coefficient $h$. The heat transfer rate may be increased by increasing the convection coefficient $h$, reducing the fluid temperature $T_\infty$, or adding materials to the area $A$.

![Figure 5.1 Use of extended surface or fin to enhance heat transfer.](image)

Look on the plane side-view of the surface and the surface with fin. The heat transfer rate without the fin from area $A$ to the surrounding fluid is

$$q_c = hA(T_s - T_\infty)$$

With the fin attached to the area $A$, the heat transfer to the surrounding fluid must first be transferred by conduction from area $A$ to the fin

$$q_t = -kA \frac{\partial T}{\partial x} \bigg|_{x=0} = \int_0^{L_s} h(T(x) - T_\infty) dA_s$$

where $dA_s = Pdx$ and $P = $ perimeter of the fin.
For the extended surface to enhance the heat transfer rate, the ratio of heat transfer with and without the fin must be greater than one

\[ \varepsilon_f \equiv \frac{q_f}{q_c} = \frac{-k \frac{\partial T}{\partial x}_{x=0}}{h(T_s - T_\infty)} = \int_0^A h(T(x) - T_\infty) dA_x \]

\[ h(T_s - T_\infty) \]

\( \varepsilon_f \) is called the fin effectiveness. For the fin to be cost effective, the fin effectiveness should be greater than 2. The temperature profile along the fin must be determined before the fin effectiveness can be calculated. Consider the extended surface of Figure 5.1. To simplify the analysis, we will assume one-dimensional heat transfer in the x direction, steady state, no heat generation, no radiation, constant heat transfer coefficient, and constant physical properties.

![Figure 5.2 Plane fin with length L, width W, and thickness δ.](image)

An energy balance will be applied to a differential control volume, \( \delta \Delta x W \), shown in Figure 5.2. The thickness \( \delta \) of the plane fin is much smaller than the length and width of the fin. Since temperature is dependent on \( x \), a differential distance along \( x \) must be chosen. The surface area of the control volume is

\[ \Delta A_s = 2\Delta x W + 2\Delta x \delta \approx 2\Delta x W \text{ since } \delta \ll W \]

From the energy balance applied to the control volume \( \Delta x LW \)

\[ q_x - (q_{x+\Delta x} + \Delta q_c) = 0 \]

Divide the equation by \( \Delta x \) and take the limit as \( \Delta x \to 0 \)

\[ \lim_{\Delta x \to 0} \left[ q_x - q_{x+\Delta x} \right] = \lim_{\Delta x \to 0} \left[ \frac{\Delta q_c}{\Delta x} \right] = 0 \]

\[ \lim_{\Delta x \to 0} \left[ \frac{q_x}{\Delta x} \right] - \lim_{\Delta x \to 0} \left[ \frac{\Delta q_c}{\Delta x} \right] = 0 \]

\[ - \frac{dq_x}{dx} - \frac{dq_c}{dx} = 0 \]

\[ dq_c = h\Delta A_s(T(x) - T_\infty) \]
The energy equation becomes

\[- \frac{dq_x}{dx} - h \frac{dA}{dx} (T(x) - T_\infty) = 0\]

From the Fourier's law

\[q_x = -kA \frac{dT}{dx}\]

where \(A\) is the cross-sectional area normal to the \(x\)-direction. The energy equation becomes

\[\frac{d}{dx} \left[ kA \frac{dT}{dx} \right] - h \frac{dA}{dx} (T(x) - T_\infty) = 0\]

since \(A_x = P \cdot x\), \(\frac{dA_x}{dx} = P\)

For constant \(k\) and \(A\), the energy equation becomes a second order ordinary differential equation (ODE) with constant coefficients.

\[\frac{d^2T}{dx^2} - \frac{hP}{kA} (T(x) - T_\infty) = 0\]

The above equation is a non-homogeneous ODE which can be made homogeneous by introducing a new variable \(\theta = T(x) - T_\infty\)

\[\frac{d^2\theta}{dx^2} - \frac{hP}{kA} \theta = 0\]

Let \(m^2 = \frac{hP}{kA}\), the solution to the homogenous ODE has two forms

1) \(\theta = C_1 e^{mx} + C_2 e^{mx}\)

2) \(\theta = B_1 \sinh(mx) + B_2 \cosh(mx)\)

The first exponential form (1) is more convenient if the domain of \(x\) is infinite: \(0 \leq x \leq \infty\) while the second form using hyperbolic functions (2) is more convenient if the domain of \(x\) is finite: \(0 \leq x \leq L\). The constants of integration \(C_1, C_2, B_1,\) and \(B_2\) are to be determined from the two boundary conditions.
5.2 Heat Transfer for Fins of Uniform Cross-Sectional Area

Example 5.2-1

A long cylindrical fin is in a convection environment with a heat transfer coefficient $h$ and an ambient temperature $T_\infty$. Determine the temperature profile along the fin and the heat dissipated by the fin if the base temperature of the fin is $T_o$.

![Figure 5.3 A long cylindrical fin with base temperature $T_o$.](image)

Solution

The $x$-coordinate is assigned in the direction along the fin with $x = 0$ at the base or left surface where the temperature is $T_o$. Since this is a long cylinder $L \to \infty$, the exponential form of the temperature profile is more convenient.

$$\theta = C_1 e^{-mx} + C_2 e^{mx}$$

The two boundary conditions required solving for $C_1$ and $C_2$ are

a) $x = 0$, $T = T_o \Rightarrow \theta = \theta_o = T_o - T_\infty$

b) $x \to \infty$, $T \to T_\infty \Rightarrow \theta = 0$

From (b), $C_2 = 0$ or else $\theta$ is infinite, therefore

$$\theta = C_1 e^{-mx}$$

From the first boundary condition (a): $\theta_0 = C_1$, therefore

$$\theta = \theta_0 e^{-mx}$$

The rate of heat transfer $q_f$ from the fin to the surrounding fluid can be obtain from the temperature distribution $\theta$. Since the fin is at steady state, the heat transfer rate $q_f$ is also the rate of heat transfer from the wall to the base of the fin as shown in Figure 5.4. Therefore

$$q_f = -kA \frac{dT}{dx} \bigg|_{x=0} = \int_0^A h(T(x) - T_\infty) dA$$
Figure 5.4 A long cylindrical fin at steady state.

\[ q_f = -kA \frac{d\theta}{dx} \bigg|_{x=0} = \int_0^\infty hP \theta dx \]

The rate of heat transfer into the fin is \(-kA \frac{d\theta}{dx} \bigg|_{x=0}\) and the rate of heat transfer out of the fin is \(\int_0^\infty hP \theta dx\). It is usually easier to evaluate \(q_f\) from the first expression \(-kA \frac{d\theta}{dx} \bigg|_{x=0}\).

\[ \theta = \theta_0 e^{mx} \Rightarrow \frac{d\theta}{dx} \bigg|_{x=0} = -m \theta_0 \Rightarrow q_f = kA m \theta_0 \]

\[ m = \sqrt{hP / kA} \Rightarrow q_f = \sqrt{hPA} \theta_0 \]

We can also determine \(q_f\) from the expression \(\int_0^\infty hP \theta dx\)

\[ q_f = \int_0^\infty hP \theta dx = hP \theta_0 \int_0^\infty \exp(-mx)dx = hP \theta_0 \left( -\frac{\exp(-mx)}{m} \right)_0^\infty = hP \theta_0 \frac{1}{\sqrt{hP / kA}} \]

\[ q_f = \sqrt{hPA} \theta_0 \]

Example 5.2-2

A cylindrical fin with length \(L\) is in a convection environment with a heat transfer coefficient \(h\) and an ambient temperature \(T_\infty\). Determine the temperature profile along the fin and the heat dissipated by the fin if the base temperature of the fin is \(T_0\) and the end of the fin is insulated.

Figure 5.3 A cylindrical fin with insulated tip and base temperature \(T_0\).
Solution

The x-coordinate is assigned in the direction along the fin with \( x = 0 \) at the base or left surface where the temperature is \( T_o \). Since this is a finite cylinder with length \( L \), the hyperbolic form of the temperature profile is more convenient.

\[
\theta = B_1 \sinh(mx) + B_2 \cosh(mx)
\]

The two boundary conditions required solving for \( B_1 \) and \( B_2 \) are

a) \( x = 0, \quad T = T_o \Rightarrow \theta = \theta_o = T_o - \infty \)

b) \( x = L, \quad \frac{dT}{dx}_{x=L} = 0 \)

From the first boundary condition: \( \theta_o = B_2 \)

From the second boundary condition:

\[
B_1 = -\frac{\sinh(mL)}{\cosh(mL)} \quad B_2 = -\frac{\sinh(mL)}{\cosh(mL)} \theta_o
\]

Therefore

\[
\theta = -\frac{\sinh(mL)}{\cosh(mL)} \theta_o \sinh(mx) + \theta_o \cosh(mx)
\]

\[
\theta = \theta_o \frac{\cosh(mL) \cosh(mx) - \sinh(mL) \sinh(mx)}{\cosh(mL)} = \theta_o \frac{\cosh[(mL-x)]}{\cosh(mL)}
\]

The heat dissipated by the fin \( q_f \) is evaluated from

\[
q_f = -kA \frac{d\theta}{dx}_{x=0}
\]

\[
\frac{d\theta}{dx}_{x=0} = -m \theta_o \frac{\sinh[(mL-0)]}{\cosh(mL)} = -m \theta_o \frac{\sinh(mL)}{\cosh(mL)} = -m \theta_o \tanh(mL)
\]

\[
q_f = kA m \theta_o \tanh(mL) = \sqrt{hP \kappa A} \theta_o \tanh(mL)
\]

With the base temperature of the fin at \( T_o \), any boundary condition at the tip can be used to solve for the two constants of integration.
**Example 5.2-3**

A cylindrical fin with length $2L$ is in a convection environment with a heat transfer coefficient $h$ and an ambient temperature $T_\infty$. Determine the temperature profile along the fin and the heat dissipated by the fin if both side of the fin is at temperature $T_o$.

![Figure 5.4 A cylindrical fin with two ends at temperature $T_o$.](image)

**Solution**

The $x$-coordinate is assigned in the direction along the fin with $x = 0$ at the base or left surface. The surface at $x = L$ is a plane of symmetry therefore $\frac{dT}{dx}_{x=L} = \frac{d\theta}{dx}_{x=L} = 0$. The problem is similar to the case of insulated tip.

$$\theta = \theta_o \frac{\cosh[(mL - x)]}{\cosh(mL)}$$

$\theta$ is plotted in Figure 5.5 for the case where $\theta_o = 100^\circ C$, $m = 3$ cm$^{-1}$, and $L = 1$ cm.

![Figure 5.5 A cylindrical fin with two ends at the same temperature.](image)

The Matlab program used for the plot is listed in Table 5.1.
Table 5.1
% Fin with both sides at the same temperature
theta0=100;
m=3;L=1;
x=0:.02:2;
theta=theta0*cosh(m*(L-x))/cosh(m*L);
plot(x,theta)
xlabel('x(cm)');ylabel('Theta(C)');
grid

The heat dissipation for the fin in this case is simply

\[ q_t = 2 \sqrt{hPkA} \theta_o \tanh(mL) \]

Figure 5.6 A cylindrical fin with convection heat transfer at the tip.

The formula for heat dissipation of a fin with insulated tip can be used for a fin with convective heat transfer at the tip if \( L \) is replaced by the corrected length \( L_c \).

\[ q_t = \sqrt{hPkA} \theta_o \tanh(mL_c) \]

where \( L_c = L + D/4 \). The corrected lengths \( L_c \) for different fin geometries are listed in Table 5.2

Table 5.2 Corrected length \( L_c \) for different fin geometries

<table>
<thead>
<tr>
<th>Fin geometry</th>
<th>( L_c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cylindrical fin with diameter ( D )</td>
<td>( L + D/4 )</td>
</tr>
<tr>
<td>Square fin with side ( \delta )</td>
<td>( L + \delta/2 )</td>
</tr>
<tr>
<td>Rectangular fin with thickness ( t )</td>
<td>( L + t/2 )</td>
</tr>
<tr>
<td>Annular fin of rectangular profile with thickness ( t )</td>
<td>( L + t/2 )</td>
</tr>
<tr>
<td>Fin of triangular profile with base thickness ( t )</td>
<td>( L )</td>
</tr>
</tbody>
</table>
Chapter 5

5.3 Fin Performance

![Diagram of fin performance](image)

**Figure 5.3-1** Use of extended surface or fin to enhance heat transfer.

Consider the area $A_b$ on the surface shown in Figure 5.3-1 where heat is being transferred from the surface at a fixed temperature $T_s$ to the surrounding fluid at a temperature $T_\infty$ with a heat transfer coefficient $h$. The heat transfer rate may be increased by adding fin to the area $A_b$. Let $q_f$ denote the rate of heat transfer from area $A_b$ with a fin that might or might not have a uniform cross-sectional area. $A_b$ is the cross-sectional area at the base of the fin. The heat transfer rate from area $A_b$ without a fin is

$$q_c = h A_b (T_s - T_\infty) = h A_b \theta_o$$

The fin performance is measured by *fin effectiveness* defined as

$$\varepsilon_f = \frac{q_f}{q_c} = \frac{q_f}{h A_b \theta_o}$$

The use of fins may be justified if $\varepsilon_f > 2$.

**Figure 5.3-2** $q_f$ is a maximum if $T(x) = T_o$ along the fin.

The fin performance is also measured by *fin efficiency* $\eta_f$ defined as the ratio $q_f/q_{max}$.

5-9
\[ \eta_f = \frac{q_f}{q_{\text{max}}} \]

\( q_{\text{max}} \) is the maximum possible heat transfer of a fin when the temperature \( T(x) \) along the fin remains constant at the base temperature \( T_o \). This condition can only occur when the thermal conductivity of the fin \( k \) approach infinity. \( q_{\text{max}} \) can be evaluated from the heat leaving the fin surface \( A_f \).

\[ q_{\text{max}} = \int_0^L h(T(x) - T_\infty) dA_f = h(T_o - T_\infty) \int_0^L dA_f = h\theta_o A_f \]

\[ \eta_f = \frac{q_f}{h\theta_o A_f} \]

From the definition of the fin efficiency: \( 0 < \eta_f < 1 \). A good fin design should have \( \eta_f > 0.8 \). After \( \eta_f \) is known for any fin geometry, then the heat dissipation by the fin \( q_f \) is simply determined from

\[ q_f = \eta_f q_{\text{max}} = \eta_f h\theta_o A_f \]

**Example 5.3-1**

A rectangular fin with length \( L \), width \( W \), and thickness \( t \ll W \) is in a convection environment with a heat transfer coefficient \( h \) and an ambient temperature \( T_\infty \). Determine the fin efficiency if the base temperature of the fin is \( T_o \) and the end of the fin is insulated.

![Figure 5.3-3 A rectangular fin with insulated tip and base temperature \( T_o \).](image)

**Solution**

The \( x \)-coordinate is assigned in the direction along the fin with \( x = 0 \) at the base or left surface where the temperature is \( T_o \). Since this is a finite fin with length \( L \), the hyperbolic form of the temperature profile is more convenient.

\[ \theta = B_1 \sinh(mx) + B_2 \cosh(mx) \]

The two boundary conditions required solving for \( B_1 \) and \( B_2 \) are

a) \( x = 0, \ T = T_o \Rightarrow \theta = \theta_o = T_o - T_\infty \)
b) \( x = L \), \( \left. \frac{dT}{dx} \right|_{x=L} = \left. \frac{d\theta}{dx} \right|_{x=L} = 0 \)

From the first boundary condition: \( \theta_0 = B_2 \)

From the second boundary condition: \( \left. \frac{d\theta}{dx} \right|_{x=L} = mB_1 \cosh(mx) + mB_2 \sinh(mx) = 0 \)

\[
B_1 = -\frac{\sinh(mL)}{\cosh(mL)} \quad B_2 = -\frac{\sinh(mL)}{\cosh(mL)} \theta_0
\]

Therefore

\[
\theta = -\frac{\sinh(mL)}{\cosh(mL)} \theta_0 \sinh(mx) + \frac{\sinh(mL)}{\cosh(mL)} \theta_0 \cosh(mx)
\]

\[
\theta = \theta_0 \frac{\cosh(mL) \cosh(mx) - \sinh(mL) \sinh(mx)}{\cosh(mL)} = \frac{\theta_0}{\cosh(mL)} \cosh[(m(L-x)]
\]

The heat dissipated by the fin \( q_f \) is evaluated from

\[
q_f = -kA \left. \frac{d\theta}{dx} \right|_{x=0}
\]

\[
\left. \frac{d\theta}{dx} \right|_{x=0} = -m \theta_0 \frac{\sinh(m(L-0))}{\cosh(mL)} = -m \theta_0 \frac{\sinh(mL)}{\cosh(mL)} = -m \theta_0 \tanh(mL)
\]

\[
q_f = kA \left. \frac{d\theta}{dx} \right|_{x=0} = \sqrt{hPA} \theta_0 \tanh(mL)
\]

The fin efficiency is given as

\[
\eta_f = \frac{q_f}{h \theta_0 A_f} = \frac{\sqrt{hPA} \theta_0 \tanh(mL)}{h \theta_0 A_f} = \frac{kPA}{hA_f^2} \tanh(mL)
\]

\[
m = \sqrt{hP/kA}; \quad A = Wt
\]

\[
P = 2W + 2t \approx 2W
\]

\[
A_f = 2WL + 2tL \approx 2WL \Rightarrow \frac{P^2}{A_f^2} = \frac{1}{L^2}
\]
\[ \eta_f = \frac{\sqrt{kp^2A}}{hPA_f} \tanh(mL) = \frac{1}{L} \frac{\sqrt{KA}}{hP} \tanh(mL) \]

\[ \eta_f = \frac{\tanh(mL)}{mL} \]

---

**Example 5.3-2**

A triangular profile fin with length \( L \), width \( W \), and base thickness \( t \ll W \) is in a convection environment with a heat transfer coefficient \( h \) and an ambient temperature \( T\infty \). Determine the fin effectiveness and the heat dissipated per unit width by a single fin. The fin efficiency is given by

\[ \eta_f = \frac{1}{mL} \frac{I_i(2mL)}{I_0(2mL)} \]

where \( I_0 \) is modified Bessel function of the first kind with order \( n \). \( L = 6 \text{ mm} \), \( t = 2 \text{ mm} \), \( k = 240 \text{ W/m-K} \), \( h = 40 \text{ W/m}^2\text{-K} \), the base temperature \( T_o = 250^\circ\text{C} \), and the ambient temperature \( T\infty = 25^\circ\text{C} \).

![Figure 5.3-4](image-url) A triangular profile fin with base temperature \( T_o \).

**Solution**

The \( x \)-coordinate is assigned in the direction along the fin with \( x = 0 \) at the base or left surface where the temperature is \( T_o \). The parameter \( m \) is first evaluated

\[ m = \sqrt{hP/kA} \]

For triangular profile fin \( P/A = 2W/(Wt) = 2/t \)

\[ m = \frac{\sqrt{40 \times 2}}{\sqrt{240 \times 0.002}} = 12.91 \text{ m}^{-1} \]

The fin efficiency \( \eta_f = \frac{1}{mL} \frac{I_i(2mL)}{I_0(2mL)} \) can be determined from the following Matlab statements.
Matlab script

L = 0.006; t = 0.002;
h = 40; k = 240;
PoA = 2/t;
m = sqrt(h*PoA/k);
I1=besseli(1,2*m*L);I0=besseli(0,2*m*L);
fin_e=I1/(m*L*I0)

\[ mL = 0.0775; \quad 2mL = 0.1549 \]

\[ I_1(0.1549) = 0.0777; \quad I_0(0.1549) = 1.0060 \]

\[ \eta_f = \frac{1}{mL} \frac{I_1(2mL)}{I_0(2mL)} = \frac{1}{0.0775} \frac{0.0777}{1.006} = 0.997 \]

Fin effectiveness is defined as

\[ \varepsilon_f = q_f/q_c = \frac{q_f}{hA_b\theta_o} = \frac{\eta_f q_{max}}{hA_b\theta_o} = \frac{\eta_f hA_f\theta_o}{hA_b\theta_o} = \frac{\eta_f A_f}{A_b} \]

\[ A_f = 2W\left[L^2 + (t/2)^2\right]^{1/2}; \quad A_b = Wt \]

\[ \varepsilon_f = \frac{2\eta_f \left[L^2 + (t/2)^2\right]^{1/2}}{t} = \frac{(2)(.997)[.006^2 + .001^2]^{1/2}}{0.002} = 6.1 \]

Heat dissipation per unit width of the fin is given as

\[ q'_f = \frac{q_f}{W} = \frac{\eta_f q_{max}}{W} = \frac{\eta_f hA_f\theta_o}{W} = 2\eta h\left[L^2 + (t/2)^2\right]^{1/2} \theta_o \]

\[ q'_f = (2)(0.997)(40)[0.006^2 + 0.001^2]^{1/2}(250 - 25) = 108 \text{ W/m} \]

---

**Example 5.3-3**

Consider a single stack of rectangular fins of length \(L\) and thickness \(t\), with convection conditions corresponding to \(h\) and \(T_{\infty}\). In a specific application, a stack that is 200 mm wide and 100 mm deep (let \(D = \text{depth} = 100 \text{ mm}\)) contains 50 fins, each of length \(L = 12 \text{ mm}\). The entire stack is made from aluminum, which is everywhere 1.0 mm thick. Data: \(T_o = 400 \text{ K}, \quad T_L = 320 \text{ K}, \quad T_{\infty} = 290 \text{ K}, \) and \(h = 80 \text{ W/m}^2\cdot\text{K}\). **Note:** The top surface of the upper plate and the bottom surface of the lower plate are not exposed to the convection conditions. The thermal conductivity of the fin material is 200 W/m·K.
a) Determine the differential equation that can be used to solve for the temperature profile along the fin.
b) Determine the two boundary conditions required to solve the differential equation.
c) Determine the rate of heat transfer from the top surface of the lower plate to the air.
d) Determine the rate of heat transfer from all the fins to the air.

**Solution**

**a)** Determine the differential equation that can be used to solve for the temperature profile along the fin.

Making an energy balance around the control volume \( tDdx \) we have

\[
-kT \frac{dT}{dx} |_{x} + kT \frac{dT}{dx} |_{x+dx} - 2h(D + t)dx(T - T_{\infty}) = 0
\]

Letting \( dx \to 0 \), we obtain the differential equation

\[
\frac{d^2T}{dx^2} - \frac{2h(D + t)}{kD} (T - T_{\infty}) = 0
\]

**b)** Determine the two boundary conditions required to solve the differential equation.

The two boundary conditions required to solve the differential equation are

1) \( x = 0, \ T = 400 \) K 
2) \( x = L, \ T = 320 \) K

**c)** Determine the rate of heat transfer from the top surface of the lower plate to the air.

The rate of heat transfer, \( q \), from the top surface of the lower plate to the air is

\[
q = (0.1 \ \text{m})(0.2 - 50 \times 0.001)(\text{m})(80 \ \text{W/m}^2\cdot\text{K})(320 - 290)(\text{K}) = 36 \ \text{W}
\]
d) Determine the rate of heat transfer from all the fins to the air.

The rate of heat transfer from all the fins, \( q_{f,\text{total}} \) to the air is given by

\[
q_{f,\text{total}} = -50kDt \left[ \frac{dT}{dx}_0 - \frac{dT}{dx}_L \right]
\]

We first need to find the solution of the differential equation

\[
\frac{d^2T}{dx^2} - \frac{2h(D + t)}{kDt} (T - T_\infty) = 0
\]

Substituting the numerical values and letting \( \theta = (T - T_\infty) \) we have

\[
m^2 = \frac{2h(D + t)}{kDt} = \frac{(2)(80)(0.1 + 0.001)}{(200)(0.1)(0.001)} = 808 \; \text{m}^2 = (28.425 \; \text{m}^{-1})^2
\]

\[
\frac{d^2\theta}{dx^2} - m^2 \theta = 0
\]

Since this is a finite fin with length \( L \), the hyperbolic form of the temperature profile is more convenient.

\[
\theta = B_1 \sinh(mx) + B_2 \cosh(mx)
\]

The two boundary conditions required solving for \( B_1 \) and \( B_2 \) are

1) \( x = 0, \; T = T_0 \Rightarrow \theta = \theta_0 = T_0 - T_\infty = B_2 \)

2) \( x = L, \; T = 320^\circ\text{K} = T_L \Rightarrow \theta = \theta_L = T_L - T_\infty = B_1 \sinh(mL) + \theta_0 \cosh(mL) \)

Solving for \( B_1 \) gives

\[
B_1 = \frac{\theta_L - \theta_0 \cosh(mL)}{\sinh(mL)}
\]

Therefore

\[
\theta = \frac{\theta_L - \theta_0 \cosh(mL)}{\sinh(mL)} \sinh(mx) + \theta_0 \cosh(mx)
\]

\[
\theta = \frac{\theta_L \sinh(mx) - \theta_0 \cosh(mL) \sinh(mx) + \theta_0 \sinh(mL) \cosh(mx)}{\sinh(mL)}
\]
\[ \theta = \frac{\theta_i \sinh(mx) + \theta_o \sinh(m(L - x))}{\sinh(mL)} \]

The slope of the temperature profile is

\[ \frac{dT}{dx} = \frac{d\theta}{dx} = \frac{m\theta_i \cosh(mx) - m\theta_o \cosh(m(L - x))}{\sinh(mL)} \]

At \( x = 0 \),

\[ \frac{dT}{dx}_{x=0} = \frac{m\theta_i - m\theta_o \cosh(mL)}{\sinh(mL)} \]

At \( x = L \),

\[ \frac{dT}{dx}_{x=L} = \frac{m\theta_i \cosh(mL) - m\theta_o}{\sinh(mL)} \]

\[ q_{\text{f, total}} = -50kD_t \left[ \frac{dT}{dx}_{x=0} - \frac{dT}{dx}_{x=L} \right] \]

\[ q_{\text{f, total}} = -50kD_t \frac{m\theta_i - m\theta_o \cosh(mL) - m\theta_i \cosh(mL) + m\theta_o}{\sinh(mL)} \]

\[ q_{\text{f, total}} = -50kD_t \frac{m(\theta_i - \theta_o) - m \cosh(mL)(\theta_i - \theta_o)}{\sinh(mL)} = -50kD_t \frac{m(\theta_i - \theta_o)[1 - \cosh(mL)]}{\sinh(mL)} \]

For \( mL = (28.425)(0.012) = 0.3411 \Rightarrow \cosh(mL) = 1.0587 \); \( \sinh(mL) = 0.34776 \)

\[ q_{\text{f, total}} = -(50)(200)(0.1)(0.001) \frac{(28.425)(140)(1-1.0587)}{0.34776} \]

\[ q_{\text{f, total}} = 671.7 \text{ W} \]
Chapter 6

Two dimensional, Steady-State Conduction

6.1 The Energy Balance Method

The general heat conduction equation is given as

$$\rho c_p \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) + q''$$

For steady state, no heat generation, and constant $k$, the heat conduction equation is simplified to

$$\nabla^2 T = 0$$

For 2-dimensional heat transfer in Cartesian coordinate

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

We only consider the numerical solution to the above equation in this course. We divide the medium of interest into a number of small regions and apply the conservation of energy to these regions. Each sub-region is assigned a reference point called a node or a nodal point. The average temperature of a nodal point is then calculated by solving the resulting equations from the energy balance. Accurate solutions can be obtained by choosing a fine mesh with a large number of nodes. We will discuss an example from Incropera’s text to illustrate the method.

Example 6.1-1

A long column with thermal conductivity $k = 1$ W/m-K is maintained at 500 K on three surfaces while the remaining surface is exposed to a convective environment with $h = 10$ W/m$^2$-K and fluid temperature $T_\infty$. The cross sectional area of the column is 1 m by 1 m. Using a grid spacing $\Delta x = \Delta y = 0.25$ m, determine the steady-state temperature distribution in the column and the heat flow to the fluid per unit length of the column.

Solution

The cross sectional area of the column is divided into many sub-areas called a grid or nodal network with 25 nodes as shown in Figure 6.1-1. There are 12 nodal points with unknown temperature, however only 8 unknowns need to be solved due to symmetry so that the nodes to the left of the centerline are the same as those to the right.
The energy balance is now applied to the control volume $\Delta x \times \Delta y \times 1$ belongs to node 1 which is an interior node. To make the derivation general, node 1 can be considered as a node with index $(m, n)$ in a two-dimensional grid as shown in Figure 6.1-1. The directions of conduction heat flow are assumed to be the positive $x$ and $y$ directions. For steady state with no heat generation

$$q_{(m-1, n)\rightarrow (m, n)} + q_{(m, n-1)\rightarrow (m, n)} = q_{(m, n)\rightarrow (m+1, n)} + q_{(m, n)\rightarrow (m, n+1)}$$

(6.1-1)

where $q_{(m-1, n)\rightarrow (m, n)}$ is the conduction heat flow between nodes $(m-1, n)$ and $(m, n)$. Fourier’s law can be used to obtain

$$q_{(m-1, n)\rightarrow (m, n)} = k(\Delta y \times 1) \frac{T_{m-1, n} - T_{m, n}}{\Delta x}$$

where $(\Delta y \times 1)$ is the heat transfer area with a unit depth and $\frac{T_{m-1, n} - T_{m, n}}{\Delta x}$ is the finite-difference approximation to the temperature gradient at the boundary between the two nodes. Applying Fourier’s law to each term in Equation (6.1-1) yields

$$k(\Delta y \times 1) \frac{T_{m-1, n} - T_{m, n}}{\Delta x} + k(\Delta x \times 1) \frac{T_{m, n-1} - T_{m, n}}{\Delta y} = k(\Delta y \times 1) \frac{T_{m, n} - T_{m+1, n}}{\Delta x} + k(\Delta x \times 1) \frac{T_{m, n} - T_{m, n+1}}{\Delta y}$$

The equation is divided by $k(\Delta x \times \Delta y \times 1)$ and simplified to

$$\frac{T_{m-1, n} - 2T_{m, n} + T_{m+1, n}}{\Delta x^2} + \frac{T_{m, n-1} - 2T_{m, n} + T_{m, n+1}}{\Delta y^2} = 0$$

(6.1-2)

For $\Delta x = \Delta y$, Eq. (6.1-2) becomes

$$T_{m, n} = \frac{T_{m+1, n} + T_{m, n+1} + T_{m-1, n} + T_{m, n-1}}{4}$$

(6.1-3)
The above result shows that the temperature of an interior node is just the average of the temperatures of the four adjoining nodal points. Using this formula, the temperatures for the first six nodes are

\[
T_1 = \frac{1}{4} (T_2 + T_3 + 500 + 500) \\
T_2 = \frac{1}{4} (T_1 + T_4 + T_1 + 500) \\
T_3 = \frac{1}{4} (T_1 + T_4 + T_5 + 500) \\
T_4 = \frac{1}{4} (T_2 + T_3 + T_6 + T_3) \\
T_5 = \frac{1}{4} (T_3 + T_6 + T_7 + 500) \\
T_6 = \frac{1}{4} (T_4 + T_5 + T_8 + T_5)
\]

Nodes 7 and 8 are not interior points, therefore Eq. (6.1-3) is not applicable.

Making energy balance for node 7 yields

\[
k(\frac{\Delta y}{2} \times 1) \frac{500 - T_7}{\Delta x} + k(\Delta x \times 1) \frac{T_5 - T_7}{\Delta y} = k(\frac{\Delta y}{2} \times 1) \frac{T_7 - T_8}{\Delta x} + h(\Delta x \times 1)(T_7 - 300)
\]

Multiplying the above equation by \( \frac{2}{k} \) we obtain
\[ 500 - T_7 + 2T_5 - 2T_7 = T_7 - T_8 + \frac{2h\Delta x}{k}(T_7 - 300) \]

\[ \frac{2h\Delta x}{k} = \frac{2\times10\times0.25}{1} = 5.0 \]

\[ T_7 = \frac{1}{9}(2T_5 + T_8 + 2000) \]

Similarly an energy balance for node 8 yields

\[ k\left(\frac{\Delta y}{2}\times1\right)\frac{T_7 - T_8}{\Delta x} + k(\Delta x\times1)\frac{T_6 - T_8}{\Delta y} = k\left(\frac{\Delta y}{2}\times1\right)\frac{T_8 - T_7}{\Delta x} + h(\Delta x\times1)(T_8 - 300) \]

Multiplying the above equation by \( \frac{2}{k} \) we obtain

\[ T_7 - T_8 + 2T_6 - 2T_8 = T_8 - T_7 + \frac{2h\Delta x}{k}(T_8 - 300) \]

\[ T_8 = \frac{1}{9}(2T_6 + 2T_7 + 1500) \]

We have 8 linear equations with 8 unknowns that can be solved by matrix method or iterations. Table 6.1-1 lists the Matlab program using Gauss-Seidel iteration to solve for the temperatures.

**Table 6.1-1**

% Initial guesses for the temperatures
T=400*ones(8,1);
for i=1:100
    Tsave=T;
    T(1)=.25*(T(2)+T(3)+1000);
    T(2)=.25*(2*T(1)+T(4)+500);
    T(3)=.25*(T(1)+T(4)+T(5)+500);
    T(4)=.25*(T(2)+2*T(3)+T(6));
    T(5)=.25*(T(3)+T(6)+T(7)+500);
    T(6)=.25*(T(4)+2*T(5)+T(8));
    T(7)=(2*T(5)+T(8)+2000)/9;
    T(8)=(2*T(6)+2*T(7)+1500)/9;
    eT=abs(T-Tsave);
    if max(eT)<.01, break, end
end
fprintf('# of iteration = %g
',i)
for i=1:8
    fprintf('Node %g, Temperature = %8.2f
',i,T(i))
end
>> finite1
# of iteration = 13
Node 1, Temperature = 489.30
Node 2, Temperature = 485.15
Node 3, Temperature = 472.06
Node 4, Temperature = 462.00
Node 5, Temperature = 436.95
Node 6, Temperature = 418.73
Node 7, Temperature = 356.99
Node 8, Temperature = 339.05

The heat transfer rate per unit depth from the column to the fluid is given as
\[
q'/L = 2h\left[\frac{\Delta x}{2}(500 - 300) + \Delta x(T_7 - 300) + \frac{\Delta x}{2}(T_8 - 300)\right]
\]
\[
q'/L = (2)(10)(0.25)[100 + (356.99 - 300) + (0.5)(339.05 - 300)]
\]
\[
q'/L = 883 \text{ W/m}
\]

Example 6.1-2

The steady-state temperature (°C) associated with selected nodal points of a two-dimensional system having a thermal conductivity of 2.0 W/m·°K are shown. The ambient fluid is at 40°C with a heat transfer coefficient of 30 W/m²·°K. The isothermal surface is at 200°C.

(a) Determine the temperature, \(T_1\), \(T_2\), and \(T_3\).

(b) Calculate the heat transfer rate per unit thickness normal to the page from the right surface to the fluid.

Solution

(a) Determine the temperature, \(T_1\), \(T_2\), and \(T_3\).
Since \( \Delta x \) is not equal to \( \Delta y \). We need to solve for the temperature at each node by making an energy balance around that node. For node 1 we obtain

\[
\left( \frac{2}{\Delta x^2} + \frac{2}{\Delta y^2} \right) T_1 = \frac{182.9 + 142.8}{0.3^2} + \frac{147 + 200}{0.2^2}
\]

\[72.222 \quad T_1 = 1.2294 \times 10^4 \Rightarrow T_1 = 170.22^\circ C\]

You should note that the temperature \( T_1 \) is the average of the four neighbouring temperatures with a higher weight for the closer points in the \( y \)-direction.

The temperature at node 2 is given by

\[72.222 \quad T_2 = \frac{139.4 + 55.8}{0.09} + \frac{2 \times 113.5}{0.04} = 7.8439 \times 10^3\]

\[T_2 = 108.6^\circ C\]

The temperature at node 3 is given by

\[
k\Delta y \frac{113.5 - T_3}{\Delta x} + k(\Delta x/2) \frac{77 - T_3}{\Delta y} = k(\Delta x/2) \frac{T_3 - 55.8}{\Delta y} + h\Delta y(T_3 - 40)
\]

\[
\frac{2}{3} (113.5 - T_3) + \frac{3}{4} (77 - T_3) = \frac{3}{4} (T_3 - 55.8) + \frac{30}{2} \times 0.2(T_3 - 40)
\]

\[T_3 = \frac{\frac{2}{3} \times 113.5 + \frac{3}{4} \times 77 + \frac{3}{4} \times 55.8 + 3 \times 40}{\frac{2}{3} + \frac{3}{4} + \frac{3}{4} + 3} = 57.15^\circ C\]

(b) Calculate the heat transfer rate per unit thickness normal to the page from the right surface to the fluid.

\[q' = 30 \times 0.2 \{0.5(200 - 40) + (77 - 40) + (57.15 - 40) + 0.5(55.8 - 40)\} = 852.3 \text{ W/m}\]
6.2 Partial Differential Equation Solver Software

The heat equation could be solved by a partial differential equation solver such as COMSOL Multiphysics developed by COMSOL Inc. The ability of this software can be found from the COMSOL User’s Guide:

“COMSOL Multiphysics is designed to make it as easy as possible to model and simulate physical phenomena. With COMSOL Multiphysics, you can perform free-form entry of custom partial differential equations (PDEs) or use specialized physics application modes. These physics modes consist of predefined templates and user interfaces that are already set up with equations and variables for specific area of physics. Further, by combining any number of these application modes into a single problem description, you can model a multiphysics problem such as one involving simultaneous mass and momentum transfer.”

COMSOL Multiphysics can solve a variety of problems beside heat transfer, including those in fluid dynamics, chemical reactions, electromagnetics, fuel cells, transport phenomena, polymer processing and so on. COMSOL is very flexible with a friendly interface. The Graphical User Interface consists of the following steps:

1. **Model Navigator**: Use this dialog box to choose the model equation, its dimensions, stationary or time-dependent analysis.
2. **Geometry Modeling**: Use this interface to draw or specify object and its dimensions and coordinates.
3. **Physics Settings**: The boundary and initial conditions are specified in the Boundary Settings. The physical properties in the model equations are specified in the Subdomain Settings.
4. **Mesh Generation and Solution**: The mesh can be initialized and refined if necessary. The problem is then solved with various options for solver like: stationary linear, stationary nonlinear, time dependent and so on. The solver can also be auto selected.
5. **Postprocessing and Visualization**: The results can be displayed in different graphical or numerical formats.

COMSOL Multiphysics will be used to solve many problems discussed in the previous sections of chapter 6.
Example 6.2-1

A long column with thermal conductivity $k = 1 \text{ W/m} \cdot \text{°K}$ is maintained at $500\text{°K}$ on three surfaces while the remaining surface is exposed to a convective environment with $h = 10 \text{ W/m}^2 \cdot \text{°K}$ and fluid temperature $T_\infty = 300\text{°K}$. The cross sectional area of the column is 1 m by 1 m. Determine the steady-state temperature distribution in the column and the heat flow to the fluid per unit length of the column.

**Solution**
When you start the COMSOL MULTIPHYSIC, the Model Navigator dialog box appears.

Select 2D in the **Space dimension** list.
In the list of application modes, open the COMSOL Multiphysics > Heat Transfer folder and then the Conduction node.
Select Steady-state analysis and click OK to go to Geometry Modeling.

The COMSOL Multiphysics – Geom1 dialog box appears.
On the Draw menu point to **Specify Objects** and click **Rectangle**.
In the **Rectangle** dialog box, find the **size** area and specify 1 in both the **Width** and **Height** edit fields.

Click **OK** and click the **Zoom Extents** button.

The next step is to set the boundary conditions.

Go to the **Physics** menu and choose **Boundary Settings**. In the **Boundary Settings** dialog box select boundary 1 (the left side). In the **Boundary condition** list select **Temperature** and type 500 in the **Temperature** edit field.
Select boundaries 3 and 4 (top and right sides) by click on 3 then shift click or control (Ctrl) click on 4. In the **Boundary condition** list select **Temperature** and type 500 in the **Temperature** edit field.

Select boundary 2 (bottom sides). In the **Boundary condition** list select **Heat flux** and type 10 in the **Heat transfer coefficient** edit field and 300 in the **External Temperature** edit field.

Click **OK**.

Next go to the **Physics** menu and choose **Subdomain Settings**.

In the **Subdomain Settings** dialog box choose subdomain 1 and enter the thermal properties in the domain according to the following screen.
Click **OK**.

Initialize the mesh by choosing **Mesh** on the Main toolbar and clicking the **Initialize Mesh** button.

Choose **Solve** and click the **Solve Problem** button. The following screen appears.

To get a plot showing the numerical value at various points, use a cross-section plot:
Go to the Postprocessing menu and choose Cross-Section Plot Parameters. In the Cross-Section Plot Parameters dialog box enter the following information:

![Cross-Section Plot Parameters dialog box]

Click Apply and the following screen appears.

![Figure 1 - COMSOL]

The heat flux leaving the bottom surface can be determined by clicking on the Boundary Integration in the Postprocessing menu. Choose Normal heat flux and boundary 2 then click Apply. A value of 580.96 W/m is displayed at the bottom of the screen.
Chapter 7

Unsteady State Conduction

7.1 Introduction

![Figure 7.1-1 An isotropic solid with surface \( A_s \).](image)

An energy balance can be applied to a control \( V \) volume to obtain

\[
V \rho c_p \frac{dT}{dt} = q_{in} - q_{out} + q_{gen}
\]

In this chapter we will consider the situation when the system is time dependent or \( V \rho c_p \frac{dT}{dt} \neq 0 \).

Two approaches can be used to derive the appropriate equation for the system: microscopic or macroscopic balance. The microscopic or differential energy balance will yield a partial differential equation that describes the temperature as a function of time and position. This approach was used in Chapter 3 to obtain the following differential conduction equation

\[
\rho c_p \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) + q''
\]

There are many situations where a macroscopic balance or Lumped Capacitance is sufficient to describe the transient behavior of a system. The temperature of system is only a function of time when the dominant thermal resistance is in the ambience and negligible thermal resistance resides within the system. The energy equation for this case is

\[
V \rho c_p \frac{dT}{dt} = q_{in} - q_{out} + q_{gen}
\]

where \( q_{in} \) and \( q_{out} \) may be due to convection or radiation. Consider a system shown in Figure 7.1-1 where \( q_{in} = q_{gen} = 0 \) and the solid is in a convective environment with \( h \) and \( T_\infty \), then the energy equation becomes

\[
\frac{dE}{dt} = V \rho c_p \frac{dT}{dt} = -q_{out}
\]
\[ V \rho c_p \frac{dT}{dt} = -hA_s(T - T_\infty) \]

The ordinary differential equation derived from the Lumped Capacitance method can easily be solved. However, it is much more involved to obtain the solution to the differential conduction equation. The methods of solution can be analytical, semi-analytical or graphical, and numerical.

### 7.2 The Lumped Capacitance Method

The variation of temperature within the solid is negligible when the internal thermal resistance is much smaller than the external thermal resistance. We need to establish the criteria when the macroscopic energy balance is valid. We will consider a steady state system to illustrate the relative magnitude of the internal and external thermal resistances. Figure 7.2-1 shows a solid plate where the heat \( q \) is being conducted from the left to the right surface and then to the surrounding fluid by convection.

![Figure 7.2-1 Steady-state temperature distribution in a plate.](image)

Applying the energy balance to the right surface of the plate yields

\[ q = q_c \Rightarrow kA \frac{(T_1 - T_2)}{L} = hA(T_2 - T_\infty) \]

Solving for the temperature ratio yields

\[ \frac{T_1 - T_2}{T_2 - T_\infty} = \frac{L}{kA} = \frac{hL}{k} \equiv Bi \]

where \( Bi = Biot \) number is the ratio of resistance to conduction within the solid over the resistance to convection. If \( Bi \ll 1 \), \( T_1 \approx T_2 \), the temperature gradient within the solid can be neglected. If \( Bi \neq \ll 1 \), \( T_1 \neq T_2 \), the temperature gradient within the solid cannot be neglected. For engineering approximation, the Lumped Capacitance method can be used when \( Bi < 0.1 \). For a general system, the Biot number is defined as

\[ Bi \equiv \frac{hL_c}{k} \], \text{where} \ L_c = \frac{V}{A_s} \]

where \( V \) = volume of the system and \( A_s \) = surface area for heat transfer.
Table 7.2-1 lists the characteristic length $L_c$ for some common system shown in Figure 7.2-1. It should be noted that for Lumped Capacitance method the Biot number is defined as

$$Bi \equiv \frac{hL_c}{k}, \text{ where } L_c = \frac{V}{A_s}$$

However in the solution of the differential conduction equation the Biot number is defined for cylindrical and spherical system as

$$Bi \equiv \frac{hR}{k}, \text{ where } R \text{ is the radius of the cylinder or sphere.}$$

**Example 7.2-1**

As part of a space experiment, a small instrumentation package is release from a space vehicle. It can be approximate by a solid aluminum sphere with a radius of 10 cm. If we take the surrounding space to be 0 K, how long will it take for the temperature of the package to change from 300 K to 30 K. Physical properties of aluminum: thermal conductivity $k = 211 \text{ W/m-K}$, emissivity $\varepsilon = 0.05$, density $\rho = 2702 \text{ kg/m}^3$, and heat capacity $c_p = 798 \text{ J/kg-K}$.

**Solution**

$$2R$$

**Figure 7.2-3** Heat leaving the sphere by radiation.
The heat flow from the sphere by radiation is given by

\[ q_r = A \varepsilon \sigma (T_s^4 - T_{\text{sur}}^4) = A \varepsilon \sigma (T_s^2 + T_{\text{sur}}^2) (T_s + T_{\text{sur}}) (T_s - T_{\text{sur}}) = Ah (T_s - T_{\text{sur}}) \]

where the radiation heat transfer coefficient \( h_r \) is defined as

\[ h_r = \varepsilon \sigma (T_s^2 + T_{\text{sur}}^2) (T_s + T_{\text{sur}}) \]

To check for the validity of the Lumped Capacitance method, we can use \( h_r \) to estimate the Biot number

\[ Bi \equiv \frac{h_r R}{3k} \]

Since \( T_{\text{sur}} = 0 \text{ K} \), \( h_r \) can be estimated at the highest surface temperature

\[ h_r = \varepsilon \sigma T_s^3 = 0.05 \times 5.67 \times 10^{-8} \times 300^3 = 0.0765 \text{ W/m}^2 \cdot \text{K} \]

\[ Bi = \frac{0.0765 \times 0.1}{3 \times 211} = 1.21 \times 10^{-5} \]

The Lumped Capacitance method can be used to estimate the time it takes for the temperature of the package to change from 300 K to 30 K.

\[ V \rho c_p \frac{dT}{dt} = q_{\text{in}} - q_{\text{out}} + q_{\text{gen}} = - q_r = - A \varepsilon \sigma T_s^4 = - A \varepsilon \sigma T^4 \]

\[ \int_{300}^{30} \frac{dT}{T^4} = - \frac{\varepsilon \sigma A}{\rho c_p V} \int_0^t dt \Rightarrow - \frac{T^{-3}}{3} \bigg|_{300}^{30} = - \frac{\varepsilon \sigma A}{\rho c_p V} t = - \frac{3 \varepsilon \sigma}{\rho c_p R} t \]

\[ t = \frac{\rho c_p (R/3)}{\varepsilon \sigma} \left( \frac{1}{30^3} - \frac{1}{300^3} \right) \]

\[ t = \frac{2702 \times 798 \times (0.1/3)}{0.05 \times 5.67 \times 10^{-8}} \left( \frac{1}{30^3} - \frac{1}{300^3} \right) = 1.56 \times 10^8 \text{ s} = 1809 \text{ days} \]

Figure 7.2-4 A Lumped Capacitance system.
Consider a *Lumped Capacitance* system shown in Figure 7.2-4 for which \( q_{in} = q_{gen} = 0 \) and the ambient temperature \( T_\infty(t) \) can be time dependent. Applying the energy balance over the system yields

\[
V \rho c_p \frac{dT}{dt} = q_{in} - q_{out} + q_{gen} = -A h(T - T_\infty) \tag{7.2-1}
\]

\[
\frac{\rho V c_p}{h A} \frac{dT}{dt} = - T + T_\infty \tag{7.2-2}
\]

In term of the time constant \( \tau \) defined as \( \tau = \frac{\rho V c_p}{h A} \), equation (7.1-2) can be written as

\[
\frac{dT}{dt} + \frac{T}{\tau} = \frac{T_\infty}{\tau} \tag{7.2-3}
\]

Equation (7.1-3) is a first order linear differential equation that can be solved with the initial condition \( t = 0, T = T_i \) to give (see Review)

\[
T = e^{\frac{t}{\tau}} \left[ \frac{1}{\tau} \int_0^t T_\infty(t) e^{\frac{t'}{\tau}} dt' + T_i \right] \tag{7.2-4}
\]

For \( T_\infty = \) constant

\[
T = T_\infty + (T_i - T_\infty) e^{\frac{t}{\tau}} \tag{7.2-5}
\]

Equation (7.1-5) can be solved for the time it takes the system to reach a temperature \( T \)

\[
t = \tau \ln \left( \frac{T_i - T_\infty}{T - T_\infty} \right)
\]

When \( t = \tau \), \( T \) is the temperature at \( e^{-1} = 0.3679 \) from the initial value as shown in Figure 7.2-5.

![Figure 7.2-5 Plot of \( T = T_\infty + (T_i - T_\infty) e^{-t/\tau} \) versus \( t/\tau \)](image-url)
**Human Body Temperature Regulation**

Humans are homeotherms, or warm-blood animals, and can regulate body temperature rather than have it adjusted by the external environment. The average internal temperature of the human body (core temperature) is maintained to within ±0.5°C around the average temperature of 37°C. The surface skin temperature is lower and under normal conditions is around 31°C. The body exchanges heat with its ambient by surface radiation \( Q_r \) (W), by conduction through direct contact with solid surfaces \( Q_k \) (W), by surface convection heat transfer \( Q_{ku} \) (W), and by surface evaporation energy conversion \( S_{lg} = -\dot{M}_{lg} \Delta h_{lg} \) where \( \dot{M}_{lg} \) (kg/s) is the evaporation rate and \( \Delta h_{lg} \) (J/kg) is the heat of evaporation for water.

When ambient is at a temperature \( T_\infty \) lower than the skin, heat flows out of the body into the ambient. In addition to the surface heat losses, heat is loss through gas, liquid, and solid discharges. This heat loss is sustained by the generation of heat by conversion of chemical bonds to thermal energy in metabolic reactions, i.e. chemical reactions, and temporarily by the reducing the body temperature. When ambient is at a higher temperature, heat flows out of the body by surface evaporation or sweating since the body must use its energy to break the physical bonds of the liquid.

The fat tissues have a thermal conductivity of one third of the other tissues, and therefore, act as an insulator. The blood flow to the surface is controlled by increasing it for heating purposes (vasodilatation) or by decreasing it for reducing heat losses (vasoconstriction). The conversion of energy, blood flow rate, and sweating are controlled by the nervous system feedback mechanisms and this control originates from the hypothalamus in the brain. There are temperature sensors throughout the body that are heat-sensitive neurons, which send higher frequency signals to the brain as the temperature increases.

**Example 7.2-2.**

Consider a person under a condition of hypothermia that is generating a maximum heat under severe shivering of 400 W. However, the total heat loss due to convection and radiation is 800 W. The body energy content \((\rho c_v V)\) is assumed to be \(5\times10^5\) J/K. Determine how long it will take for the body temperature to drop by 10°C.

**Solution**

Neglect any other energy loss, the heat balance for the body is

\[
\frac{dE}{dt} = \text{accumulated energy change (W)} = \rho c_v V \frac{dT}{dt} = -q_{out} + q_{gen}
\]

\[
5\times10^5 \frac{dT}{dt} = -800 + 400 = -400 \Rightarrow t = 1.25\times10 = 12,500\ s = 3.47\ hr.
\]

---

Chapter 7

Unsteady State Conduction

Review: First Order Linear Ordinary Differential Equations

The following differential equation

\[ \frac{dT}{dt} + \frac{T}{\tau} = \frac{T_{\infty}}{\tau} \]  

(7.2-6)

is a first order linear differential equation with the general form

\[ \frac{dT}{dt} + \alpha(t) T = f(t) \]  

(7.2-7)

where

\[ \alpha(t) = \frac{1}{\tau} \]

\[ f(t) = \frac{T_{\infty}}{\tau}, f(t) \] is called the forcing function.

To solve the first order equation, we need to find a function \( I(t) \) called the integrating factor. When the integrating factor is multiplied to Eq. (7.2.7)

\[ I(t) \left[ \frac{dT}{dt} + \alpha(t) T \right] = I(t) f(t) \]  

(7.2-8)

The left hand side of Eq. (7.2-8) will become an exact derivative

\[ \frac{d}{dt}[I(t)T] = I(t)f(t) \]  

(7.2.9)

Compare Eq. (7.2-8) to Eq. (7.2.9)

\[ I(t) \left[ \frac{dT}{dt} + \alpha(t) T \right] = \frac{d}{dt}[I(t)T] \]

\[ I(t) \frac{dT}{dt} + I(t)\alpha(t)T = I(t) \frac{dT}{dt} + T \frac{dI(t)}{dt} \]

\[ I(t)\alpha(t) = \frac{dI(t)}{dt} ; \text{ or } \frac{dI(t)}{I(t)} = \alpha(t)dt \]

The integrating factor \( I(t) \) is then
\[ I(t) = \exp \left[ \alpha(t) dt \right] \]  
(7.2.10)

Equation (7.2.9) can be integrated to obtain

\[ I(t) T = \int I(t) f(t) dt + C; \text{ or} \]

\[ T = \frac{1}{I(t)} \int I(t) f(t) dt + \frac{C}{I(t)} \]  
(7.2-11)

where \( C \) is an arbitrary constant of integration.

The integrating factor for Eq. (7.2-6) is

\[ I(t) = \exp \left[ \int \alpha(t) dt \right] = \exp \left[ \int \left( \frac{1}{\tau} \right) dt \right] = \exp(t/\tau) \]

Hence

\[ T = \exp(-t/\tau) \frac{1}{\tau} \int_{0}^{t} T_{\infty}(t)e^{t/\tau} dt + \exp(-t/\tau) \]  
(7.2-12)

Applying the initial condition \( t = 0, T = T_{i} \) we obtain \( C = T_{i} \)

The final solution is

\[ T = e^{-t/\tau} \left[ \frac{1}{\tau} \int_{0}^{t} T_{\infty}(t)e^{t/\tau} dt + T_{i} \right] \]

---

**Example 7.2-3**

A liquid droplet rises in heavier, immiscible host liquid and is exposed to vertically increasing temperatures as shown in Figure 7.2-6. The rising velocity of the liquid drop \( v \) is assumed to be constant. The temperature gradient of the host liquid is also a constant \( \frac{dT_{\infty}}{dx} = \beta \). If the liquid droplet is initially at the same temperature as that of the host liquid \( T_{i} \), determined the temperature of the liquid droplet as it rises up the column. You can assume the Biot number is much less than 1.

**Solution**

Applying the energy balance over the liquid droplet yields

\[ V \rho c_{p} \frac{dT}{dt} = q_{\text{in}} - q_{\text{out}} + q_{\text{gen}} = -Ah(T - T_{\infty}) \]
Rearranging the equation

\[ \frac{\rho Vc_p}{hA} \frac{dT}{dt} = -T + T_\infty \]

In term of the time constant \( \tau \) defined as \( \tau = \frac{\rho Vc_p}{hA} \), the equation can be written as

\[ \frac{dT}{dt} + \frac{T}{\tau} = \frac{T_\infty}{\tau} \]

\( T_\infty \) is not a constant as the liquid drop rises up the column.

\[ \frac{dT_\infty}{dt} = \frac{dT_\infty}{dx} \frac{dx}{dt} = \beta v \Rightarrow T_\infty = T_i + \beta vt \]

The solution for the droplet temperature is given as

\[ T = e^{\beta vt} \left[ \frac{1}{\tau} \int_0^t T_\infty(t)e^{\beta vt}dt + T_i \right] \]

\[ \frac{1}{\tau} \int_0^t T_\infty(t)e^{\beta vt}dt = \frac{1}{\tau} \int_0^t (T_i + \beta vt)e^{\beta vt}dt = \frac{1}{\tau} \int_0^t T_i e^{\beta vt}dt + \frac{1}{\tau} \int_0^t \beta vt e^{\beta vt}dt \]
\[
\frac{1}{\tau} \int_0^\infty T_i(t)e^{t} \, dt = T_i(e^{t} - 1) + \frac{\beta v}{\tau} \int_0^\infty te^{t} \, dt
\]

The integral \( \int_0^\infty te^{t} \, dt \) can be evaluated using integration by parts:

\[
d(uv) = udv + vdu \Rightarrow \int udv = \int d(\nu v) - \int vdu
\]

\[
\int udv = \int d(\nu v) - \int vdu
\]

Let \( u = t \Rightarrow du = dt \), \( dv = e^{t} \, dt \Rightarrow v = \tau e^{t} \)

\[
\int_0^\infty te^{t} \, dt = t \tau e^{t} \bigg|_0^\infty - \tau \int_0^\infty e^{t} \, dt = t \tau e^{t} - \tau e^{t} \bigg|_0^\infty = t \tau e^{t} - \tau(e^{t} - 1)
\]

\[
\frac{\beta v}{\tau} \int_0^\infty te^{t} \, dt = \beta v[te^{t} - \tau(e^{t} - 1)]
\]

\[
\frac{1}{\tau} \int_0^\infty T_i(t)e^{t} \, dt = T_i(e^{t} - 1) + \frac{\beta v}{\tau} \int_0^\infty te^{t} \, dt = T_i(e^{t} - 1) + \beta vt e^{t} - \beta v(\tau(e^{t} - 1))
\]

\[
T = e^{t} \left[ T_i(e^{t} - 1) + \beta vt e^{t} - \beta v(\tau(e^{t} - 1)) \right]
\]

\[
T = T_i + \beta vt - \beta v(\tau(e^{t} - 1)) = T_\infty - \beta v(1 - e^{t} - \tau)
\]

For \( t \to \infty \), \( T \to T_\infty - \beta v \) as shown in Figure 7.2-7

---

**Figure 7.2-7** The liquid droplet is cooler than the host liquid.
Chapter 7

Unsteady State Conduction

7.3 Differential Energy Balance

When the internal temperature gradient is not negligible or \( Bi \neq << 1 \), the microscopic or differential energy balance will yield a partial differential equation that describes the temperature as a function of time and position. This approach was used in Chapter 3 to obtain the following differential conduction equation

\[
\rho c_p \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) + q''
\]  

(7.3-1)

For one-dimensional heat transfer in a slab with convective conditions of \( h \) and \( T_\infty \), equation (7.3-1) is simplified to

\[
\rho c_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} \Rightarrow \frac{1}{\alpha} \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}
\]  

(7.3-2)

Figure 7.3-1 One-dimensional unsteady heat transfer in a slab.

Equation (7.3-2) can be solved with the following initial and boundary conditions

I. C.  
\( t = 0, \ T(x, 0) = T_i \)

B. C.  
\( x = 0, \ \frac{\partial T}{\partial x} \bigg|_{x=0} = 0; \quad x = L, \ -k \frac{\partial T}{\partial x} \bigg|_{x=L} = h(T - T_\infty) \)

In general, the temperature within the slab depends on many parameters besides time \( t \) and position \( x \).
\[ T = T(x, t, T_i, T_x, L, k, \alpha, h) \]

The differential equation and its boundary conditions are usually changed to the dimensionless forms to simplify the solutions. We define the following dimensionless variables

Dimensionless temperature: \[ \theta^* = \frac{T - T_\infty}{T_i - T_\infty} \Rightarrow T = T_\infty + \theta^*(T_i - T_\infty) \]

Dimensionless distance: \[ x^* = \frac{x}{L} \Rightarrow x = L x^* \]

Dimensionless time or Fourier number: \[ t^* = Fo = \frac{\alpha}{L^2} \Rightarrow t = \frac{L^2}{\alpha} Fo \]

Substituting \( T, x, \) and \( t \) in terms of the dimensionless quantities into equation (7.3-2) yields

\[ (T_i - T_\infty) \frac{1}{\alpha} \frac{\partial \theta^*}{\partial Fo} = (T_i - T_\infty) \frac{1}{L^2} \frac{\partial^2 \theta^*}{\partial x^*^2} \]

\[ \frac{\partial \theta^*}{\partial Fo} = \frac{\partial^2 \theta^*}{\partial x^*^2} \]

Similarly, the initial and boundary conditions can be transformed into dimensionless forms

\[ \theta^*(x^*, 0) = 1 \]

\[ \left. \frac{\partial \theta^*}{\partial x^*} \right|_{x^*=0} = 0; \quad \left. \frac{\partial \theta^*}{\partial x^*} \right|_{x^*=1} = -Bi* \theta^*(1, t^*), \text{ where } Bi = \frac{hL}{k} \]

Therefore \( \theta^* = f(x^*, Fo, Bi) \)

The dimensionless temperature \( \theta^* \) depends only on \( x^*, Fo, \) and \( Bi \). Equation (7.3-3) can be solved by the method of separation of variables to obtain

\[ \theta^* = \sum_{n=1}^{\infty} C_n \exp(-\zeta_n^2 Fo) \cos(\zeta_n x^*) \]  

(7.3-4)

where the coefficients \( C_n \) are given by

\[ C_n = \frac{4 \sin \zeta_n}{2 \zeta_n + \sin(2 \zeta_n)} \]

and \( \zeta_n \) are the roots of the equation: \( \zeta_n \tan(\zeta_n) = Bi \).
Table 7.3-1 lists the Matlab program that evaluates the first ten roots of equation \( \zeta_n \tan(\zeta_n) = Bi \) and the dimensionless temperatures given in equation (7.3-4). The program uses Newton’s method to find the roots (see Review).

**Table 7.3-1** Matlab program to evaluate and plot \( \theta = \sum_{n=1}^{\infty} C_n \exp(-\zeta_n^2 Fo) \cos(\zeta_n x^*) \)

```matlab
% Plot the dimensionless temperature within a slab
% The guess for the first root of equation z*tan(z)=Bi depends on the Biot number
% Biot=[0 .01 .1 .2 .5 1 2 5 10 inf];
alfa=[0 .0998 .3111 .4328 .6533 .8603 1.0769 1.3138 1.4289 1.5707];
zeta=zeros(1,10);cn=zeta;
Bi=1;
fprintf('Bi = %g, New ',Bi)
Bin=input('Bi = ');
if length(Bin)>0;Bi=Bin;end
% Obtain the guess for the first root
if Bi>10
    z=alfa(10);
else
    z=interp1(Biot,alfa,Bi);
end
% Newton method to solve for the first 10 roots
for i=1:10
    for k=1:20
        ta=tan(z);ez=(z*ta-Bi)/(ta+z*(1+ta*ta));
        z=z-ez;
        if abs(ez)<.00001, break, end
    end
    % Save the root and calculate the coefficients
    zeta(i)=z;
    cn(i)=4*sin(z)/(2*z+sin(2*z));
    fprintf('Root # %g  =%8.4f, Cn = %9.4e
',i,z,cn(i))
% Obtain the guess for the next root
    step=2.9+i/20;
    if step>pi; step=pi;end
    z=z+step;
end
% Evaluate and plot the temperatures
hold on
Fop=[.1 .5 1 2 10];
xs=-1:.05:1;
cosm=cos(cn'*xs);
for i=1:5
    Fo=Fop(i);
```
\[
\theta = cn \ast \exp(-Fo \ast zeta^2) \ast \cos \theta;
\]

```matlab
plot(xs,theta)
end
```

grid

xlabel('x*'); ylabel('Theta*')

\[Bi = .5\]

Root # 1  =  0.6533,  \[Cn = 1.0701e+000\]

Root # 2  =  3.2923,  \[Cn = -8.7276e-002\]

Root # 3  =  6.3616,  \[Cn = 2.4335e+000\]

Root # 4  =  9.4775,  \[Cn = -1.1056e-002\]

Root # 5  = 12.6060,  \[Cn = 6.2682e-003\]

Root # 6  = 15.7397,  \[Cn = -4.0264e-003\]

Root # 7  = 18.8760,  \[Cn = 2.8017e-003\]

Root # 8  = 22.0139,  \[Cn = -2.0609e-003\]

Root # 9  = 25.1526,  \[Cn = 1.5791e-003\]

Root # 10  = 28.2920,  \[Cn = -1.2483e-003\]

Figure 7.3-2 shows a plot of dimensionless temperature \(\theta\) versus dimensionless distance \(x^*\) at various Fourier number for a \(Biot\) number of 0.5.

---

**Figure 7.3-2** Dimensionless temperature distribution at various Fourier number.

---

**Review: The Newton-Raphson Method**

The Newton-Raphson method and its modification is probably the most widely used of all root-finding methods. Starting with an initial guess \(x_1\) at the root, the next guess \(x_2\) is the intersection of the tangent from the point \([x_1, f(x_1)]\) to the \(x\)-axis. The next guess \(x_3\) is the intersection of the tangent from the point \([x_2, f(x_2)]\) to the \(x\)-axis as shown in Figure 7.3-3. The process can be repeated until the desired tolerance is attained.
The Newton-Raphson method can be derived from the definition of a slope

\[ f'(x_1) = \frac{f(x_1) - 0}{x_1 - x_2} \Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \]

In general, from the point \([x_n, f(x_n)]\), the next guess is calculated as

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \]

The derivative or slope \(f(x_n)\) can be approximated numerically as

\[ f'(x_n) = \frac{f(x_n + \Delta x) - f(x_n)}{\Delta x} \]

**Example**

Solve \(f(x) = x^3 + 4x^2 - 10\) using the the Newton-Raphson method for a root in the interval \([1, 2]\).

**Solution**

From the formula \(x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\)

\[ f(x_n) = x_n^3 + 4x_n^2 - 10 \Rightarrow f'(x_n) = 3x_n^2 + 8x_n \]
\[ x_{n+1} = x_n - \frac{x_n^3 + 4x_n^2 - 10}{3x_n^2 + 8x_n} \]

Using the initial guess, \( x_n = 1.5 \), \( x_{n+1} \) is estimated as

\[ x_{n+1} = 1.5 - \frac{1.5^3 + 4 \times 1.5^2 - 10}{3 \times 1.5^2 + 8 \times 1.5} = 1.3733 \]

For the roots of equation \( \zeta_n \tan(\zeta_n) = Bi \), let

\[ f = \zeta \tan(\zeta) - Bi \]

Then \( f' = \tan(\zeta) + \zeta(1 + \tan(\zeta)^2) \);

The differential conduction equation for heat transfer in the radial direction of an infinite cylinder with radius \( R \) is

\[ \rho c_p \frac{\partial T}{\partial t} = k \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) \quad \text{or} \quad \frac{\partial T}{\partial t} = \alpha \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) \quad (7.3-5) \]

The differential conduction equation for heat transfer in the radial direction of a sphere with radius \( R \) is

\[ \rho c_p \frac{\partial T}{\partial t} = k \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) \quad \text{or} \quad \frac{\partial T}{\partial t} = \alpha \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) \quad (7.3-6) \]

Equations (7.3-5) and (7.3-6) can be solved with the following initial and boundary conditions

I. C. \quad t = 0, \; T(r, 0) = T_i

B. C. \quad r = 0, \; \frac{\partial T}{\partial r} \bigg|_{r=0} = 0; \quad r = R, \; -k \frac{\partial T}{\partial r} \bigg|_{r=R} = h(T - T_\infty)

The solution of equation (7.3-5) for the infinite cylinder is given as

\[ \Theta = \sum_{n=1}^{\infty} C_n \exp(-\xi_n^2 Fo) J_0(\zeta_n x^*) \quad (7.3-7) \]

where \( J_0(\zeta_n x^*) \) is Bessel function of the first kind, order zero. The coefficient \( C_n \) are not the same as those in a slab. The solution of equation (7.3-6) for a sphere is given as

\[ \Theta = \sum_{n=1}^{\infty} C_n \exp(-\xi_n^2 Fo) \frac{\sin(\zeta_n r^*)}{\zeta_n r^*} \quad (7.3-8) \]
Since \( \lim_{r^* \to 0} \sin(\zeta_n r^*) = \lim_{r^* \to 0} \frac{\zeta_n \cos(\zeta_n r^*)}{\zeta_n} = 1 \), it should be noted that at \( r^* = 0 \)

\[
\theta = \sum_{n=1}^{\infty} C_n \exp(-\zeta_n^2 \text{Fo})
\]

For one-dimensional heat transfer in a semi-infinite solid as shown in Figure 7.3-4, the differential equation is the same as that in one-dimensional heat transfer in a slab

\[
\rho c_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} \Rightarrow \frac{1}{\alpha} \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}
\]

**Figure 7.3-4** One-dimensional heat transfer in a semi-infinite solid.

We consider three cases with the following initial and boundary conditions

**Case 1:**

I. C.: \( T(x, 0) = T_i \)

B. C.: \( T(0, t) = T_s \), \( T(x \to \infty, t) = T_i \)

**Case 2:**

I. C.: \( T(x, 0) = T_i \)

B. C.: \( -k \frac{\partial T}{\partial x} \bigg|_{x=0} = q_0^*, \quad T(x \to \infty, t) = T_i \)

**Case 3:**

I. C.: \( T(x, 0) = T_i \)

B. C.: \( -k \frac{\partial T}{\partial x} \bigg|_{x=0} = h(T - T_\infty), \quad T(x \to \infty, t) = T_i \)

All three cases have the same initial condition \( T(x, 0) = T_i \) and the boundary condition at infinity \( T(x \to \infty, t) = T_i \). However the boundary condition at \( x = 0 \) is different for each case, therefore the solution will be different and will be summarized in a table later.
Example 7.3-1. (Incropera, Heat and Mass Transfer, problem 5.77)
Irreversible cell damage occurs in living tissue maintained at temperature greater than 48°C for a duration greater than 10 seconds. When a portion of a worker’s body comes into contact with machinery that is at elevated temperatures in the range of 50 to 100°C, the worker can suffer serious burn injuries. You can assume that living tissue has a normal temperature of 37°C, is isotropic, and has constant properties equivalent to those of liquid water. Compute locations in the tissue at which the temperature will reach 48°C after 10 s of exposure to machinery at 50°C and 100°C.

Solution

The unsteady state temperature distribution in a semi-infinite medium is listed in Table 7.4-3.

Table 7.4-3 Semi-infinite medium

Constant Surface Temperature: \( T(0, t) = T_s \)

\[
\frac{T(x, t) - T_s}{T_i - T_s} = \text{erf} \left( \frac{x}{2\sqrt{\alpha t}} \right) ; \quad q_i(t) = \frac{k(T_s - T_i)}{\sqrt{\pi \alpha t}}
\]

Constant Surface Heat Flux: \( q_s = q_0 \)

\[
T(x, t) - T_i = \frac{2q_0(\alpha t / \pi)^{1/2}}{k} \exp \left( -\frac{x^2}{4\alpha t} \right) - \frac{q_s x}{k} \text{erfc} \left( \frac{x}{2\sqrt{\alpha t}} \right)
\]

The complementary error function, \( \text{erfc}(w) \), is defined as \( \text{erfc}(w) = 1 - \text{erf}(w) \)

Surface Convection: \[-k \frac{\partial T}{\partial x} \bigg|_{x=0} = h[T_\infty - T(0, t)]\]

\[
\frac{T(x, t) - T_s}{T_\infty - T_s} = \text{erfc} \left( \frac{x}{2\sqrt{\alpha t}} \right) - \left[ \exp \left( \frac{hx}{k} + \frac{h^2 \alpha t}{k^2} \right) \right] \left[ \text{erfc} \left( \frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k} \right) \right]
\]
The equation applied to this problem is given by the expression for constant surface temperature

\[
\frac{T(x, t) - T_s}{T_i - T_s} = \text{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)
\]

The thermal diffusivity of water at 37°C is 1.513×10⁻⁷ m²/s. We will determine the location \(x_b\) from the skin surface where the temperature is 48°C at 10 seconds. \(x_b\) is the depth of the tissue layer that will be damaged if the surface temperature is maintained at elevated temperature for an additional 10 seconds. If \(T_s = 50°C\)

\[
\frac{T(x, t) - T_s}{T_i - T_s} = \frac{48 - 50}{37 - 50} = 0.154
\]

Therefore \(\left(\frac{x}{2\sqrt{\alpha t}}\right) = \text{erf}^{-1}(0.154)\)

The inverse of the error function can be evaluated using Matlab command \texttt{erfinv}

\[
>> \text{erfinv}(0.154)
\]

\[
\text{ans} =
\]

\[
0.1373
\]

\[
>> \frac{x}{2\sqrt{\alpha t}} = 0.1373 \Rightarrow x_b = 0.1373\times 2\times (1.513\times 10^{-7}\times 10)^{1/2} = 3.3786\times 10^{-4} m
\]

\(x_b = 0.34 \text{ mm}\)

If \(T_s = 50°C\), 0.34 mm layer of tissue will suffer cell damage for a contact period of 20s.

If \(T_s = 100°C\), \(x_b = 2.36 \text{ mm}\)

The temperature history at a location \(x_b = 2 \text{ mm}\) can be plotted using the following Matlab program.

\[
\% \text{ Temperature history at } x = 2 \text{ mm from the skin surface maintained at 100 C}
\]
\[
t=1:30;
\]
\[
\text{tp}=[0 t];
\]
\[
\text{Ti}=37;\text{T}_s=100;
\]
\[
\alpha=1.513\times 10^{-7};
\]
\[
x=.002;
\]
\[
T=\text{T}_s+(\text{Ti}-\text{T}_s)\times \text{erf}(x./2\times \text{sqrt}(\alpha\times \text{t}));
\]
\[
\text{Tp}=[\text{Ti} \text{T}];
\]
\[
\text{plot(tp,Tp);grid on}
\]
\[
\text{xlabel('Time, t(s)');ylabel('Temperature, T(C)')}
\]
From Figure 7.3-5, the critical temperature of 48°C is reach within approximately 7 seconds at 2 mm from the skin surface where a temperature of 100°C is maintained.
Chapter 7

Unsteady State Conduction

7.4 Approximate Solutions

The summation in the series solution for transient conduction such as equation (7.3-4) can be terminated after the first term for $Fo > 0.2$. The full series solution is

$$\theta = \sum_{n=1}^{\infty} C_n \exp(-\zeta_n^2 Fo) \cos(\zeta_n x)$$ (7.3-4)

The first term approximation is

$$\theta' = C_1 \exp(-\zeta_1^2 Fo) \cos(\zeta_1 x)$$ (7.4-1)

where $C_1$ and $\zeta_1$ can be obtained from Table 7.4-1 for various value of Biot number. Table 7.4-2 lists the first term approximation for a slab, an infinite cylinder, and a sphere. Table 7.4-3 lists the solution for one-dimensional heat transfer in a semi-infinite medium for three different boundary conditions at the surface $x = 0$. Table 7.4-4 shows the combination of one-dimensional solutions to obtain the multi-dimensional results.

Total Energy Transfer

The total heat transfer $Q$ from a system to the ambience up to any $t$ in the transient process can be evaluated from the temperature distribution since

$$Q = E_i - E_t$$

where $E_i$ is the initial energy of the system, $E_i = \rho \c_p \int_V T_i dV$, and $E_t$ is the energy of the system at time $t$, $E_t = \rho \c_p \int_V T_t dV$. Therefore

$$Q = \rho \c_p \int_V (T_i - T_t) dV = \rho \c_p \int_V [(T_i - T_\infty) - (T - T_\infty)]dV$$

$$Q = \rho \c_p V(T_i - T_\infty) - \rho \c_p \int_V (T - T_\infty) dV$$

The maximum amount of energy transfer, $Q_0$, between the system and surrounding is $\rho \c_p V(T_i - T_\infty)$. Therefore

$$\frac{Q}{Q_0} = 1 - \frac{1}{V} \int_V \frac{(T - T_\infty)}{(T_i - T_\infty)} dV = 1 - \frac{1}{V} \int_V \theta^* dV$$

Using the approximate solution for heat transfer in a slab, we have
\[ \theta^* = \frac{T - T_\infty}{T_i - T_\infty} = C_1 \exp(-\zeta_1^2 Fo) \cos(\zeta_1 x) \]

\[ \frac{Q}{Q_0} = 1 - C_1 \exp(-\zeta_1^2 Fo) \frac{1}{L_A} \int_0^L \cos(\zeta_1 x^*) dx = 1 - C_1 \exp(-\zeta_1^2 Fo) \int_0^L \cos(\zeta_1 x^*) dx \]

\[ \frac{Q}{Q_0} = 1 - C_1 \exp(-\zeta_1^2 Fo) \left. \frac{\sin(\zeta_1 x^*)}{\zeta_1} \right|_0^L = 1 - C_1 \exp(-\zeta_1^2 Fo) \frac{\sin(\zeta_1)}{\zeta_1} \]

With the definition of \( \theta_0^* = C_1 \exp(-\zeta_1^2 Fo) \), we obtain the formula listed in Table 7.4-2

\[ \frac{Q}{Q_0} = 1 - \frac{\sin(\zeta_1)}{\zeta_1} \theta_0^* \]

For lumped capacitance system, \( T \) is not a function of position, therefore

\[ Q = \rho c_p \int_V (T_i - T) dV = \rho c_p V (T_i - T) = \rho c_p V [(T_i - T_\infty) - (T - T_\infty)] = \rho c_p V (\theta_i - \theta) \]

If the surrounding is convective with \( h \) and \( T_\infty \) then

\[ T = T_\infty + (T_i - T_\infty) e^{-t/\tau} \text{ or } \theta = \theta_i e^{-t/\tau} \]

The total amount of heat transfer up to any time \( t \) is then

\[ Q = \rho c_p V \theta_i (1 - e^{-t/\tau}) \]

The above formula can also be obtained from the relation

\[ Q = \int_0^t q \, dt = \int_0^t h A_s (T - T_\infty) \, dt = h A_s \int_0^t \theta_i e^{-t/\tau} \, dt = h A_s \theta_i \tau (1 - e^{-t/\tau}) \]

Since \( \tau = \frac{\rho V c_p}{h A_s} \), the final result for the total heat transfer in Lumped Capacitance system is

\[ Q = \rho c_p V \theta_i (1 - e^{-t/\tau}) \]
Table 7.4-1 Coefficients used in the one-term approximation to the series solutions for transient one-dimensional conduction or diffusion

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<th>( C_1 )</th>
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Table 7.4-2 Approximate solutions for diffusion and conduction (valid for \(Fo>0.2\))

\[
F_O = \frac{\alpha t}{L^2} = \frac{\alpha t}{r_0^2}, \quad \theta^* = \frac{T - T_\infty}{T_i - T_\infty}, \quad \theta_0^* = C_1 \exp(-\zeta^2 F_O)
\]

**Conduction in a slab**

\(L\) is defined as the distance from the center of the slab to the surface. If one surface is insulated, \(L\) is defined as the total thickness of the slab.

\[
\theta^* = \theta_0^* \cos(\zeta_1 x^*); \quad \frac{Q}{Q_o} = 1 - \frac{\sin(\zeta_1)}{\zeta_1} \theta_0^*
\]

**Conduction in an infinite cylinder**

\[
\theta^* = \theta_0^* J_0(\zeta_1 r^*); \quad \frac{Q}{Q_o} = 1 - \frac{2\theta_0^*}{\zeta_1} J_1(\zeta_1)
\]

**Conduction in a sphere**

\[
\theta^* = \frac{1}{\zeta_1 r^*} \theta_0^* \sin(\zeta_1 r^*); \quad \frac{Q}{Q_o} = 1 - \frac{3\theta_0^*}{\zeta_1^3} \left[ \sin(\zeta_1) - \zeta_1 \cos(\zeta_1) \right]
\]

If the temperature at the surface \(T_s\) is known \(T_\infty\) will be replaced by \(T_s\) and \(C_1\) will be obtained from table at \(Bi = \infty\)

---

**Table 7.4-3 Semi-infinite medium**

**Constant Surface Temperature:** \(T(0, t) = T_s\)

\[
\frac{T(x, t) - T_s}{T_i - T_s} = \text{erf} \left( \frac{x}{2\sqrt{\alpha t}} \right); \quad q_s^* (t) = \frac{k(T_s - T_i)}{\sqrt{\pi \alpha t}}
\]

**Constant Surface Heat Flux:** \(q_s^* = q_0^*\)

\[
T(x, t) - T_i = \frac{2q_s^* (\alpha t / \pi)^{1/2}}{k} \exp \left( -\frac{x^2}{4\alpha t} \right) - \frac{q_s^* x}{k} \text{erfc} \left( \frac{x}{2\sqrt{\alpha t}} \right)
\]

The complementary error function, \(\text{erfc}(w)\), is defined as \(\text{erfc}(w) = 1 - \text{erf}(w)\)

**Surface Convection:** \(- \frac{k \partial T}{\partial x} \bigg|_{x=0} = h[T_\infty - T(0, t)]\)

\[
\frac{T(x, t) - T_s}{T_\infty - T_s} = \text{erfc} \left( \frac{x}{2\sqrt{\alpha t}} \right) - \left[ \exp \left( \frac{hx + h^2 \alpha t}{k^2} \right) \right] \left[ \text{erfc} \left( \frac{x}{2\sqrt{\alpha t}} + \frac{h \sqrt{\alpha t}}{k} \right) \right]
\]
Table 7.4-4 Multidimensional Effects

The temperature profiles for a finite cylinder and a parallelepiped can be obtained from the temperature profiles of infinite cylinder and slabs.

\[
\text{[ finite cylinder ]} = \text{[ infinite cylinder ]} \times \text{[ slab } 2L \text{ ]}
\]

\[
\text{[ parallelepiped ]} = \text{[ slab } 2L_1 \text{ ]} \times \text{[ slab } 2L_2 \text{ ]} \times \text{[ slab } 2L_3 \text{ ]}
\]

\[
S(x, t) \equiv \frac{T(x,t) - T_\infty}{T_i - T_\infty} \quad \text{Semi–infinite solid}
\]

\[
P(x, t) \equiv \frac{T(x,t) - T_\infty}{T_i - T_\infty} \quad \text{Plane wall}
\]

\[
C(r, t) \equiv \frac{T(r,t) - T_\infty}{T_i - T_\infty} \quad \text{Infinite cylinder}
\]
Example 7.4-1

A slab with a thickness of 0.050 m is at an initial temperature of 25°C. The slab is heated by passing a hot gas over its surfaces, with the gas temperature and the convection coefficient assumed to have constant values of $T_\infty = 600$°C and $h = 200$ W/m²°K. Slab is made from a materials with $k = 0.50$ W/m°K and $\alpha = 3.5 \times 10^{-7}$ m²/s.

1) Determine the time required to achieve 75% of the maximum possible energy transfer.

2) Determine the highest and lowest temperatures in the slab after 30 minutes.

Solution

![Figure 7.4-1 One-dimensional unsteady heat transfer in a slab.](image)

1) We use the approximate solution listed in Table 7.4-2 for heat transfer in a slab.

$$\frac{Q}{Q_o} = 1 - \frac{\sin(\zeta_1)}{\zeta_1} \theta_0$$

For 75% of the maximum possible energy transfer $\frac{Q}{Q_o} = 0.75$. Heat transfer occurs on both sides of the slab, therefore $2L = 0.05$ m or $L = 0.025$ m.

$$Bi = \frac{hL}{k} = \frac{200 \times 0.025}{0.5} = 10$$
Chapter 7

Unsteady State Conduction

From Table 7.4-1, $\zeta_1 = 1.4289$, $C_1 = 1.262$, $\theta_0^*$ can now be solved

$$\theta_0^* = \frac{0.25\zeta_1}{\sin(\zeta_1)} = 0.36085 = C_1\exp(-\zeta_1^2 Fo)$$

$$Fo = -\frac{\ln(\theta_0^*/C_1)}{\zeta_1^2} = 0.61319$$

Since $Fo > 0.2$, the approximate solution is valid. The time required to achieve 75% of the maximum possible energy transfer is then

$$Fo = \frac{\alpha t}{L^2} \Rightarrow t = \frac{0.61319 \times 0.025^2}{3.5 \times 10^{-7}} = 1095 \text{ s}$$

2) Since heat is transferred into the slab, the highest temperature at any time is at the surface where $x^* = 1$ and the lowest temperature is at the center where $x^* = 0$.

At $t = 30 \text{ min} = 1800 \text{ s} \Rightarrow Fo = \frac{\alpha t}{L^2} = \frac{3.5 \times 10^{-7} \times 1800}{0.025^2} = 1.008$

$$\theta_0^* = \frac{T(x^* = 0, t) - T_x}{T_i - T_x} = C_1\exp(-\zeta_1^2 Fo) = 1.262 \exp(-1.4289^2 \times 1.008)$$

$$\theta_0^* = 0.16115$$

$$T(x^* = 0, 30 \text{ min}) = 600 + 0.16115 \times (25 - 600) = 507.3^\circ \text{C}$$

$$\theta^* = \theta_0^* \cos(\zeta_1 x^*) \Rightarrow \frac{T(x^* = 1, t) - T_x}{T_i - T_x} = 0.16115 \times \cos(1.4289) = 0.022791$$

$$T(x^* = 1, 30 \text{ min}) = 600 + 0.022791 \times (25 - 600) = 586.9^\circ \text{C}$$
Example 7.4-2

Asphalt pavement may achieve temperatures as high as 50°C on a hot summer day. Assume that such a temperature exists throughout the pavement, when suddenly a rainstorm reduces the surface temperature to 20°C. Calculate the total amount of energy (J/m²) that will be transferred from the asphalt over a 30-min period in which the surface is maintained at 20°C. Asphalt: \( k = 0.062 \text{ W/m·K} \), \( \alpha = 3.2 \times 10^{-8} \text{ m}^2/\text{s} \).

Solution

Asphalt pavement may be treated as a semi-infinite medium with constant surface temperature. The formula for heat flux is given from Table 7.4–3 as

\[
q_s(t) = \frac{k(T_s - T_i)}{\sqrt{\pi \alpha t}}
\]

This formula assumes heat transfer into the solid as indicated by the \( x \) direction shown in Figure 7.4-2. The total amount of energy \( Q \) (J/m²) that will be transferred from the asphalt over a 30-min period is then

\[
Q = \int_0^t q_s(t)dt = \int_0^t \frac{k(T_s - T_i)}{\sqrt{\pi \alpha t}} dt = \frac{2k(T_s - T_i)}{\sqrt{\pi \alpha}} t^{1/2}
\]

\[
Q = \frac{2 \times 0.062 \times (20 - 50)}{\sqrt{\pi \times 3.2 \times 10^{-8}}} \times 1800^2 = -4.98 \times 10^5 \text{ J/m}^2
\]

The minus sign indicates heat transfers in the negative \( x \) direction.
Example 7.4-3

A one-dimensional plane wall with a thickness of 0.4 m initially at a uniform temperature of 250°C is suddenly immersed in an oil bath at 30°C. One side of the wall is insulated and the heat transfer coefficient for the other side is 500 W/m²·K. The properties of the wall are $k = 50$ W/m·K, density $= 7835$ kg/m³, and $C_p = 465$ J/kg·K. Determine the temperature at the center of the wall after 60 minutes.

Solution

Since heat transfers through only one side of the slab, $L$ is the thickness of the slab: $L = 0.4$ m. Using the approximate solution we have

$$
\theta^* = C_1 \exp(-\zeta_1^2 Fo) \cos(\zeta_1 x^*)
$$

$$
Bi = \frac{hL}{k} = \frac{500 \times 0.4}{50} = 4
$$

From Table 7.4-1, $\zeta_1 = 1.2646$, $C_1 = 1.2287$.

$$
\alpha = \frac{k}{\rho C_p} = \frac{50}{7835 \times 465} = 1.3724 \times 10^{-5} \text{ m}^2/\text{s}
$$

$$
Fo = \frac{\alpha L}{L^2} = \frac{1.3724 \times 10^{-5} - 5 \times 60 \times 60}{.4^2} = 0.3088
$$

Since $Fo > 0.2$, the approximate solution is valid. At the center of the plate $x^* = 0.5$, therefore
\[ T(x^* = 0.5, t) - T_x = \theta^* = C_1 \exp(-\zeta_1^2 Fo) \cos(\zeta_1 x^*) \]

\[ T(x^* = 0.5, t) - T_x = 1.2287 \exp(-1.2646^2 \times 0.3088) \cos(1.2646 \times 0.5) = 0.6049 \]

\[ T(x^* = 0.5, 60 \text{ min}) = 30 + 0.6049 \times (250 - 30) = 163.1^\circ \text{C} \]

**Example 7.4-4**

Two large blocks of different materials, such as copper and concrete, have been sitting in a room at 23°C for a long time. When you touch the concrete block your heat loss is 0.8 W. Estimate the heat loss when you touch the copper block. Assume your hand temperature is constant at 37°C.

Data:

<table>
<thead>
<tr>
<th>Materials</th>
<th>( k ) (W/m·K)</th>
<th>( \alpha ) (m²/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Copper</td>
<td>401</td>
<td>1.166 × 10⁻⁴</td>
</tr>
<tr>
<td>Concrete</td>
<td>1.4</td>
<td>6.917 × 10⁻⁷</td>
</tr>
</tbody>
</table>

**Solution**

Large block of material may be treated as a semi-infinite medium with constant surface temperature boundary condition. The formula for heat flux is given from Table 7.4 -3 as

\[ q_x(t) = \frac{k(T_s - T_i)}{\sqrt{\pi \alpha t}} \]

Let subscript 1 denotes copper and subscript 2 denotes concrete. The ratio of heat loss is then

\[ \frac{q_1}{q_2} = \frac{k_1(T_s - T_i)}{\sqrt{\pi \alpha_1 t}} \frac{\sqrt{\pi \alpha_2 t}}{k_2(T_s - T_i)} = \frac{k_1}{k_2} \left( \frac{\alpha_2}{\alpha_1} \right)^{1/2} \]
The heat loss when you touch the copper block is

\[ q_1 = (0.8) \frac{401}{1.4} \left( \frac{6.917 \times 10^{-7}}{1.166 \times 10^{-4}} \right)^{1/2} = 18 \text{ W} \]

**Example 7.4-5**

A short cylinder with height of 0.080 m and radius of 0.030 m is at an initial temperature of 40°C. The slab is plunged into fluid with \( h = 300 \text{ W/m}^2\text{oK} \) and \( T_\infty = 200\text{°C} \). Cylinder is made from a materials with \( k = 26 \text{ W/m-K} \) and \( \alpha = 8.6 \times 10^{-6} \text{ m}^2/\text{s} \). Determine the center temperature of the cylinder after 5 minutes.

**Solution**

![Figure 7.4-4 Heat transfer in a finite cylinder.](image)

For heat transfer in a short cylinder, temperature is a function of \( r, x, \) and \( t \). The differential equation

\[ \frac{1}{\alpha} \frac{\partial T}{\partial t} = \nabla^2 T \]
in cylindrical coordinate with dimensionless temperature \( \theta = \frac{T - T_{x}}{T_{i} - T_{w}} \) is given as

\[
\frac{1}{\alpha} \frac{\partial \theta^*}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \theta^*}{\partial r} \right) + \frac{\partial^2 \theta^*}{\partial x^2}
\]

The differential equation can be solved with the following initial and boundary conditions

**I. C.** \( t = 0, \ \theta^* = 1 \)

**B. C. on** \( x: \) \( x = 0, \ \left. \frac{\partial \theta^*}{\partial x} \right|_{x=0} = 0; \ \ x = L, \ \left. \frac{\partial \theta^*}{\partial x} \right|_{x=L} = -\frac{h}{k} \theta(x = L, r, t) \)

**B. C. on** \( r: \) \( r = 0, \ \left. \frac{\partial \theta^*}{\partial r} \right|_{r=0} = 0; \ \ r = R, \ \left. \frac{\partial \theta^*}{\partial r} \right|_{r=R} = -\frac{h}{k} \theta(x, r = R, t) \)

We assume that the solution can be separated into a product of functions \( P(x, t) \) and \( C(r, t) \) where

\[
P(x, t) \equiv \frac{T(x, t) - T_{x}}{T_{i} - T_{w}} \quad \text{and} \quad C(r, t) \equiv \frac{T(r, t) - T_{x}}{T_{i} - T_{w}}
\]

\( \theta(x, r, t) = P(x, t) \times C(r, t) \)

Taking the partial derivative of \( \theta(x, r, t) \) with respect to each of the independent variable we obtain

\[
\frac{\partial \theta^*}{\partial t} = P(x, t) \frac{\partial C}{\partial t} + C(r, t) \frac{\partial P}{\partial t}
\]

\[
\frac{\partial \theta^*}{\partial x} = C \frac{\partial P}{\partial x} \Rightarrow \frac{\partial^2 \theta^*}{\partial x^2} = C \frac{\partial^2 P}{\partial x^2}
\]

\[
\frac{\partial \theta^*}{\partial r} = P \frac{\partial C}{\partial r} \Rightarrow \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \theta^*}{\partial r} \right) = P \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial C}{\partial r} \right)
\]

In terms of the new variables, the differential conduction equation becomes

\[
\frac{1}{\alpha} P(x, t) \frac{\partial C}{\partial t} + \frac{1}{\alpha} C(r, t) \frac{\partial P}{\partial t} = P \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial C}{\partial r} \right) + C \frac{\partial^2 P}{\partial x^2}
\]
Dividing the above equation by $PC$ and rearranging we obtain

$$\frac{1}{C} \left[ \frac{1}{\alpha} \frac{\partial C}{\partial t} - \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial C}{\partial r} \right) \right] + \frac{1}{P} \left[ \frac{1}{\alpha} \frac{\partial P}{\partial t} - \frac{\partial^2 P}{\partial x^2} \right] = 0$$

The differential equation is satisfied if

$$\frac{1}{\alpha} \frac{\partial P}{\partial t} - \frac{\partial^2 P}{\partial x^2} = 0 \quad \text{and} \quad \frac{1}{\alpha} \frac{\partial C}{\partial t} - \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial C}{\partial r} \right) = 0$$

The differential equation in $P$ can be transformed into dimensionless form

$$\frac{1}{\alpha} \frac{\partial P}{\partial t} = \frac{\partial^2 P}{\partial x^2} \Rightarrow \frac{\partial P}{\partial t} = \frac{\partial^2 P}{\partial x^2}$$

Similarly, the initial and boundary conditions can be transformed into dimensionless forms

$$P(x^*, 0) = 1$$

$$\left. \frac{\partial P}{\partial x^*} \right|_{x^*=0} = 0; \quad \left. \frac{\partial P}{\partial x^*} \right|_{x^*=1} = -Bi^*P(1, t^*), \text{ where } Bi = \frac{hL}{k}$$

Since the differential equation and its initial and boundary conditions are the same as those of heat transfer in a slab, the solution must be the same.

$$P = \sum_{n=1}^{\infty} C_n \exp(-\zeta_n^2 Fo) \cos(\zeta_n x^*)$$ \hspace{1cm} (7.3-4)

The differential equation in $C$ is

$$\frac{1}{\alpha} \frac{\partial C}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial C}{\partial r} \right)$$

with the following initial and boundary conditions

I. C. \quad $t = 0, \ C(r, 0) = 1$

B. C. \quad $r = 0, \left. \frac{\partial C}{\partial r} \right|_{r=0} = 0; \quad r = R, \left. \frac{\partial C}{\partial r} \right|_{r=R} = -Bi^*C(1, t^*), \text{ where } Bi = \frac{hR}{k}$

The solution for $C$ is the same as that of an infinite cylinder given as

$$C = \sum_{n=1}^{\infty} C_n \exp(-\zeta_n^2 Fo) J_0(\zeta_n x^*)$$ \hspace{1cm} (7.3-7)
In summary, the solution for a finite cylinder with radius $R$ and height $2L$ may be obtained from the product of the solution for an infinite cylinder with radius $R$ and the solution for a slab with thickness $2L$. The solution for many multi-dimensional systems may be obtained from the product solutions of one-dimensional systems as listed in Table 7.4-4.

We now get back to the numerical example:

A short cylinder with height of 0.080 m and radius of 0.030 m is at an initial temperature of 40°C. The slab is plunged into fluid with $h = 300$ W/m²·K and $T_\infty = 200$°C. Cylinder is made from a materials with $k = 26$ W/m·°K and $\alpha = 8.6 \times 10^{-6}$ m²/s. Determine the center temperature of the cylinder after 5 minutes.

For the slab: $2L = 0.08$ m or $L = 0.04$ m.

$$Bi = \frac{hL}{k} = \frac{300 \times 0.04}{26} = 0.4615$$

$$Fo = \frac{\alpha t}{L^2} = \frac{8.6 \times 10^{-6} \times 5 \times 60}{.04^2} = 1.6125$$

From Table 7.4-1, $\zeta_1 = 0.6313$, $C_1 = 1.0656$. At the center $x^* = 0$.

$$P = C_1 \exp(-\zeta_1^2 Fo) \cos(\zeta_1 x^*) = 0.560$$

For the infinite cylinder

$$Bi = \frac{hR}{k} = \frac{300 \times 0.03}{26} = 0.3462$$

$$Fo = \frac{\alpha t}{R^2} = \frac{8.6 \times 10^{-6} \times 5 \times 60}{.03^2} = 2.8667$$

From Table 7.4-1, $\zeta_1 = 0.7974$, $C_1 = 1.0814$. At the center $r^* = 0$.

$$C = C_1 \exp(-\zeta_1^2 Fo) J_0(\zeta_1 r^*) = 0.1748$$

$$\theta(x, r, t) = \frac{T - T_\infty}{T_i - T_\infty} = P(x, t) \times C(r, t) = 0.560 \times 0.1748 = 0.0979$$

$$T = 200 + 0.0979 \times (40 - 200) = 184.3°$$

$7-34$
Chapter 8

Convective Heat Transfer

8.1 Introduction

Convective heat transfer is usually classified as forced convection or free convection. Forced convection occurs when the flow is caused by an external means, such as a fan or blower, a compressor, a pump, or atmospheric winds. Free convection occurs when the flow is caused only by the density differences due to temperature variation in the fluid. It should be noted that free convection also exists in forced convection, however the contribution of free convection in this situation is negligible. Convection can occur with or without a phase change. When there is no change in phase, the energy that is being transferred is the sensible energy of the fluid. When a phase change occurs such as boiling or condensation there is an additional heat exchange due to the latent heat of the fluid from the change in physical molecular bonds.

8.2 Boundary Layer Concept

Consider a flow over a flat plate where the free stream velocity is $u_\infty$. The fluid flow can be divided into two regions: a velocity boundary layer region next to the solid surface in which momentum transfer exists and a region outside the boundary layer in which momentum transfer is negligible or the viscosity of the fluid can be considered to be zero.

The thickness of the velocity boundary layer $\delta$ may be defined as the distance from the surface of the plate to the location where the velocity in the $x$ direction $u$ is within 1% of the free stream velocity $u_\infty$ or $u/u_\infty = 0.99$. The equations used to solve for the steady-state velocity profile within the boundary layer are given as

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad \text{(Continuity)} \tag{8.2-1}
\]

\[
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2}, \quad \text{(x-momentum)} \tag{8.2-2}
\]
\( \nu \) is the velocity in the \( y \) direction. The energy equation for convection heat transfer requires the velocity therefore the above two equations must be solved for \( u \) and \( \nu \). \( \frac{\partial p}{\partial x} \) is a known quantity since within the boundary layer \( \frac{\partial p}{\partial y} = 0 \) due to the assumption that the thickness of the boundary layer \( \delta \) is small. Therefore \( \frac{\partial p}{\partial x} = \frac{dp}{dx} \) which is a known quantity from solving the momentum equation outside the boundary layer where viscosity is equal to zero.

A thermal boundary layer also exists when the fluid flows over a surface if the fluid free stream temperature \( T_\infty \) is not the same as the surface temperatures \( T_s \). Heat transfer is significant within the thermal boundary layer region.

![Figure 8.2-2 Thermal boundary layer on an isothermal flat plate [1].](image)

The thickness of the thermal boundary layer \( \delta_t \) may be defined as the distance from the surface of the plate to the location where the fluid temperature \( T \) is within 1\% of the free stream temperature \( T_\infty \) or

\[
\frac{T_s - T}{T_\infty - T_\infty} = 0.99
\]

The thickness of the velocity boundary layer is normally not the same as the thickness of the thermal boundary layer: \( \delta \neq \delta_t \) in general. The equation used to solve for the steady-state temperature profile within the boundary layer is given as

\[
u \frac{\partial T}{\partial x} + \nu \frac{\partial T}{\partial y} = \alpha \left( \frac{\partial^2 T}{\partial y^2} + \frac{\nu}{C_p} \left( \frac{\partial u}{\partial y} \right)^2 \right)
\] (8.2-3)

The left hand side of equation (8.2-3) denotes the convection heat transfer, the first term on the right hand side denotes conduction heat transfer, and the second term denotes heat generated by viscous dissipation.

For mass transfer, the equation used to solve for the steady-state concentration profile within the boundary layer is given as

\[
u \frac{\partial C_A}{\partial x} + \nu \frac{\partial C_A}{\partial y} = D_{AB} \frac{\partial^2 C_A}{\partial y^2}
\] (8.2-4)
Equation (8.2-4) is the species continuity or convective-diffusion equation. The left hand side of equation (8.2-4) denotes the convection mass transfer and the right hand side denotes diffusion heat transfer. The conservation equations (8.2-2)-(8.2-4) can be rearranged into dimensionless forms with the following dimensionless variables

\[ x^* = x/L, \quad y^* = y/L, \quad u^* = u/u_\infty, \quad v^* = v/u_\infty, \quad p^* = p/(\rho u_\infty^2), \quad T^* = \frac{T - T_s}{T_\infty - T_s}, \quad C_A^* = \frac{C_A - C_{A,s}}{C_{A,\infty} - C_{A,s}} \]

Neglecting the viscous dissipation term, the boundary layer equations in terms of the dimensionless variables are

\[ u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{dp^*}{dx^*} + \nu \frac{\partial^2 u^*}{\partial y^*^2} \quad (8.2-5) \]
\[ u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} = \frac{\alpha}{u_\infty L} \frac{\partial^2 T^*}{\partial y^*^2} \quad (8.2-6) \]
\[ u^* \frac{\partial C_A^*}{\partial x^*} + v^* \frac{\partial C_A^*}{\partial y^*} = \frac{D_{AB}}{u_\infty L} \frac{\partial^2 C_A^*}{\partial y^*^2} \quad (8.2-7) \]

We can use the following dimensionless numbers to further rearrange the conservation equations

\[ Re_L = \text{Reynold number} = \frac{u_\infty L}{\nu}, \]
\[ Pr = \text{Prandtl number} = \frac{\nu}{\alpha}, \]
\[ Sc = \text{Schmidt number} = \frac{\nu}{D_{AB}} \]
\[ \frac{\alpha}{u_\infty L} = \frac{\nu}{\alpha} = \frac{1}{Re_L Pr}, \]
\[ \frac{D_{AB}}{u_\infty L} = \frac{\nu}{u_\infty L} \frac{D_{AB}}{\nu} = \frac{1}{Re_L Sc} \]

\[ u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{dp^*}{dx^*} + \frac{1}{Re_L} \frac{\partial^2 u^*}{\partial y^*^2} \quad (8.2-8) \]

The dimensionless velocity depends only on the variables and dimensionless number contained within the equation, therefore

\[ u^* = f_1(x^*, y^*, Re_L, \frac{dp^*}{dx^*}) \]

The shear stress at the solid surface is given as
\[ \tau_s = \mu \frac{\partial u}{\partial y} (y = 0) = \frac{\mu}{u_sL} \frac{\partial u^*}{\partial y^*} (y^* = 0) \]

The friction coefficient is defined as the ratio of the surface shear stress to the kinetic energy

\[ C_f = \frac{\tau_s}{\rho u_s^2/2} = \frac{2}{Re_L} \frac{\partial u^*}{\partial y^*} (y^* = 0) \]

The velocity gradient at the surface \( \frac{\partial u^*}{\partial y^*} (y^* = 0) \) does not depend on \( y^* \) since \( y^* \) is already specified (\( y^* = 0 \)).

\[ \frac{\partial u^*}{\partial y^*} (y^* = 0) = f_2 (x^*, Re_L, \frac{dp^*}{dx^*}) \]

In this expression \( p^*(x) \) depends on the surface geometry and may be obtained from a separate consideration of flow conditions. For a prescribed geometry

\[ C_f = \frac{2}{Re_L} f(x^*, Re_L) \]

The energy equation in terms of the dimensionless numbers becomes

\[ u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} = \frac{1}{Re_L Pr} \frac{\partial^2 T^*}{\partial y^*^2} \]

Therefore, the dimensionless temperature is a function given by

\[ T^* = f_3 (x^*, y^*, Re_L, Pr, \frac{dp^*}{dx^*}) \]

\( T^* \) depends on \( \frac{dp^*}{dx^*} \) since \( u^* \) and \( v^* \) depend on \( \frac{dp^*}{dx^*} \). The heat transfer coefficient \( h \) can then be obtained from the energy balance at the surface

\[ h(T_s - T_\infty) = - k_f \frac{\partial T}{\partial y} (y = 0) \]

Since \( T^* = \frac{T - T_\infty}{T_s - T_\infty} \), the heat transfer coefficient in terms of the dimensionless variables is

\[ h = - \frac{k_f}{L} \frac{(T_s - T_\infty) \frac{\partial T^*}{\partial y^*} (y^* = 0)}{L(T_s - T_\infty) \frac{\partial y^*}{\partial y^*} (y^* = 0)} \]
The heat transfer coefficient is usually expressed in terms of Nusselt \( Nu \) number that is essentially a dimensionless heat transfer coefficient.

\[
Nu = \frac{hL}{k_f} = \frac{\partial T^*}{\partial y^*}(y^* = 0).
\]

For a prescribed geometry

\[
Nu = f_4(x^*, Re_L, Pr)
\]

The average Nusselt number is defined as

\[
\overline{Nu} = \frac{\overline{hL}}{k_f} = f_5(Re_L, Pr)
\]

The average Nusselt number does not depend on \( x^* \) since it is the average property over the surface area. Once the local heat transfer coefficient \( h \) is known, the heat transfer between the surface and the fluid is calculated from

\[
q = \int_{A_s} h(T_s - T_\infty) dA_s
\]

In the special case where \( T_s \) is a constant

\[
q = (T_s - T_\infty) \int_{A_s} h dA_s
\]

The average heat transfer coefficient \( \overline{h} \) is defined as \( \overline{h} = \frac{1}{A_s} \int_{A_s} h dA_s \) so that the heat transfer \( q \) can also be calculated from

\[
q = \overline{h} A_s(T_s - T_\infty)
\]

Similarly, the mass transfer coefficient \( h_m \) is usually expressed in terms of Sherwood \( Sh \) number that is essentially a dimensionless mass transfer coefficient

\[
Sh = f_6(x^*, Re_L, Sc)
\]

The average Sherwood number is defined over the surface area

\[
\overline{Sh} = \frac{h_mL}{D_{4B}} = f_7(Re_L, Sc)
\]

Once the local mass transfer coefficient \( h_m \) is known, the mole transfer of species \( A \) between the surface and the fluid is calculated from
\[ N_A = \int_{A_s} h_m (C_{A,s} - C_{A,\infty}) dA_s \]

\( N_A \) is the moles of \( A \) transferred per unit time. In the special case where \( C_{A,s} \) is a constant

\[ N_A = (C_{A,s} - C_{A,\infty}) \int_{t_0} h_m dA_s \]

The average mass heat transfer coefficient \( \bar{h}_m \) is defined as

\[ \bar{h}_m = \frac{1}{A_s} \int_{t_0} h_m dA_s \]

so that the mass transfer \( N_A \) can also be calculated from

\[ N_A = \bar{h}_m A_s (C_{A,s} - C_{A,\infty}) \]

In general the momentum, heat, and mass boundary thickness are not the same. They are related by the following expressions

\[ \frac{\delta}{\delta_t} \approx Pr^n \text{ and } \frac{\delta}{\delta_c} \approx Sc^n \]

The dimensional analysis of the heat and mass conservation equations indicates that

\[ Nu = f_4(x^*, Re_L, Pr) \text{ and } Sh = f_6(x^*, Re_L, Sc) \]

However experimental data suggest that the dependence of \( Nu \) on \( Pr \) and the dependence of \( Sh \) on \( Sc \) can be factored out so that

\[ Nu = f(x^*, Re_L) Pr^n \]

\[ Sh = f(x^*, Re_L) Sc^n \]

The above formulas indicate that in forced convection Reynolds number is the driving force for the transfer. Heat transfer also depends on Prandtl number and mass transfer depends on Schmidt number.
Chapter 8

Convection Heat Transfer

8.3 Correlations for Heat Convection

Many correlations are available for the estimation of the heat transfer coefficients. In this section we will outline a procedure that may be used to find the formula that is most suitable for a given system. The focus is on the application of the formula, therefore only a few basic formulas will be listed. You will need to find in the literatures the formula that is best for your system if necessary. Table 8.3-1 lists the outline that may be used to find the correlation needed for a given system.

---------- Table 8.3-1 Outline of heat transfer correlations available. ----------

External forced convection flow
   Geometry of the flow: flat plate, cylinder, sphere, ...
   Laminar or turbulence

Internal forced convection flow
   Geometry of the flow: Circular or noncircular tubes.
   Laminar or turbulence

External free convection flow
   Geometry of the flow: flat plate, cylinder, sphere, ...
   Laminar or turbulence

Free convection within enclosure
   Geometry of the flow: rectangular cavities, concentric cylinders, or concentric spheres.

Boiling
   Pool boiling: nucleate or film boiling
   Forced convection boiling: external forced convection boiling or two-phase flow.

Condensation
   Film condensation: geometry of the system: vertical plate, horizontal tube,...
   Dropwise condensation

Once the correlation is found that is valid for the conditions and ranges of the dimensionless numbers required for your system, you will need to know at what temperature, pressure, and/or concentration the physical properties are needed.
External forced convection flow

Table 8.3-2 lists some correlations to determine the heat transfer coefficient for external forced convection flow. The expressions for the flat plate are obtained from the solutions of the boundary layer equations. The other formulas are experimental correlations.

<table>
<thead>
<tr>
<th>Correlations</th>
<th>Geometry</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \overline{Nu}_x = 0.332 Re_x^{1/2} Pr^{1/3} )</td>
<td>Flat plate</td>
<td>Laminar, local, ( T_f, 0.6 \leq Pr \leq 50 )</td>
</tr>
<tr>
<td>( \overline{Nu}_x = 0.664 Re_x^{1/2} Pr^{1/3} )</td>
<td>Flat plate</td>
<td>Laminar, average, ( T_f, 0.6 \leq Pr \leq 50 )</td>
</tr>
<tr>
<td>( Nu_x = 0.0296 Re_x^{4/5} Pr^{1/3} )</td>
<td>Flat plate</td>
<td>Turbulent, local, ( T_f, Re_x \leq 10^8, 0.6 \leq Pr \leq 50 )</td>
</tr>
<tr>
<td>( \overline{Nu}_x = (0.037 Re_x^{4/5} - 871) Pr^{1/3} )</td>
<td>Flat plate</td>
<td>Mixed, average, ( T_f, Re_{x,c} = 5 \times 10^5 )</td>
</tr>
<tr>
<td>( \overline{Nu}_D = 0.3 + [0.62 Re_D^{1/2} Pr^{1/3}] [1 + (0.4/Pr)^{3/4}] \times [1 + (Re_D/282,000)^{5/8}]^{4/5} )</td>
<td>Cylinder</td>
<td>Average, ( T_f, Re_D Pr &gt; 0.2 )</td>
</tr>
<tr>
<td>( \overline{Nu}_D = 2 + (0.4 Re_D^{1/2} + 0.06 Re_D^{2/3} Pr^{0.4}) \times (\mu/\mu_s)^{1/4} )</td>
<td>Sphere</td>
<td>Average, ( T_x, 3.5 &lt; Re_D &lt; 7.6 \times 10^4, 0.71 &lt; Pr &lt; 380, 1.0 &lt; (\mu/\mu_s) &lt; 3.2 )</td>
</tr>
</tbody>
</table>

The expression for the flat plate

\[
Nu_x = 0.332 Re_x^{1/2} Pr^{1/3}
\]

provides the local heat transfer coefficient \( h \) for laminar flow at a location \( x \) from the leading edge of the flat plate. The Reynolds number \( Re_x \) is defined as

\[
Re_x = \frac{u_x x}{v}
\]

The critical Reynolds number, \( Re_{x,c} \), for transition to turbulent flow is taken to be \( 5 \times 10^5 \) unless specified otherwise from the problem. The physical properties should be evaluated at the film temperature \( T_f \) defined as \( T_f = 0.5(T_s + T_\infty) \).

The formula

\[
\overline{Nu}_x = 0.664 Re_x^{1/2} Pr^{1/3}
\]
provides an average heat transfer coefficient for the heat transfer from the leading edge to position $x$.

The formula

$$Nu_x = 0.0296 Re_x^{4/5} Pr^{1/3}$$

provides the local heat transfer coefficient $h$ for turbulent flow at a location $x$ from the leading edge of the flat plate.

The formula

$$\overline{Nu}_x = (0.037 Re_x^{4/5} - 871) Pr^{1/3}$$

provides a mixed average heat transfer coefficient for the heat transfer from the leading edge to position $x$. The first part of the flat plate has laminar flow up to position $x_c$ determined from

$$5 \times 10^5 = \frac{u_x x_c}{v}$$

the later part from $x_c$ to $x$ has turbulent flow. If the transition from laminar to turbulent flow occurs at a critical Reynolds number different than $5 \times 10^5$ the mixed average can be determined from the expression

$$\overline{h}_x = \frac{1}{x} \left( \int_0^x h_{\text{lam}} \, dx + \int_{x_c}^x h_{\text{turb}} \, dx \right)$$

Substituting $h_{\text{lam}} = \frac{k}{x} x 0.332 Re_x^{1/2} Pr^{1/3}$ and $h_{\text{turb}} = \frac{k}{x} 0.0296 Re_x^{4/5} Pr^{1/3}$ we obtain

$$\overline{h}_x = \frac{k}{x} \left[ 0.332 \left( \frac{u_x}{v} \right)^{1/2} \int_0^{x_c} \frac{dx}{x^{1/2}} + 0.0296 \left( \frac{u_x}{v} \right)^{4/5} \int_{x_c}^x \frac{dx}{x^{1/3}} \right]$$

Integrating, we then obtain

$$\overline{Nu}_x = [0.664 Re_{x,c}^{1/2} + 0.037 (Re_x^{4/5} - Re_{x,c}^{1/2})] Pr^{1/3}$$

If $Re_{x,c} = 500,000$, then

$$\overline{Nu}_x = (0.037 Re_L^{4/5} - 871) Pr^{1/3}$$

The formula $\overline{Nu}_D = 2 + (0.4 Re_D^{1/2} + 0.06 Re_D^{2/3}) Pr^{0.4} (\mu/\mu_s)^{1/4}$ provides an average heat transfer coefficient for heat transfer to or from a sphere where all the properties except $\mu_s$ are to be evaluated at the fluid temperature $T_\infty$. Therefore $\mu$ is evaluated at $T_\infty$ while $\mu_s$ is evaluated at $T_s$. 
Internal forced convection flow

Table 8.3-3 listed some correlations for forced convection flow in a circular tube.

--------------- Table 8.3-3 Correlations for forced convection flow in a circular tube ---------------

<table>
<thead>
<tr>
<th>Correlation</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \overline{Nu_D} = 3.66 + 0.0499 \frac{Re_D Pr}{(L/D)} )</td>
<td>Laminar, Entrance Effect, ( T_s ) uniform ( 0.03 Re_D Pr &lt; L/D ) ( \text{Properties at} \ 0.5(T_{m,i} + T_{m,o}) )</td>
</tr>
<tr>
<td>( \overline{Nu_D} = 0.023 Re_D^{4/5} Pr^n )</td>
<td>Turbulent, fully developed, ( T_s ) uniform ( 0.7 &lt; Pr &lt; 160 ) ( n = 0.3 ) for ( T_s &lt; T_m ), ( n = 0.4 ) for ( T_s &gt; T_m ) ( \text{Properties at} \ T_m )</td>
</tr>
<tr>
<td>( (\overline{Nu_D})^{10} = (\overline{Nu_{D,l}})^{10} + \left( \frac{\exp[(2,200 - Re_D)/365]}{(\overline{Nu_{D,l}})^2} + \frac{1}{(\overline{Nu_{D,t}})^5} \right) )</td>
<td>Transitional flow ( 2,300 &lt; Re_D &lt; 10,000 ) ( \overline{Nu_{D,l}} ) and ( \overline{Nu_{D,t}} ) are the laminar and turbulent Nusselt number given in this table.</td>
</tr>
</tbody>
</table>

\( T_m \) is the bulk fluid temperature that is the temperature averaged over the cross sectional area of the tube. For non-circular tube, the correlations in Table 8.3-3 may be used with the diameter \( D \) replaced by the effective or hydraulic diameter \( D_h \) defined as

\[
D_h = \frac{4A}{P} = \frac{4(\text{flow cross-sectional area})}{\text{wetted perimeter}}
\]

For flow in an annular space, the effective diameter is

\[
D_h = \frac{\pi(D_o^2 - D_i^2)}{\pi(D_i + D_o)} = D_o - D_i
\]

---

\[ \text{Figure 8.3-2 The concentric tube annulus.} \]
External free convection flow

Table 8.3-4 listed some correlations for external free convection flow. For free convection, the driving force for flow is the density difference caused by temperature gradient within the fluid. Therefore the important dimensionless number for the correlation is the *Grashof* number that indicates the ratio of the buoyancy force to the viscous force acting on the fluid.

\[
Gr_L = \frac{g \beta (T_s - T_x) L^3}{\nu^2}
\]

where \( \beta \) is the expansion coefficient that depends on the fluid. For an ideal gas, \( \rho = p/RT \), the expansion coefficient can be determined

\[
\beta = -\frac{1}{\rho} \left( \frac{\partial \rho}{\partial T} \right) \rho = \frac{1}{\rho} \frac{p}{RT^2} = \frac{1}{T}
\]

where \( T \) must be the absolute temperature. The *Grashof* number is usually combined with the *Prandtl* number to become the *Rayleigh* number.

\[
Ra_L = Gr_L Pr = \frac{g \beta (T_s - T_x) L^3}{\nu \alpha}
\]

<table>
<thead>
<tr>
<th>Correlations</th>
<th>Geometry</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \overline{Nu}_L ) = [ \frac{0.387 Ra_L^{1/6}}{[1 + (0.949 / Pr)^{9/16}]^{8/27}} ] ( \overline{Nu}_L ) 2</td>
<td>Vertical plate</td>
<td>Properties evaluated at ( T_f ) May be applied to a vertical cylinder if ( (D/L) \geq (35/Gr_L^{1/4}) )</td>
</tr>
<tr>
<td>( \overline{Nu}_L = 0.54 Ra_L^{1/4} ) ( (10^4 &lt; Ra_L &lt; 10^7) )</td>
<td>Horizontal plate</td>
<td>( T_i ), Hot surface up or cold surface down ( L = A_s / P )</td>
</tr>
<tr>
<td>( \overline{Nu}_L = 0.15 Ra_L^{1/3} ) ( (10^7 &lt; Ra_L &lt; 10^{11}) )</td>
<td>Horizontal plate</td>
<td>( T_i ), Cold surface up or hot surface down ( L = A_s / P )</td>
</tr>
<tr>
<td>( \overline{Nu}_L = 0.27 Ra_L^{1/4} ) ( (10^5 &lt; Ra_L &lt; 10^{10}) )</td>
<td>Horizontal cylinder</td>
<td>( Ti ), ( Ra_D &lt; 10^{12} )</td>
</tr>
<tr>
<td>( \overline{Nu}_D ) = [ 0.60 + \frac{0.387 Ra_D^{1/6}}{[1 + (0.559 / Pr)^{9/16}]^{8/27}} ] ( \overline{Nu}_D ) 2</td>
<td>Sphere</td>
<td>( Ti ), ( Ra_D &lt; 10^{12}, Pr \geq 0.7 )</td>
</tr>
<tr>
<td>( \overline{Nu}_D = 2 + \frac{0.589 Ra_D^{1/6}}{[1 + (0.469 / Pr)^{9/16}]^{4/9}} )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

---

Table 8.3-4 Correlations for external free convection flow

8-11
Convection Heat Transfer

8.4 Correlations for Boiling and Condensation

Film Pool Boiling

The correlation for film boiling on a cylinder or sphere of diameter $D$ is of the form

$$\frac{Nu_D}{k_v} = C \left[ \frac{g(\rho_l - \rho_v)h'_{fg} D^4}{\nu_v k_v (T_s - T_{sat})} \right]^{1/4}, \quad C = 0.62 \text{ for horizontal cylinders and } C = 0.67 \text{ for spheres}$$

The corrected latent heat $h'_{fg}$ accounts for the sensible energy required to maintain temperatures within the vapor blanket above the saturation temperature. $h'_{fg}$ may be approximated as

$$h'_{fg} = h_{fg} + 0.80 c_{p,v}(T_s - T_{sat})$$

Vapor properties are evaluated at the film temperature, $T_f = 0.5(T_s + T_{sat})$, $h_{fg}$ and the liquid density is evaluated at the saturation temperature.

Radiation must also be considered for the evaluation of the heat transfer $q$ between the solid and the surrounding fluid

$$q = \overline{h} A_s (T_s - T_{sat})$$

where the total heat transfer coefficient $\overline{h}$ can be estimated from

$$\overline{h}^{4/3} = \overline{h}_{\text{conv}}^{4/3} + \overline{h}_{\text{rad}} 1/3$$

If $\overline{h}_{\text{rad}} < \overline{h}_{\text{conv}}$, a simpler form may be used
The effective radiation coefficient is defined as

\[
\overline{h} = \overline{h}_{\text{con}} + \frac{3}{4} \overline{h}_{\text{rad}}
\]

The effective radiation coefficient is defined as

\[
\overline{h}_{\text{rad}} = \varepsilon \sigma (T_s^2 + T_{\text{sat}}^2)(T_s + T_{\text{sat}})
\]

where \( \varepsilon \) is the emissivity of the solid and \( \sigma \) is the Stefan-Boltzman constant, \( \sigma = 5.67 \times 10^{-8} \text{ W/m}^2\text{K}^4 \).

**Film Condensation on a Vertical Plate**

![Figure 8.4-2 Film condensation on a vertical surface.](image-url)

The correlation for laminar film condensation on a vertical surface with height \( L \) is of the form

\[
\frac{N_u}{L} = \frac{\overline{h}_L L}{k_f} = 0.943 \left[ \frac{g(\rho_l - \rho_v)h'_{fg}L}{\nu_f k_f(T_{\text{sat}} - T_s)} \right]^{1/4}
\]

The modified latent heat \( h'_{fg} \) accounts for the effects of thermal advection. \( h'_{fg} \) may be approximated as

\[
h'_{fg} = h_{fg} + 0.68 c_p \rho_v (T_{\text{sat}} - T_s)
\]

Liquid properties are evaluated at the film temperature, \( T_f = 0.5(T_s + T_{\text{sat}}) \), \( h_{fg} \) and the vapor density is evaluated at the saturation temperature.
Film Condensation on Radial Systems

Figure 8.4-3 Film condensation on radial systems.

The correlation for film condensation on a cylinder or sphere of diameter $D$ is of the form

$$\overline{h_D} = C \left[ \frac{g(d - d_v)k_T^{3/4}h'_{fg}}{v_l(T_{sat} - T_s)D} \right]^{1/4}, \quad C = 0.729 \text{ for horizontal cylinders and } C = 0.826 \text{ for spheres}$$

For a vertical tier of $N$ horizontal tubes, the average heat transfer coefficient over $N$ tube is given as

$$\overline{h_{D,N}} = 0.729 \left[ \frac{g(d - d_v)k_T^{3/4}h'_{fg}}{NV_l(T_{sat} - T_s)D} \right]^{1/4}$$

The modified latent heat $h'_{fg}$ accounts for the effects of thermal advection. $h'_{fg}$ may be approximated as

$$h'_{fg} = h_{fg} + 0.68c_{p_l}(T_{sat} - T_s)$$

Liquid properties are evaluated at the film temperature, $T_f = 0.5(T_s + T_{sat})$, $h_{fg}$ and the vapor density is evaluated at the saturation temperature.
Chapter 8

Convection Heat Transfer

8.5 Examples of Convection Systems

Example 8.5-1

On a summer day the air temperature is 28°C and the relative humidity is 25%. Water evaporates from the surface of a lake at a rate of 0.10 kg/hr per square meter of the water surface area. If the water temperature is also 28°C, determine the value of the convection mass transfer coefficient.

Solution

![Figure 8.5-1 Water evaporating in air at 28°C.](image)

The density of saturated water vapor $\rho_{w,\text{sat}}$ at 28°C can be obtained from a steam table.
\[ \rho_{w,\text{sat}} = \frac{1}{v_g} = \frac{1}{36.69 \text{m}^3 / \text{kg}} = 0.0273 \text{ kg/m}^3 \]

Since the air temperature is the same as the saturation temperature, at 25% relative humidity

\[ \rho_{w,\infty} = 0.25 \rho_{w,\text{sat}} = 0.25 \times 0.0273 = 8.81 \times 10^{-3} \text{ kg/m}^3 \]

The mass transfer coefficient \( \bar{h}_m \) is calculated by the equation

\[ n''_w = \bar{h}_m (\rho_{w,s} - \rho_{w,\infty}) = \bar{h}_m (\rho_{w,\text{sat}} - 0.25 \rho_{w,\text{sat}}) = \bar{h}_m (0.75 \rho_{w,\text{sat}}) \]

\[ \bar{h}_m = \frac{n''_w}{0.75 \rho_{w,\text{sat}}} = \frac{0.10}{(0.75)(0.0273)} = 4.884 \text{ m/hr} = 1.36 \times 10^{-3} \text{ m/s} \]

---

**Example 8.5-2**

Insulating “wet” suits worn by scuba divers are usually made of \( \frac{1}{8} \) in. thick foam neoprene (\( k = 0.025 \text{ Btu/hr ft} \cdot \text{°F} \)) which traps next to the skin a layer of stagnant water about \( \frac{1}{8} \) in. thick. Determine the rate of heat loss from a 200 lb, 6 ft tall diver swimming at 3 miles per hour in 55°F sea water if his skin temperature does not fall below 74°F.

**Solution**

We treat the swimmer as a cylinder to estimate the surface area for heat transfer. However we will use the equation for flat plate to determine the heat transfer coefficient since the available correlation is for flow normal to the cylinder.

Volume of swimmer \( V = \frac{200}{62.4} = \pi \frac{D^2}{4} \times 6 \Rightarrow D = \sqrt[6]{\frac{800}{\pi \times 6 \times 62.4}} = 0.82 \text{ ft} \)

Surface area for heat loss: \( A_s \approx \pi \times 0.82 \times 6 = 15 \text{ ft}^2 \)
From Figure 8.5-2b, the heat loss can be approximated by

\[ q = A_s \frac{\Delta T}{\sum R_i} = A_s \frac{74 - 55}{\frac{L_1}{k_1} + \frac{L_2}{k_2} + \frac{1}{h}} \]

where \( k_1 \) = thermal conductivity of neoprene, \( k_2 \) = thermal conductivity of sea water, and \( h \) = heat transfer coefficient from the outside surface of neoprene to sea water at 55°F.

Properties of sea water at 60°F: \( \nu = 1.22 \times 10^{-5} \text{ ft}^2/\text{s} \), \( k_2 = 0.336 \text{ Btu/hr} \cdot \text{ft} \cdot ^\circ\text{F} \), \( Pr = 7.88 \)

\[ Re_L = \frac{L u_w}{\nu} = \frac{6 \times 3 \times (5280/3600)}{1.22 \times 10^{-5}} = 2.16 \times 10^6 \]

The equation needed to evaluate \( h \) is

\[ \overline{Nu_L} = [0.664 Re_{x,c}^{1/2} + 0.037(Re_L^{4/5} - Re_{x,c}^{1/2})]Pr^{1/3} \]

If \( Re_{x,c} = 500,000 \) then

\[ \overline{Nu_L} = (0.037 Re_L^{4/5} - 871)Pr^{1/3} = (0.037 \times (2.16 \times 10^6)^{4/5} - 871) \times 7.88^{1/3} = 6880 \]

\[ \overline{h} = \frac{k_2}{L} \overline{Nu_L} = \frac{0.336 \times 6880}{6} = 385 \text{ Btu/hr} \cdot \text{ft}^2 \cdot ^\circ\text{F} \]

The heat loss is then

\[ q = A_s \frac{\Delta T}{\sum R_i} \]

\[ \sum R_i = \frac{L_1}{k_1} + \frac{L_2}{k_2} + \frac{1}{h} = \frac{1}{0.25} + \frac{1}{0.36} + \frac{1}{385} = 0.45 \text{ hr} \cdot \text{ft}^2 \cdot ^\circ\text{F}/\text{Btu} \]

\[ q = 15 \times \frac{74 - 55}{0.45} = 15 \times 42.2 = 630 \text{ Btu/hr} \]

Figure 8.5-2b Schematic for heat loss evaluation.
Example 8.5-3

A horizontal copper rod 8 mm in diameter and 100 mm long is in the airspace between surfaces of an electronic device to enhance heat dissipation. The ends of the rod are at 90°C, while air at 25°C is in cross flow over the cylinder with a velocity of 20 m/s. Determine the temperature at the midplane of the rod and the rate of heat transfer from the rod.

Solution

\[ \frac{dT}{dx} = 0 \]

\[ \text{Figure 8.5-3 Heat loss from a cylinder.} \]

\[ \overline{Nu}_D = 0.3 + \{0.62 \ Re_D^{1/2} Pr^{1/3} \times [1 + (0.4/Pr)^{2/3}]^{-1/4}\} \times [1 + (Re_D/282,000)^{5/8}]^{4/5} \]

Properties of air at \( T_f \approx 60°C, \nu = 15.7 \times 10^{-6} \text{ m}^2/\text{s}, k = 0.0257 \text{ W/m-K}, Pr = 0.704 \)

Thermal conductivity of copper at 80°C: \( k = 398 \text{ W/m·°K} \)

\[ Re_D = \frac{u_x D}{\nu} = \frac{20 \times 0.008}{15.7 \times 10^{-6}} = 10,200 \]

\[ \overline{Nu}_D = 0.3 + \{0.62 \times 10,200^{1/2} \times 0.704^{1/3} \times [1 + (0.4/0.704)^{2/3}]^{-1/4}\} \times [1 + (10,200/282,000)^{5/8}]^{4/5} = 23.2 \]

\[ \overline{h} = \frac{k}{D} \overline{Nu}_D = \frac{0.0257}{0.008} \times 23.2 = 75 \text{ W/m·°K} \]

The \( x \)-coordinate is assigned in the direction along the cylinder with \( x = 0 \) at the base or left surface. The surface at \( x = L \) is a plane of symmetry therefore \( \frac{dT}{dx} \bigg|_{x=L} = \frac{d\theta}{dx} \bigg|_{x=L} = 0 \). The problem is similar to the case of a cylindrical fin with insulated tip. Therefore

\[ \frac{\theta}{\theta_0} = \frac{T - T_x}{T_b - T_x} = \frac{\cosh[(m(L - x))]}{\cosh(mL)} \]
where 

\[ m = \sqrt{\frac{hP}{kA}} = \sqrt{\frac{4h}{kD}} = \left( \frac{4 \times 75}{0.008 \times 398} \right)^{1/2} = 9.71 \text{ m}^{-1} \]

At \( x = L \),

\[ \frac{T - T_\infty}{T_h - T_\infty} = \frac{\cosh(0)}{\cosh(9.71 \times 0.5)} = \frac{1}{1.12} = 0.893 \]

\[ T(L) = 25 + 0.893(90 - 25) = 83^\circ C \]

The rate of heat transfer from the rod is given by

\[ q_t = 2 \sqrt{hPkA} \theta_0 \tanh(mL) \]

\[ \sqrt{hPkA} = [(75)(\pi \times 0.008)(398)(\pi \times 0.004^2)]^{1/2} = 0.194 \text{ W/}^\circ C \]

\[ q_t = 2 \times 0.194 \times \tanh(9.71 \times 0.5) = 25.2 \text{ W} \]
Example 8.5-4

A square, horizontal plate of pure aluminum, initially at 300°C, is allowed to cool in a large chamber. The plate, 0.5 m by 0.5 m and 8 mm thick, is insulated at the bottom. The walls of the chamber and the enclosed air are each maintained at 26°C.

1) If the surface emissivity of the plate is 0.25, what is the initial cooling rate? Neglect heat transfer from the sides of the plate.

2) Determine the effective Biot number to show that the assumption of uniform plate temperature is valid.

Solution

![Figure 8.5-4 Free convection from a horizontal plate.](image)

1) Determine initial cooling rate \( \frac{dT}{dt} \)

Assuming constant physical properties, a macroscopic energy balance over the plate yields

\[
\rho V c_p \frac{dT}{dt} = - \left[ h A_s (T_s - T_\infty) + \varepsilon \sigma A_s (T_s^4 - T_{\text{sur}}^4) \right]
\]

where \( V = A_s w \), \( w \) is the plate thickness. The initial cooling rate is then

\[
\frac{dT}{dt} = - \frac{1}{\rho V c_p} \left[ h (T_s - T_\infty) + \varepsilon \sigma (T_s^4 - T_{\text{sur}}^4) \right]
\]

We need to evaluate the heat transfer coefficient \( h \) for free convection of a hot surface facing up.
The physical properties of air at $T_f = 0.5(T_s + T_\infty) = 0.5(300 + 26) = 163^\circ C = 436$ K: $\nu = 30.72 \times 10^{-6}$ m$^2$/s, $k = 0.0363$ W/m-K, $\alpha = 44.7 \times 10^{-6}$ m$^2$/s, $Pr = 0.687$. Physical properties of aluminum at 300$^\circ$C: $k = 232$ W/m-K, $\rho = 2702$ kg/m$^3$, $c_p = 1022$ J/kg$\cdot$K.

\[ L = A_s/P = (0.5 \times 0.5)/(4 \times 0.5) = 0.125 \text{ m} \]

\[ Ra_L = Gr_L Pr = \frac{g \beta (T_s - T_\infty) L^3}{\nu \alpha} = \frac{9.81 \times (1/436)(300 - 26) \times 0.125^3}{(30.72 \times 10^{-6})(44.7 \times 10^{-6})} = 6.77 \times 10^6 \]

\[ \overline{N}_u = 0.54 Ra_L^{1/4} \quad (10^4 < Ra_L < 10^7) \]

\[ \overline{N}_u = 0.54 \times (8.77 \times 10^6)^{1/4} = 29.4 \]

\[ \overline{h} = \frac{k}{L} \overline{N}_u = \frac{0.0363}{0.125} \times 29.4 = 8.53 \text{ W/m}^2\cdot\text{K} \]

\[ \frac{dT}{dt} = -\frac{1}{2702 \times 0.008 \times 1022} [8.53 (300 - 26) + 0.25(5.67 \times 10^{-8})(573^4 - 299^4)] \]

\[ \frac{dT}{dt} = -0.167 \text{ ^\circ C/s} \]

2) Check the assumption of uniform plate temperature.

\[ \overline{h}_\text{rad} = \varepsilon \sigma (T_s^2 + T_\text{sur}^2)(T_s + T_\text{sur}) \]

\[ \overline{h}_\text{rad} = 0.25 (5.67 \times 10^{-8})(573^2 + 299^2)(573 + 299) = 5.16 \text{ W/m}^2\cdot\text{K} \]

Since $\overline{h}_\text{rad} < \overline{h}_\text{con}$, the total heat transfer coefficient is given by

\[ \overline{h} = \overline{h}_\text{con} + \frac{3}{4} \overline{h}_\text{rad} = 8.53 + \frac{3}{4} \times 5.16 = 12.4 \text{ W/m}^2\cdot\text{K} \]

\[ Bi = \frac{\overline{h}_w}{k} = \frac{12.4(0.008)}{232} = 4.3\times10^{-4} \ll 1 \]

Assumption of uniform plate temperature is justified.
Chapter 9

Heat Exchangers

9.1 Introduction

Heat exchangers are devices for transferring heat between two fluid streams. Heat exchangers can be classified as indirect contact type and direct contact type. Indirect contact type heat exchangers have no mixing between the hot and cold streams, only energy transfer is allowed as shown in Figure 9.1-1.

![Figure 9.1-1 Indirect contact type heat exchanger.](image)

Direct contact type heat exchangers have no wall to separate the cold from the hot streams as shown in Figure 9.1-2.

![Figure 9.1-2 Direct contact type heat exchanger.](image)

Heat exchangers can be classified according to 1) transfer process, 2) number of fluids, 3) surface compactness, 4) construction, 5) flow arrangements, and 6) heat transfer mechanism. Table 9.1-1 shows the classification for the first three types. The complete table can be found from the Heat Transfer Handbook page 17.3.
Table 9.1-1. Classification of heat exchangers

Heat exchangers can be found in automotives (radiators) or in power cycles. Figure 9.1-3b shows a schematic diagram of an air-standard gas turbine with directions for principal heat transfers indicated by arrows. Gas turbines are usually lighter and more compact than the vapor power system even though a larger portion of work developed by the gas turbine is required to drive the compressor.

An open gas turbine engine is shown in Figure 9.1-3a. Air is continuously drawn into the compressor of this engine, where it is compressed to a high pressure. The air then enters a combustor, a combustion chamber, where it is mixed with fuel and combustion occurs, resulting
in combustion products at an elevated temperature. The combustion products do work by expanding through the turbine and are subsequently discharged to the surroundings. Part of the turbine work developed is used to drive the compressor. An air-standard study analysis is used to study the open gas turbine engine with the assumptions that air is the working ideal gas and the energy generated by combustion is accomplished by a heat transfer source.

With the air-standard idealization, ambient air enters the compressor at state 1 and later returns to the surrounding at state 4 with a temperature higher than the ambient temperature. The discharged air would eventually return to the same state as the air entering the compressor so we can consider the air passing through the gas turbine engine as undergoing a thermodynamic cycle. The air-standard Brayton cycle represents the states visited by the gas with an additional heat exchanger for the air to release heat to the surroundings and return to its original state 1. The air-standard Brayton cycle consists of two heat exchangers, a compressor, and a turbine.

The condenser and evaporator shown in Figure 9.1-4 are the heat exchangers. This figure depicts the most common refrigeration. In step 4 → 1, heat is removed at the temperature $T_L$ from the system being refrigerated by the evaporation of a liquid under the pressure $P_L$. In step 1 → 2, saturated vapor at $P_L$ is compressed isentropically to $P_H$ where it becomes superheated vapor. In step 2 → 3, heat $Q_H$ is transferred to the surrounding by condensation at $T_H$. In step 3 → 4, the cycle is closed by throttling the liquid to the lower pressure $P_L$. There is no change in enthalpy during this step.

![Figure 9.1-4 A vapor-compression refrigeration cycle and its Ts diagram.](image-url)
9.2 Heat Exchanger types.

We will discuss three types of tubular heat exchangers: *concentric tube*, *cross flow*, and *shell-and-tube* heat exchangers. A *concentric tube* or *double pipe* heat exchanger is the simplest heat exchanger for which the hot and cold fluids move in the same or opposite directions as shown in Figure 9.2-1.

![Figure 9.2-1 Concentric tube heat exchangers.](image)

Figure 9.2-2 shows cross flow heat exchangers where fluid flows perpendicular across the tube bank rather than parallel with it. There is no mixing of the fluid outside the tube in the y-direction for the arrangement in the left side of Figure 9.2-2, while there is mixing of the fluid outside the tube in the y-direction for the arrangement in the right side of Figure 9.2-2. The fluid inside the tubes is considered unmixed since it is confined and cannot mix with other stream. Cross flow heat exchangers are usually used to heat or cool a gas such as air.

![Figure 9.2-2 Cross flow heat exchangers.](image)

*Shell-and-tube* heat exchanger is the most common configuration. There are many different forms of *shell-and-tube* heat exchangers according to the number of shell-and-tube passes. A common form with one shell pass and two tube passes is shown in Figure 9.2-3. Baffles are usually installed to increase the heat transfer coefficient of the fluid by introducing turbulence and cross-flow in the shell side.
The advantages of shell and tube exchangers are as follows:\(^1\):

1) Large surface area in a small volume;
2) Good mechanical layout: a good shape for pressure operation;
3) Well-established fabrication techniques;
4) Construction from a wide range of materials;
5) Easy maintenance;
6) Well-established design procedures.

In general, the tube side is used for fluid which is more corrosive or dirtier or at a higher pressure. The shell side is used for liquid of high viscosity or gas. It is usually easier to clean the inside of tubes than to clean the shell side. Heat exchanger shutdowns are most often caused by fouling, corrosion, and erosion. Figure 9.2-4 shows type designation for shell-and-tube heat exchangers by the American Tubular Heat Exchanger Manufacturers Association, TEMA. The TEMA standards cover three classes of exchanger: class R for exchangers in the generally severe duties of the petroleum and related industries, class C for exchangers in moderate duties of

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\(^1\) Tower and Sinnott, *Chemical Engineering Design*, Elsevier, 2008, pg. 801
commercial and general process application, and class B for exchangers in the chemical process industries².

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Table: TEMA Heat Exchanger Types

<table>
<thead>
<tr>
<th>Front End Stationary Head Types</th>
<th>Shell Types</th>
<th>Rear End Head Types</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>F</td>
<td>L</td>
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<td>B</td>
<td>G</td>
<td>M</td>
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<td>X</td>
<td>R</td>
</tr>
</tbody>
</table>

Figure 9.2-4 TEMA Heat Exchanger Types

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² Tower and Sinnott, *Chemical Engineering Design*, Elsevier, 2008, pg. 803
The TEMA specifies the standards related to the mechanical design features, materials of construction, and testing of the shell and tube exchangers with a unique nomenclature. The TEMA nomenclature is a three-letter designation based on the mode of differential thermal expansion between the shell and the tubes, their degree of disassembly, and the shell-side flow arrangement. The first letter of the three-letter designation indicates the front end head type. The second letter of the three relates to the shell type. The last letter of the three indicates the rear end head type.

![Diagram of baffle types](image)

Figure 9.2-5 Types of baffle used in shell and tube heat exchangers.

Baffles improve the rate of heat transfer by increasing the fluid velocity and directing the fluid stream across the tubes. The most commonly used type of baffle is the single baffle shown in Figure 9.2-5a; two other types are shown in Figure 9.2-5a and b.

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3 Tower and Sinnott, *Chemical Engineering Design*, Elsevier, 2008, pg. 809
The tubes in exchangers are usually arranged in an equilateral, square, or rotated square patterns as shown in Figure 9.2-6 where $P_t$ denotes tube pitch. The triangular and rotated square patterns have higher heat transfer rates and higher pressure drop than the square pattern. A square or rotated square pattern is used for heavily fouling fluids, where mechanical cleaning of the outside tubes is necessary. The tube pitch (distance between tube centers) is normally 1.25 times the tube outside diameter unless process requirements dictate otherwise.
Chapter 9

9.3 Analysis of Heat Exchangers

All heat exchangers analyses require energy balances between the fluids. These balances will be performed for steady state systems with the following assumptions:

a) Heat capacity $c_p$ is not a function of temperature;
b) Heat transfer coefficient $h$ is constant and does not vary along tube length. This is the case for fully developed flow where effect of property variations is not important;
c) The system is well insulated so that there is no heat loss to surroundings;
d) Longitudinal heat conduction in the fluid and walls is negligible. There is no conduction along the flow direction;
e) Tube walls are smooth without scale of dirt or oxidation;
f) Fluid potential and kinetic energies are negligible;
g) Flow is characterized by bulk or mean velocity and mean temperature at any cross section.

$\Delta T = \frac{dq}{\int V dA}$

The mean velocity $u_m$ and mean temperature $T_m$ is defined by the following equations:

$$u_m = \frac{1}{A} \int_A V_s dA \quad \text{and} \quad T_m = \frac{1}{u_m A} \int_A V_s T dA$$

We will analyze a simple parallel flow heat exchanger depicted in Figure 9.3-1.

Figure 9.3-1 Simple parallel flow heat exchanger.
Let $T_h$ and $T_c$ be the mean fluid temperatures of the tube and shell sides respectively. The energy change of the tube side fluid along the $x$-direction is equal to the energy transferred through the tube wall

$$q_0''dx = q_0''dA_s = \rho_h u_{m,h} c_{p,h} A dT_h$$

(9.3-1)

In this equation $P$ is the tube perimeter, $A$ is the tube cross sectional area, and $q_0''$ is the heat flux through the tube wall. In terms of the mass flow rate $\dot{m}_h = \rho_h u_{m,h} A$, Eq. (9.3-1) becomes

$$dq_h = q_0''dA_s = \dot{m}_h c_{p,h} dT_h$$

(9.3-2)

We have a similar equation for the energy received by the cold fluid

$$dq_c = -q_0''dA_s = \dot{m}_c c_{p,c} dT_c$$

(9.3-3)

The energy given up by the hot fluid is absorbed by the cold fluid,

$$dq_c = -dq_h = dq$$

(9.3-4)

Equation (9.3-2) can be integrated from the inlet to the outlet of the hot stream,

$$\int dq_h = \int_{T_{hi}}^{T_{ho}} \dot{m}_h c_{p,h} dT_h$$

$$q_h = \dot{m}_h c_{p,h}(T_{ho} - T_{hi})$$

(9.3-5)

Similarly for the cold stream

$$q_c = \dot{m}_c c_{p,c}(T_{co} - T_{ci})$$

(9.3-6)

Since $q_c = -q_h = q$, equations (9.3-5 and 6) can be written as

$$q = \dot{m}_h c_{p,h}(T_{hi} - T_{ho})$$

(9.3-7)

$$q = \dot{m}_c c_{p,c}(T_{co} - T_{ci})$$

(9.3-8)

In the differential form we have

$$dq = -\dot{m}_h c_{p,h}dT_h$$

(9.3-9)

$$dq = \dot{m}_c c_{p,c}dT_c$$

(9.3-10)

We now define the capacitance rates for the hot and cold streams $C_h$ and $C_c$ respectively

$$C_h = \dot{m}_h c_{p,h} \text{ and } C_c = \dot{m}_c c_{p,c}$$
Equations (9.3-7, 8, 9, and 10) become

\[ q = C_h(T_{hi} - T_{ho}) \]  \hspace{1cm} (9.3-11)

\[ q = C_c(T_{co} - T_{ci}) \]  \hspace{1cm} (9.3-12)

\[ dq = -C_h dT_h \]  \hspace{1cm} (9.3-13)

\[ dq = C_c dT_c \]  \hspace{1cm} (9.3-14)

The heat transfer between the hot and cold streams shown in Figure 9.3-2 is now considered. Without fouling resistance, the heat transfer between the hot and cold streams for the chosen control volume is given by

\[ dq = \frac{T_h - T_c}{h_i dA_i + \frac{1}{2\pi k} \ln(r_o/r_i)} = \frac{T_h - T_c}{h_o 2\pi r_o dx + \frac{1}{2\pi k} \ln(r_o/r_i) + \frac{1}{h_o 2\pi r_o dx}} \]

The heat transfer between the streams can be written in terms of the overall heat transfer coefficients \( U_i \) or \( U_o \) as follows:

\[ dq = \frac{(T_h - T_c) 2\pi r_i dx}{\frac{1}{h_i} + \frac{1}{k} \frac{1}{r_i} \ln(r_o/r_i) + \frac{1}{h_o} \frac{1}{r_o}} = U_i (T_h - T_c) dA_i \]

\[ dq = \frac{(T_h - T_c) 2\pi r_o dx}{\frac{1}{h_i} \frac{1}{r_i} + \frac{1}{k} \frac{1}{r_o} \ln(r_o/r_i) + \frac{1}{h_o}} = U_o (T_h - T_c) dA_o \]

The overall heat transfer coefficients \( U_i \) and \( U_o \) are defined as
\[ U_i = \frac{1}{\frac{1}{h_i} + \frac{r_i \ln(r_o/r_i)}{k} + \frac{1}{h_o r_o}} \text{ or } \frac{1}{U_i dA_i} = \frac{1}{\frac{1}{h_i} + \frac{\ln(r_o/r_i)}{h_o r_o}} \]

\[ U_o = \frac{1}{\frac{r_o}{h_i} r_i \ln(r_o/r_i)} + \frac{1}{h_o} \text{ or } \frac{1}{U_o dA_o} = \frac{1}{\frac{1}{h_i} + \frac{\ln(r_o/r_i)}{2\pi k dx} + \frac{1}{h_o dA_o}} \]

\[ dq = -C_h dT_h = U_o(T_h - T_c) dA_o \Rightarrow \frac{dT_h}{T_h - T_c} = -\frac{U_o dA_o}{C_h} \quad (9.3-15) \]

\[ dq = C_c dT_c = U_o(T_h - T_c) dA_o \Rightarrow \frac{dT_c}{T_h - T_c} = \frac{U_o dA_o}{C_c} \quad (9.3-16) \]

Subtracting Eq. (9.3-16) from Eq. (9.3-15) gives

\[ \frac{dT_h}{T_h - T_c} - \frac{dT_c}{T_h - T_c} = -\frac{U_o dA_o}{C_h} - \frac{U_o dA_o}{C_c} \]

\[ \frac{d(T_h - T_c)}{T_h - T_c} = -\left(\frac{1}{C_h} + \frac{1}{C_c}\right) U_o dA_o \quad (9.3-17) \]

**Figure 9.3-3** Parallel flow heat exchanger.

Integrating Eq. (9.3-17) over the surface area of a parallel flow heat exchanger shown in Figure 9.3-3 and assuming that \( U_o \) is independent of \( x \), the distance along the heat exchanger, we obtain

\[ -\int_{T_h}^{T_c} \frac{d(T_h - T_c)}{T_h - T_c} = \int_{A_o} \left(\frac{1}{C_h} + \frac{1}{C_c}\right) U_o dA_o \]

\[ \ln\left(\frac{T_{ci}}{T_{co}}\right) = \left(\frac{1}{C_h} + \frac{1}{C_c}\right) U_o dA_o \quad (9.3-18) \]

From Equations (9.3-11 and 12) we have
\[
\frac{1}{C_h} = \frac{T_{hi} - T_{ho}}{q} \quad \text{and} \quad \frac{1}{C_c} = \frac{T_{co} - T_{ci}}{q}
\]

Therefore
\[
\frac{1}{C_h} + \frac{1}{C_c} = \frac{T_{hi} - T_{ho}}{q} + \frac{T_{co} - T_{ci}}{q} = \frac{(T_{hi} - T_{ci}) - (T_{ho} - T_{co})}{q} = \frac{\Delta T_1 - \Delta T_2}{q}
\]

We have defined \(\Delta T_1 = T_{hi} - T_{ci}\) and \(\Delta T_2 = T_{ho} - T_{co}\) for parallel flow. Substituting \(\frac{1}{C_h} + \frac{1}{C_c} = \frac{\Delta T_1 - \Delta T_2}{q}\) into Eq. (9.3-18) gives

\[
\ln \frac{\Delta T_1}{\Delta T_2} = \frac{\Delta T_1 - \Delta T_2}{q} U_o A_o
\]

Solving for \(q\) we obtain

\[
q = U_o A_o \Delta T_{lm} \quad \text{where} \quad \Delta T_{lm} = \frac{\Delta T_1 - \Delta T_2}{\ln (\Delta T_1 / \Delta T_2)} \quad (9.3-19a)
\]

We can repeat the procedure with the following expressions

\[
dq = - C_h dT_h = U_i (T_h - T_c) dA_i \Rightarrow \frac{dT_h}{T_h - T_c} = - \frac{U_i dA_i}{C_h}
\]

\[
dQ_c = C_c dT_c = U_i (T_h - T_c) dA_i \Rightarrow \frac{dT_c}{T_h - T_c} = \frac{U_i dA_i}{C_c}
\]

We then obtain the heat transfer rate based on the inside surface area \(A_i\) of the tube.

\[
q = U_i A_i \Delta T_{lm} \quad (9.3-19b)
\]

For counter flow heat exchanger as shown in Figure 9.3-4, we will obtain a similar expression for the heat transfer rate

\[
q = U_o A_o \Delta T_{lm} = U_i A_i \Delta T_{lm} \quad (9.3-20)
\]

Figure 9.3-4 Counter flow heat exchanger.
For countercurrent flow, \( \Delta T_{lm} \) is defined by the following equation

\[
\Delta T_{lm} = \frac{(T_{hi} - T_{co}) - (T_{ho} - T_{ci})}{\ln \frac{T_{hi} - T_{co}}{T_{ho} - T_{ci}}} = \frac{\Delta T_1 - \Delta T_2}{\ln (\Delta T_1 / \Delta T_2)} \tag{9.3-21}
\]

Hence, for counter flow \( \Delta T_1 = T_{hi} - T_{co} \) and \( \Delta T_2 = T_{ho} - T_{ci} \). For other heat exchanger geometries such as cross flow and shell and tube heat exchanger, the heat transfer rate is given by

\[
q = U_c A F \Delta T_{lm}
\]

In this equation, \( \Delta T_{lm} \) is based on counter flow and is given by Equation (9.3-21). \( F \) is the correction factor to account for the configuration in a heat exchanger for which the flow is neither parallel nor counter current. The \( F \) factor for cross-flow heat exchangers with both fluids unmixed are shown in Figure 9.3-6.

Figure 9.3-6 F factor for cross flow heat exchangers with both fluids unmixed.
The $F$ factor can be obtained from a chart similar to the one shown in Figure 9.3-7 for shell-and-tube heat exchangers with one shell pass and two tube passes. $T_i$ and $T_o$ are the inlet and outlet temperatures of the fluid on the shell side, respectively. $t_i$ and $t_o$ are the inlet and outlet temperatures of the fluid on the tube side, respectively.

The two parameters $R$ and $P$ required to read the chart are defined as

$$R = \frac{T_i - T_o}{t_o - t_i} = \frac{T_{hi} - T_{ho}}{T_{co} - T_{ci}} \quad \text{and} \quad P = \frac{t_o - t_i}{T_i - t_i} = \frac{T_{co} - T_{ci}}{T_{hi} - T_{ci}}$$

The $F$ factor may also be evaluated from

$$F = \left(\frac{\sqrt{R^2 + 1}}{R - 1}\right) \ln \left\{ \frac{(1 - P)}{(1 - RP)} \right\}$$

$$\ln \left\{ \frac{2 - P(R + 1 - \sqrt{R^2 + 1})}{2 - P(R + 1 + \sqrt{R^2 + 1})} \right\}$$

**Figure 9.3-7** $F$ factor for shell-and-tube heat exchanger with one shell pass and two tube passes.
If $R = 1$, the above equation becomes indeterminate, which reduces to

$$F = \frac{P \sqrt{R^2 + 1}}{1 - P} \ln \left\{ \frac{2 - P (R + 1 - \sqrt{R^2 + 1})}{2 - P (R + 1 + \sqrt{R^2 + 1})} \right\}$$

For exchangers with $N$ shell passes, $P$ is replaced by $P_x$ where

$$P_x = \frac{1 - \left( \frac{RP - 1}{P - 1} \right)^{1/N}}{R - \left( \frac{RP - 1}{P - 1} \right)^{1/N}}$$

for $R \neq 1$ and $P_x = \frac{P}{(N - NP + P)}$ for $R = 1$

The parameter $R = \frac{T_{hi} - T_{ho}}{T_{co} - T_{ci}}$ is equal to the cold fluid flow rate times the cold fluid mean specific heat, divided by the hot fluid flow rate times the hot fluid specific heat. $R$ is simply the capacitance ratio of the cold stream to the hot stream. This definition can be obtained from the following energy balances:

$q = \dot{m}_h c_{p,h}(T_{hi} - T_{ho}) \Rightarrow T_{hi} - T_{ho} = \frac{q}{C_h}$

$q = \dot{m}_c c_{p,c}(T_{co} - T_{ci}) \Rightarrow T_{co} - T_{ci} = \frac{q}{C_c}$

Therefore $R = \frac{T_{hi} - T_{ho}}{T_{co} - T_{ci}} = \frac{q / C_h}{q / C_c} = \frac{C_c}{C_h}$

Either definition of $R$ can be used, whichever is more convenient.
Chapter 9

Example 9.3-1. A process fluid having a specific heat of 3800 J/kg·K and flowing at 4 kg/s is to be cooled from 100°C to 50°C with chilled water (specific heat of 4180 J/kg·K). Cooling water is available at 15°C with the outlet temperature limited to 35°C. Assuming an overall heat transfer coefficient $U_o$ of 2000 W/m²·K, calculate the required water flow rate and the heat transfer areas for the following exchanger configurations: (a) parallel flow, (b) counter flow, and (c) shell-and-tube, one shell pass and 2 tube passes.

Water flows in the tube side of length 9 ft and $\frac{3}{4}$ in. O.D. with wall thickness = 0.083 in. Determine the velocity of water.

Solution

Calculate the required water flow rate

$$q = m_w c_{ph}(T_{hi} - T_{ho}) = (4)(3800)(100 - 50) = 760000 \text{ J/s}$$

The required water flow rate is

$$\dot{m}_w = \frac{q}{c_{pc}(T_{co} - T_{ci})} = \frac{760000}{(4180)(35 - 15)} = 9.09 \text{ kg/s}$$

(a) Parallel flow heat exchanger

$$\Delta T_1 = T_{hi} - T_{ci} = 100 - 15 = 85\degree C$$

$$\Delta T_2 = T_{ho} - T_{co} = 50 - 35 = 15\degree C$$

$$\Delta T_{lm} = \frac{\Delta T_1 - \Delta T_2}{\ln(\Delta T_1/\Delta T_2)} = \frac{85 - 15}{\ln(85/15)} = 40.36\degree C$$

Area required for parallel flow heat exchanger

$$A_o = \frac{q}{U_o \Delta T_{lm}} = \frac{760000}{(2000)(40.36)} = 9.42 \text{ m}^2$$

(b) Counter flow heat exchanger

$$\Delta T_1 = T_{hi} - T_{co} = 100 - 35 = 65\degree C$$

$$\Delta T_2 = T_{ho} - T_{ci} = 50 - 15 = 35\degree C$$

$$\Delta T_{lm} = \frac{\Delta T_1 - \Delta T_2}{\ln(\Delta T_1/\Delta T_2)} = \frac{65 - 35}{\ln(65/35)} = 48.46\degree C$$
Area required for counter flow heat exchanger

\[ A_o = \frac{q}{U_o \Delta T_{lm}} = \frac{760,000}{(2000)(48.46)} = 7.841 \text{ m}^2 \]

(c) Shell-and-tube, one shell pass and 2 tube passes, heat exchanger

\[ \Delta T_{lm} = \frac{\Delta T_1 - \Delta T_2}{\ln(\Delta T_1 / \Delta T_2)} = \frac{65 - 35}{\ln(65/35)} = 48.46^\circ \text{C} \]

Area required for shell-and-tube heat exchanger

\[ A_o = \frac{q}{U_o \Delta T_{lm} F} = \frac{760,000}{(2000)(48.46)F} = \frac{7.841}{F} \text{ m}^2 \]

The $F$ factor can be evaluated from

\[ F = \left( \frac{\sqrt{R^2 + 1}}{R - 1} \right) \ln \left\{ \frac{(1 - P)}{(1 - R P)} \right\} \]

\[ = \frac{2 - P \left( R + 1 - \sqrt{R^2 + 1} \right)}{2 - P \left( R + 1 + \sqrt{R^2 + 1} \right)} = 0.9228 \]

Area required for shell and tube heat exchanger

\[ A_o = \frac{7.841}{0.9228} = 8.497 \text{ m}^2 \]

Outside area of one tube = $\pi(0.75)(9)/12 = 1.7671 \text{ ft}^2 = 0.1642 \text{ m}^2$ (1 ft$^2$ = .0929 m$^2$)

Number of tubes required = 8.497/0.1642 = 52 tubes

Number of tubes per pass = 52/2 = 36

Cross-sectional area of one tube = $\pi(0.75 - 2\times0.083)^2/4/144 = 1.86\times10^{-3} \text{ ft}^2 = 1.728\times10^{-4} \text{ m}^2$

Water velocity in tube = $9.09/(1000\times36\times1.728\times10^{-4}) = 1.46 \text{ m/s}$
**Example 9.3-2.**

Oil having a specific heat of 2350 J/kg·K is to be cooled from 160°C to 100°C with chilled water (specific heat of 4181 J/kg·K) which is supplied at a temperature of 15°C and a flow rate of 2.5 kg/s. Water flows on the tube side with heat transfer coefficient of 3060 W/m²·K and oil on the shell side with heat transfer coefficient of 400 W/m²·K. There are 10 thin wall multiple-passes tubes with diameter of 25 mm. If the water outlet temperature is 85°C, determine the heat transfer rate, the oil mass flow rate, and the tube length.

**Solution**

(a) The heat transfer rate

\[ q_c = \dot{m}_c c_p,c (T_{co} - T_{ci}) = (2.5)(4181)(85 - 15) = 7.312 \times 10^5 \text{ W} \]

(b) The oil mass flow rate

\[ \dot{m}_o = \frac{q}{c_p,h (T_{hi} - T_{ho})} = \frac{7.312 \times 10^5}{(2350)(160 - 100)} = 5.19 \text{ kg/s} \]

(c) The tube length

Area required for shell-and-tube heat exchanger with one shell pass and multiple of two tube passes.

\[ A = \frac{q}{U \Delta T_{lm} F} \]

\[ \Delta T_1 = T_{hi} - T_{co} = 160 - 85 = 75^\circ\text{C} \]

\[ \Delta T_2 = T_{ho} - T_{ci} = 100 - 15 = 85^\circ\text{C} \]

\[ \Delta T_{lm} = \frac{\Delta T_1 - \Delta T_2}{\ln (\Delta T_1 / \Delta T_2)} = \frac{75 - 85}{\ln (75/85)} = 79.9^\circ\text{C} \]

The \( F \) factor can be evaluated from

\[ R = \frac{T_{hi} - T_{ho}}{T_{co} - T_{ci}} = 0.86, \quad P = \frac{T_{co} - T_{ci}}{T_{hi} - T_{ci}} = 0.48 \]

\[ F = \left( \frac{\sqrt{R^2 + 1}}{R - 1} \right) \ln \left( \frac{(1-P)}{(1-RP)} \right) = 0.88 \]

\[ \ln \left( \frac{2 - P(R + 1 - \sqrt{R^2 + 1})}{2 - P(R + 1 + \sqrt{R^2 + 1})} \right) \]
For thin wall

\[
U = \frac{1}{\frac{1}{h_i} + \frac{1}{h_o}} = \frac{1}{\frac{1}{3060} + \frac{1}{400}} = 354 \text{ W/m}^2\text{-K}
\]

\[
A = \frac{q}{U\Delta T_{lm}} = \frac{7.312 \times 10^5}{(354)(79.9)(0.88)} = 29.4 \text{ m}^2
\]

Length of tube \(L = \frac{A}{n_{tube}\pi D} = \frac{29.4}{(10)(\pi)(0.025)} = 37.4 \text{ m}
\]

Equation \(q = UAF\Delta T_{lm}\) can easily be used to determine either \(q\) or \(UA\) if the inlet fluid temperatures are known and the outlet temperatures are specified or readily determined from the following energy balance

\[
q = \dot{m}_h c_{ph}(T_{hi} - T_{ho}) = \dot{m}_c c_{pc}(T_{co} - T_{ci}) \quad (9.2-3)
\]

where  
\(\dot{m}_h =\) mass flow rate of the hot stream  
\(\dot{m}_c =\) mass flow rate of the cold stream

This procedure is called the log mean temperature difference (LMTD) method. If only the inlet temperatures are known, an iterative procedure is required by the LMTD method.

**Example 9.3-3.**

A process fluid at 100\(^\circ\)C having a specific heat of 3800 J/kg\-K and flowing at 4 kg/s is to be cooled with chilled water (specific heat of 4180 J/kg\-K). Cooling water with flow rate of 9.09 kg/s is available at 15\(^\circ\)C. Assuming an overall heat transfer coefficient \(U_o\) of 2000 W/m\(^2\)-K, calculate the outlet temperatures and the heat transfer rate for a counter flow heat exchanger with heat transfer area of 7.841 m\(^2\).

**Solution**

The heat transfer rate can be obtained from the energy balances:

\[
q = \dot{m}_h c_{ph}(T_{hi} - T_{ho}) = \dot{m}_c c_{pc}(T_{co} - T_{ci})
\]

\[
q = (4)(3800)(100 - T_{ho}) = (9.09)(4180)(T_{co} - 15)
\]

\[
q = (15,200)(100 - T_{ho}) = (38,000)(T_{co} - 15)
\]
\[
q = (15,200)(100 - T_{ho}) = UA\Delta T_{lm} = (2000)(7.841) \frac{(100 - T_{co}) - (T_{ho} - 15)}{\ln \frac{100 - T_{co}}{T_{ho} - 15}}
\]

We need to solve the following two equations for two unknowns \(T_{ho}\) and \(T_{co}\):

\[
(15,200)(100 - T_{ho}) = (38,000)(T_{co} - 15)
\]

\[
(15,200)(100 - T_{ho}) = (2000)(7.841) \frac{(100 - T_{co}) - (T_{ho} - 15)}{\ln \frac{100 - T_{co}}{T_{ho} - 15}}
\]

The following Matlab codes can be used to solve the above equations: (Note: The codes between the dash line (----) must be saved under the file name “LMTD” before the statement \(p=fminsearch('LMTD',[60 40])\) can be used in the command window to find the solution).

\[\begin{align*}
\text{function } y &= \text{LMTD}(p) \\
\text{Tho} &= p(1); Tco = p(2); \\
f1 &= (15200)*(100 - Tho) - (38000)*(Tco - 15); \\
f2 &= (15200)*(100 - Tho) - 2000*7.841*((100 - Tco) - (Tho - 15))/\log((100 - Tco)/(Tho - 15)); \\
y &= f1*f1 + f2*f2;
\end{align*}\]

\[\begin{align*}
>> p &= fminsearch('LMTD',[60 40])
\end{align*}\]

\[\begin{align*}
p &= 5.0001e+001 \quad 3.5000e+001
\end{align*}\]

Therefore \(T_{ho} = 50^\circ C\) and \(T_{co} = 35^\circ C\). The heat transfer rate is then

\[
q = (15,200)(100 - T_{ho}) = (15,200)(100 - 50) = 760,000 \text{ W}
\]

In Example 9.3-3 we need to solve two nonlinear algebraic equations since the outlet temperatures are unknown. There is alternative procedure called the effectiveness-NTU method that may be used to avoid an iterative procedure by the LMTD method.
9.4 The Effectiveness-NTU Method

If only the inlet temperatures are known, the effectiveness-NTU method may be used to avoid an iterative procedure by the LMTD method. The NTU method can also be applied even when this is not the case. From the parallel flow heat exchanger analysis in section 9.3 we list equation (9.3-18)

\[
\ln \left( \frac{T_{hi} - T_{ci}}{T_{ho} - T_{co}} \right) = \left( \frac{1}{C_h} + \frac{1}{C_c} \right) U_o A_o
\]  

Equation (9.3-18) becomes

\[
\ln \frac{\Delta T_1}{\Delta T_2} = \left( \frac{1}{C_{\text{min}}} + \frac{1}{C_{\text{max}}} \right) UA
\]

Factoring out \( C_{\text{min}} \) and defining \( NTU \) to be \( \frac{UA}{C_{\text{min}}} \) we have

\[
\ln \frac{\Delta T_2}{\Delta T_1} = -\frac{UA}{C_{\text{min}}} \left( 1 + \frac{C_{\text{min}}}{C_{\text{max}}} \right) = -NTU \left( 1 + \frac{C_{\text{min}}}{C_{\text{max}}} \right)
\]

or

\[
\frac{\Delta T_2}{\Delta T_1} = \frac{T_{ho} - T_{co}}{T_{hi} - T_{ci}} = \exp \left\{ -NTU \left( 1 + \frac{C_{\text{min}}}{C_{\text{max}}} \right) \right\}
\]

We now define the effectiveness \( \epsilon \) of the heat exchanger as the ratio of the actual heat transfer rate to the maximum heat transfer rate:

\[
\epsilon = \frac{q}{q_{\text{max}}}
\]
In this equation, the actual heat transfer rate is given by

\[ q = \dot{m}_h c_{ph} (T_{hi} - T_{ho}) = \dot{m}_c c_{pc} (T_{co} - T_{ci}) \]  \hspace{1cm} (9.4-4)

The maximum heat transfer rate \( q_{\text{max}} \) is defined for the heat transfer in an infinite long counter flow heat exchanger as shown in Figure 9.4-2 for the cases \( C_h > C_c \) and \( C_h < C_c \).

**Figure 9.4-2** Infinite long counter flow heat exchanger.

In an infinite long counter flow heat exchanger, when the hot stream has more energy that can be received by the cold stream \( (C_h > C_c) \), the outlet temperature of the cold stream \( T_{co} \) will eventually reach the inlet temperature of the hot stream, i.e., \( T_{co} = T_{hi} \). Hence

\[ q_{\text{max}} = \dot{m}_c c_{pc} (T_{co} - T_{ci}) = C_c (T_{hi} - T_{ci}) = C_{\text{min}} (T_{hi} - T_{ci}) \]

On the other hand, when all the energy of the hot stream can be received by the cold stream \( (C_h < C_c) \), the outlet temperature of the hot stream \( T_{ho} \) will eventually reach the inlet temperature of the cold stream, i.e., \( T_{ho} = T_{ci} \). Hence

\[ q_{\text{max}} = \dot{m}_h c_{ph} (T_{hi} - T_{ho}) = C_h (T_{hi} - T_{ci}) = C_{\text{min}} (T_{hi} - T_{ci}) \]

The maximum heat transfer rate is therefore always given by

\[ q_{\text{max}} = C_{\text{min}} (T_{hi} - T_{ci}) \]  \hspace{1cm} (9.4-5)

The heat exchanger effectiveness is then

\[ \varepsilon = \frac{q}{q_{\text{max}}} = \frac{C_h (T_{hi} - T_{ho})}{C_{\text{min}} (T_{hi} - T_{ci})} = \frac{C_c (T_{co} - T_{ci})}{C_{\text{min}} (T_{hi} - T_{ci})} \]  \hspace{1cm} (9.4-6)

If \( C_{\text{min}} = C_h \) then \( \varepsilon = \frac{(T_{hi} - T_{ho})}{(T_{hi} - T_{ci})} \)

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If \( C_{\min} = C_c \) then \( \varepsilon = \frac{(T_{co} - T_{ci})}{(T_{hi} - T_{ci})} \)

We now want to express \( \frac{T_{ho} - T_{co}}{T_{hi} - T_{ci}} = \exp \left\{ -NTU \left( 1 + \frac{C_{\min}}{C\max} \right) \right\} \) in term of \( \varepsilon \) only. Adding and subtracting \( T_{hi} \) and \( T_{ci} \)

\[
\frac{T_{ho} - T_{co}}{T_{hi} - T_{ci}} = \frac{T_{ho} - T_{hi} + T_{hi} - T_{co} + T_{ci} - T_{ci}}{T_{hi} - T_{ci}} = \frac{(T_{ho} - T_{hi}) + (T_{hi} - T_{ci}) - (T_{co} - T_{ci})}{T_{hi} - T_{ci}}
\]

\[
\frac{T_{ho} - T_{co}}{T_{hi} - T_{ci}} = \frac{T_{ho} - T_{hi}}{T_{hi} - T_{ci}} + \frac{T_{hi} - T_{ci}}{T_{hi} - T_{ci}} = \frac{T_{co} - T_{ci}}{T_{hi} - T_{ci}}
\]

From Equation (9.4-6),

\[
\varepsilon = \frac{C_{h} (T_{hi} - T_{ho})}{C_{\min} (T_{hi} - T_{ci})} \Rightarrow \frac{T_{ho} - T_{hi}}{T_{hi} - T_{ci}} = -\varepsilon \frac{C_{\min}}{C\max}
\]

Let \( C_{\min} = C_c \) and \( C\max = C_h \) then \( \varepsilon = \frac{(T_{co} - T_{ci})}{(T_{hi} - T_{ci})} \). Hence

\[
\frac{T_{ho} - T_{co}}{T_{hi} - T_{ci}} = -\varepsilon \frac{C_{\min}}{C\max} + 1 - \varepsilon = 1 - \varepsilon \left( \frac{C_{\min}}{C\max} + 1 \right) = \exp \left\{ -NTU \left( 1 + \frac{C_{\min}}{C\max} \right) \right\}
\]

Solving for \( \varepsilon \), we obtain with the definition \( C_r = \frac{C_{\min}}{C\max} \)

\[
\varepsilon = \frac{1 - \exp \left\{ -NTU \left( 1 + \frac{C_{\min}}{C\max} \right) \right\}}{1 + \frac{C_{\min}}{C\max}} = \frac{1 - \exp \left\{ -NTU \left( 1 + C_{r} \right) \right\}}{1 + C_{r}} \quad (9.4-7)
\]

**Exercise:** If \( C_{\min} = C_h \) and \( C\max = C_c \) show that the above equation is still valid.

In general \( \varepsilon = f\left( NTU, C_r \right) \). If \( NTU, C_{\min}, C\max, T_{hi}, \) and \( T_{ci} \) are known then \( T_{ho} \) and \( T_{co} \) can be computed directly with the effectiveness-\( NTU \) method with no iteration as with the LMTD method.

\[
\varepsilon = \frac{q}{q_{\max}} = \frac{C_{h} (T_{hi} - T_{ho})}{C_{\min} (T_{hi} - T_{ci})} = \frac{C_c (T_{co} - T_{ci})}{C_h (T_{hi} - T_{ci})}
\]

Since \( \varepsilon \) is known, the above equation can be solved for \( T_{ho} \) and \( T_{co} \).
Chapter 9

Table 9.4-1 shows the expressions of effectiveness for a variety of heat exchangers. For shell-and-tube heat exchanger with \( n \) shell passes, \((NTU)\) would first be calculated using the heat transfer area for one shell, \( \varepsilon_1 \) would then be calculated from Equation (9.4-9), and \( \varepsilon \) would finally be calculated from Equation (9.4-10). For \( C_r = 0 \), as in a boiler or condenser, the heat exchanger behavior is independent of flow arrangement and \( \varepsilon \) is given by Equation (9.4-14). Figure 9.4-3 shows the effectiveness of parallel and counter flow heat exchanger.

<table>
<thead>
<tr>
<th>Table 9.4-1 Heat Exchanger Effectiveness Relations</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Concentric tube</strong></td>
</tr>
<tr>
<td>Parallel flow</td>
</tr>
<tr>
<td>[ \varepsilon = \frac{1 - \exp\left{ -NTU \left(1 + C_r \right) \right}}{1 + C_r} ] (9.4-7)</td>
</tr>
<tr>
<td>Counter flow</td>
</tr>
<tr>
<td>[ \varepsilon = \frac{1 - \exp\left{ -NTU \left(1 - C_r \right) \right}}{1 - C_r \exp\left{ -NTU \left(1 - C_r \right) \right}} ] (9.4-8)</td>
</tr>
<tr>
<td>[ \varepsilon = \frac{NTU}{1 + NTU} ]</td>
</tr>
<tr>
<td><strong>Shell-and-tube</strong></td>
</tr>
<tr>
<td>One shell pass 2, 4,…tube passes</td>
</tr>
<tr>
<td>[ \varepsilon_1 = 2 \left{ 1 + C_r + \left(1 + C_r^2\right)^{1/2} \right} \left[ 1 + \exp\left{ -NTU_1 \left(1 + C_r^2\right)^{1/2} \right} \right]^{-1} ] (9.4-9)</td>
</tr>
<tr>
<td>[ \varepsilon_1 = 2 \left{ 1 + C_r + \left(1 + C_r^2\right)^{1/2} \right} \left[ 1 - \exp\left{ -NTU_1 \left(1 + C_r^2\right)^{1/2} \right} \right]^{-1} ]</td>
</tr>
<tr>
<td>[ \varepsilon = \left[ \frac{\left(1 - \varepsilon_1 \right)^n}{1 - \varepsilon_1} \right]^{-1} ] (9.4-10)</td>
</tr>
<tr>
<td>Cross-flow (single pass)</td>
</tr>
<tr>
<td>Both fluids unmixed</td>
</tr>
<tr>
<td>[ \varepsilon = 1 - \exp\left{ \frac{1}{C_r} \right} \left(NTU\right)^{0.22} \left{ \exp\left{ -C_r \left(NTU\right)^{0.78}\right} - 1 \right} ] (9.4-11)</td>
</tr>
<tr>
<td>[ \varepsilon = \left(1 - \exp\left{ -C_r \left(NTU\right)^{0.78}\right} - 1 \right} ]</td>
</tr>
<tr>
<td><strong>Exchangers with ( C_r = 0 )</strong></td>
</tr>
<tr>
<td>[ \varepsilon = 1 - \exp\left( -NTU \right) ] (9.4-14)</td>
</tr>
</tbody>
</table>

---

Example 9.4-1.

Derive the relation $\varepsilon = \frac{1 - \exp[-NTU(1 + C_r)]}{1 + C_r}$ for a parallel-flow concentric tube heat exchanger. You can assume $\dot{m}_c c_p c < \dot{m}_h c_p h$ and first evaluate the outlet temperature of the cold fluid $T_{co}$ in terms of the known parameters: $T_{ci}$, $T_{hi}$, $\dot{m}_c$, $\dot{m}_h$, $A$, and $U$.

Solution

---

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Choose a control volume $A_c dx$ at a distance $x$ from the inlets where $A_c$ is the cross-sectional area of the tube. Applying the energy balance around this control volume gives

$$\dot{m}_c c_{p,c} T_{c|x} - \dot{m}_c c_{p,c} T_{c|x+dx} + UP dx (T_h - T_c) = 0$$

Let $C_{\text{min}} = \dot{m}_c c_{p,c}$. Rearranging the equation and dividing by $dx$ gives

$$\frac{C_{\text{min}} (T_{c|x} - T_{c|x+dx})}{dx} = - UP (T_c - T_h)$$

In the limit $dx \to 0$, we have

$$C_{\text{min}} \frac{dT_c}{dx} = - UP (T_c - T_h)$$

(E-1)

In this equation, $P$ is the perimeter of the inner tube. Before integrating the equation over the length of the heat exchanger we need to express $T_h$ as a function of $T_c$. Choose a control volume over the first part of the heat exchanger from 0 to $x$, the energy supplied to the cold stream by the hot stream is given by

$$q_x = \dot{m}_h c_{p,h} (T_{hi} - T_h)$$

$q_x$ is also the energy received by the cold stream, therefore

$$q_x = \dot{m}_h c_{p,h} (T_{hi} - T_h) = \dot{m}_c c_{p,c} (T_c - T_{ci})$$

$$C_{\text{max}} (T_{hi} - T_h) = C_{\text{min}} (T_c - T_{ci})$$

Solving for $T_h$ gives

$$T_h = T_{hi} - \frac{C_{\text{min}}}{C_{\text{max}}} (T_c - T_{ci}) = T_{hi} - C_i (T_c - T_{ci})$$
Substituting the expression for $T_h$ into Eq. (E-1) we obtain

$$C_{\min} \frac{dT_c}{dx} = -UP(T_c - T_h) = -UP[T_c - T_{hi} + C_r(T_c - T_{ci})]$$

$$C_{\min} \frac{dT_c}{dx} = -UP[(1 + C_r)T_c - T_{hi} - C_r T_{ci}]$$

Separating the variables and integrating over the length of the heat exchanger gives

$$\int_{T_{ci}}^{T_{co}} \frac{dT_c}{(1 + C_r)T_c - T_{hi} - C_r T_{ci}} = -\frac{UP}{C_{\min}} \int_0^L dx$$

$$\frac{1}{1 + C_r} \ln \frac{(1 + C_r)T_{co} - T_{hi} - C_r T_{ci}}{(1 + C_r)T_{ci} - T_{hi} - C_r T_{ci}} = -\frac{UPL}{C_{\min}} = -\frac{UA}{C_{\min}} = -NTU$$

The outlet temperature $T_{co}$ of the cold stream can then be evaluated from the following expression:

$$\frac{(1 + C_r)T_{co} - T_{hi} - C_r T_{ci}}{(1 + C_r)T_{ci} - T_{hi} - C_r T_{ci}} = \exp[-NTU(1 + C_r)] \quad (E-2)$$

We can rearrange equation (E-2) to solve for $\varepsilon$.

Since

$$\varepsilon = \frac{q}{q_{\max}} = \frac{C_{\min} (T_{co} - T_{ci})}{C_{\min} (T_{hi} - T_{ci})} = \frac{T_{co} - T_{ci}}{T_{hi} - T_{ci}}$$

The left hand side of Eq. (E-2) will be expressed in terms of $\varepsilon$

$$\frac{C_r(T_{co} - T_{ci}) + T_{co} - T_{hi}}{T_{ci} - T_{hi}} = \exp[-NTU(1 + C_r)]$$

$$-\varepsilon C_r + \frac{T_{co} - T_{ci} + T_{ci} - T_{hi}}{T_{ci} - T_{hi}} = \exp[-NTU(1 + C_r)]$$

$$-\varepsilon C_r - \varepsilon + 1 = -(1 + C_r) \varepsilon + 1 = \exp[-NTU(1 + C_r)]$$

Hence

$$\varepsilon = \frac{1 - \exp[-NTU(1 + C_r)]}{1 + C_r}$$
Example 9.4-2. Gray whales have counter-flow heat exchange in their tongues to preserve heat. The tip of the tongue is cooled with the cold sea water. The heat exchange is between the incoming warm bloodstream (entering with the deep-body temperature) flowing through the arteries and the outgoing cold bloodstream (leaving the tongue surface region) flowing through the veins. This is shown in Figure 9.4-1. In each heat exchanger unit, nine veins of diameter $D_c$ completely encircle (no heat loss to the surroundings) the central artery of diameter $D_h$. The length of the heat exchange region is $L$. Determine the exit temperature of the cold bloodstream $T_{c,o}$. The inlet temperature of the cold bloodstream $T_{c,i}$ is 2°C. The inlet temperature of the warm bloodstream $T_{h,i}$ is 36°C. $L = 55$ cm, $D_h = 3$ mm, $D_c = 1$ mm, $u_h = 1$ mm/s, $u_c = 1$ mm/s. The resistance to heat transfer by conduction through the tongues tissues is $R_{cond} = 5$°C/ W. The resistance in the bloodstreams can be obtained from the following relations

$$\frac{1}{R_{conv,h}} = A_h Nu_h \frac{k_f}{D_h}, \quad \frac{1}{R_{conv,c}} = A_c Nu_c \frac{k_f}{D_c}, \text{ for } Re < 10, Nu = 4.36$$

Use the following properties for blood: $\rho_f = 1,000$ kg/m$^3$, $k_f = 0.590$ W/m-K, $\nu_f = 1.13 \times 10^{-6}$ m$^2$/s, $c_{p,f} = 4,186$ J/kg-K

**Figure E9.4-2.** A schematic of the vascular heat exchanger in the gray whale tongue.

**Solution**

Since only the inlet temperatures are given, we will use the effectiveness-NTU method to solve for the exit temperature of the cold blood stream. First, we evaluate the Reynolds numbers in the artery and in the veins

$$Re_h = \frac{u_h D_h}{\nu_f} = \frac{10^{-3} (m/s) \times 3 \times 10^{-3} (m)}{1.13 \times 10^{-6} (m^2/s)} = 2.655$$

---

The values of the Reynolds number are less than 10. Therefore we can use the given expression to determine the thermal resistance in the blood streams.

\[ \frac{1}{R_{\text{conv},h}} = A_h Nu_h \frac{k_f}{D_h} = \pi D_h L Nu_h \frac{k_f}{D_h} = \pi L Nu_h k_f \]

\[ \frac{1}{R_{\text{conv},h}} = \pi(0.55)(4.36)(0.59) = 4.448 \text{ W/K} \Rightarrow R_{\text{conv},h} = 0.225 \text{ K/W} \]

\[ \frac{1}{R_{\text{conv},c}} = A_c Nu_c \frac{k_f}{D_c} = 9 \pi D_c L Nu_c \frac{k_f}{D_c} = 9 \pi L Nu_h k_f \]

\[ \frac{1}{R_{\text{conv},c}} = 9\pi(0.55)(4.36)(0.59) = 40.0 \text{ W/K} \Rightarrow R_{\text{conv},c} = 0.025 \text{ K/W} \]

The total resistance is the sum of the individual resistances

\[ R_t = R_{\text{conv},c} + R_{\text{cond}} + R_{\text{conv},h} = 0.025 + 5.0 + 0.225 = 5.25 \text{ K/W} \]

\[ UA = \frac{1}{R_t} = 0.1905 \text{ W/K} \]

The cold and hot blood flow rates are given by

\[ \dot{m}_c = 9 \rho_f \frac{\pi D_c^2}{4} u_c = (9)(1000) \frac{\pi(10^{-3})^2}{4} 10^{-3} = 7.07 \times 10^{-6} \text{ kg/s} \]

\[ \dot{m}_h = \rho_h \frac{\pi D_h^2}{4} u_c = (1000) \frac{\pi(3 \times 10^{-3})^2}{4} 10^{-3} = 7.07 \times 10^{-6} \text{ kg/s} \]

The hot and cold heat capacity rates are then calculated

\[ C_h = \dot{m}_h c_{p,h} = (7.07 \times 10^{-6})(4186) = 0.0296 \text{ W/K} \]
\[ C_c = \dot{m}_c c_{p,c} = (7.07 \times 10^{-6})(4186) = 0.0296 \text{ W/K} \]

Therefore \( C_t = \frac{C_{\text{min}}}{C_{\text{max}}} = 1 \) and the number of transfer unit NTU is:
\[ NTU = \frac{UA}{C_{\min}} = \frac{0.1905}{0.0296} = 6.437 \]

For counter flow heat exchanger with \( C_r = 1 \), we have
\[
\varepsilon = \frac{NTU}{1 + NTU} = 0.8656 = \frac{C_r(T_{c,o} - T_{c,i})}{C_{\min}(T_{h,i} - T_{c,i})} = \frac{(T_{c,o} - T_{c,i})}{(T_{h,i} - T_{c,i})}
\]

The exit temperature of the cold bloodstream, \( T_{c,o} \), is finally
\[
T_{c,o} = T_{c,i} + 0.8656(T_{h,i} - T_{c,i}) = 2 + 0.8656(36 - 2) = 31.43^\circ C
\]

Example 9.4-3. 

Oil having a specific heat of 2000 J/kg·K and flowing at 1 kg/s is to be cooled from 340 K to 310 K with water. Cooling water is available at 290 K with the outlet temperature limited to 300 K.

(a) Determine \( UA \) for a shell-and-tube heat exchanger with one shell pass and two tube passes.

(b) Water enters the exchanger with \( UA = 2212 \) W/K at 290 K and a capacitance rate of 6300 W/K. If oil with a specific heat of 2100 J/kg·K enters the exchanger at 370 K and 0.75 kg/s, determine the outlet temperatures of oil and water.

Solution

(a) Determine \( UA \)
\[
\Delta T_1 = T_{h,i} - T_{c,o} = 340 - 300 = 40 \text{ K}
\]
\[
\Delta T_2 = T_{h,o} - T_{c,i} = 310 - 290 = 20 \text{ K}
\]
\[
\Delta T_{lm} = \frac{\Delta T_1 - \Delta T_2}{\ln(\Delta T_1 / \Delta T_2)} = \frac{40 - 20}{\ln(40 / 20)} = 28.85 \text{ K}
\]

\( UA \) required for shell-and-tube heat exchanger
\[
AU = \frac{q}{\Delta T_{lm} F}
\]
\[
q = \dot{m}_h c_{p,h}(T_{h,i} - T_{h,o}) = (1)(2000)(340 - 310) = 60,000 \text{ J/s}
\]

The \( F \) factor can be evaluated from
\[
R = \frac{T_{h,i} - T_{h,o}}{T_{c,o} - T_{c,i}} = 3, \quad P = \frac{T_{c,o} - T_{c,i}}{T_{h,i} - T_{c,i}} = 0.2
\]
\[ F = \left( \frac{\sqrt{R^2 + 1}}{R - 1} \right) \ln \left\{ \frac{(1 - P)}{(1 - RP)} \right\} = 0.935 \]

\[ \ln \left\{ \frac{2 - P\left(R + 1 - \sqrt{R^2 + 1}\right)}{2 - P\left(R + 1 + \sqrt{R^2 + 1}\right)} \right\} \]

\[ AU = \frac{q}{\Delta T_{in} F} = \frac{60,000}{(28.85)(0.935)} = 2224 \text{ W/K} \]

(b) Outlet temperatures of oil and water

We use the effectiveness-NTU method.

\[ C_h = (2100)(0.75) = 1575 \text{ W/K} < C_c = 6300 \text{ W/K} \]

\[ NTU = \frac{UA}{C_{\min}} = \frac{2212}{1575} = 1.4044, \quad C_r = \frac{C_{\min}}{C_{\max}} = \frac{1575}{6300} = 0.25 \]

\[ \varepsilon = 2 \left\{ 1 + C_r + (1 + C_r^2)^{1/2} \frac{1 + \exp \left[ -NTU \left(1 + C_r^2\right)^{1/2} \right]}{1 - \exp \left[ -NTU \left(1 + C_r^2\right)^{1/2} \right]} \right\}^{-1} = 0.686 \]

\[ \varepsilon = 0.686 = \frac{C_h(T_{hi} - T_{ho})}{C_{\min}(T_{hi} - T_{ci})} = \frac{T_{hi} - T_{ho}}{T_{hi} - T_{ci}} \]

\[ T_{ho} = T_{hi} - 0.686(T_{hi} - T_{ci}) = 370 - 0.686(370 - 290) = 315.1 \text{ K} \]

\[ P = 0.686 = \frac{C_c(T_{co} - T_{ci})}{C_{\min}(T_{ho} - T_{ci})} \]

\[ T_{co} = T_{ci} + 0.686(T_{hi} - T_{ci}) \frac{C_{\min}}{C_c} = 290 + 0.686(370 - 290) \frac{1575}{6300} = 303.7 \text{ K} \]
10.1 Introduction

Definition of Radiation

Radiation heat transfer is the energy carried by electromagnetic waves called photon heat carriers. In thermal radiation, the energy is emitted at the surface of a solid only as a result of its temperature. If the temperature is at absolute zero, there is no thermal radiation. If \( T_s > T_{\text{sur}} \), experiments show that a solid cools even in vacuum. This cooling is a direct consequence of the emission of the thermal radiation from the surface since the vacuum prevents energy loss from the surface of the solid by conduction or convection.

Mechanism of Radiation

Radiation is due to the molecular electronic, rotational, and vibrational energy transitions of the matter. The emission of thermal radiation is associated with thermally excited conditions within the matter. 'Excited' molecules at a surface releases 'packets' of energy which, from quantum mechanics, interact with matter in discrete quanta or 'photons'. Photons travel in straight line at speed of light, \( c \), \( (2.998 \times 10^8 \text{ m/s}) \).

'Wave' with frequency of emissions \( \nu \) and wavelength \( \lambda \) can be attributed to photons, the two properties are related by
A photon has an energy $E$ given by Planck's law

$$E = \frac{\hbar}{\nu}$$

where $\hbar$ = Planck's constant = $6.625 \times 10^{-34}$ J·s/molecule. Energy exchange is by photon interaction. In this chapter mostly the surface-radiation heat transfer will be discussed.

Wavelength of photon defines kind of radiation emitted such as ultraviolet, visible light, infrared, or microwave. The wavelength of visible light is between 0.4 µm and 0.7 µm. 'White' light is the total sum of all colors or electromagnetic radiation from 0.1 µm to 10 µm. Thermal radiation is electromagnetic radiation from 0.1 µm to 100 µm. Only this range of the spectrum is relevant to heat transfer since for wavelength less than 0.1 µm the temperature will be too high and for wavelength greater than 100 µm the flux will be too low.

**Figure 10.1-1** Spectrum of electromagnetic radiation\(^1\).

### 10.2 Blackbody Radiation

The blackbody is an ideal radiator with the following properties.

---

A blackbody absorbs all incident radiation independent of wavelength and direction. No surface can emit more energy than a blackbody at the same temperature and wavelength. A blackbody is a diffuse emitter since its emission is independent of direction.

An actual blackbody does not exist; the closest approximation to a blackbody is a cavity whose inner surface is at a uniform temperature. When radiation enters the cavity through a small aperture it is almost completely absorbed by the cavity since it is very likely to be reflected many times before leaving the cavity.

Figure 10.2-1 shows the spectral emissive power $E_{\lambda,b}$ of a blackbody as a function of temperature and wavelength [2]. The term spectral is used to denote the dependence of the emissive power on wavelength. The surface emissive power is the rate at which energy is released per unit area.

![Complete absorption](image1)

![Diffuse emitter](image2)

Figure 10.2-1 Spectral blackbody emissive power$^2$.

---

\[ E_{\lambda,b} = \frac{C_1}{\lambda^5 \left[ \exp\left(\frac{C_2}{\lambda T}\right) - 1 \right]} \]  

(10.2-1)

where \( C_1 = 3.742 \times 10^8 \) W·μm\(^4\)/m\(^2\) and \( C_2 = 1.439 \times 10^4 \) μm·K. The unit of \( E_{\lambda,b} \) is W/m\(^2\)·μm.

The total emissive power of a blackbody is the rate of thermal radiation energy emitted over the entire spectrum at a given temperature,

\[ E_b = \int_{0}^{\infty} E_{\lambda,b} d\lambda = \int_{0}^{\infty} \frac{C_1}{\lambda^5 \left[ \exp\left(\frac{C_2}{\lambda T}\right) - 1 \right]} d\lambda \]  

(10.2-2)

Let \( x = \frac{C_2}{\lambda T} \) then \( \lambda = \frac{C_2}{xT} \); \( d\lambda = -\frac{C_2}{Tx^2} \, dx \); \( \frac{1}{\lambda^5} = \frac{x^5T^5}{C_2^5} \).

Eq. (10.2-2) becomes

\[ E_b = -\left( \frac{C_1}{C_2^4} \int_{x}^{\infty} \frac{x^3}{e^x - 1} \, dx \right) T^4 = \left( \frac{C_1}{C_2^4} \int_{0}^{\infty} \frac{x^3}{e^x - 1} \, dx \right) T^4 = \sigma T^4 \]  

(10.2-3)

where \( \sigma = \frac{C_1}{C_2^4} \int_{0}^{\infty} \frac{x^3}{e^x - 1} \, dx = \frac{C_1 \pi^4}{15} = 5.67 \times 10^{-8} \) W/m\(^2\)·K\(^4\).

Divide the spectral emissive power by \( T^5 \) to obtain

\[ \frac{E_{\lambda,b}}{T^5} = \frac{C_1}{\lambda^5 \left[ \exp\left(\frac{C_2}{\lambda T}\right) - 1 \right]} \]

The plot of \( \frac{E_{\lambda,b}}{T^5} \) versus \( \lambda T \) yields a single curve with a maximum at \( \lambda T = 0.028978 \) m·K.
Chapter 10

Radiation Heat Transfer

10.3 Radiation Intensity

Radiation intensity $I_{\lambda,e}$ is defined as the rate at which radiant energy is emitted at the wavelength $\lambda$ in the $(\theta, \phi)$ direction, per unit area of the emitting surface normal to this direction $dA_1 \cos \theta$, per unit solid angle $d\omega$, and per unit wavelength interval $d\lambda$.

$$I_{\lambda,e} \left[ \frac{W}{m^2 \cdot sr \cdot \mu m} \right] = \frac{dq}{(dA_1 \cos \theta)(d\omega)(d\lambda)} \quad (10.3-1)$$

The solid angle $d\omega$ is defined as the ratio of the element of area $dA_n$ on the sphere to the square of the sphere’s radius.

$$d\omega = \frac{dA_n}{r^2} = \frac{(rd\theta)(r \sin \theta d\phi)}{r^2} = \sin \theta d\theta d\phi \quad (10.3-2)$$

The unit of solid angle is steradian (sr) and the unit of wavelength is micrometer ($\mu m$).

**Emissive Power**

The spectral, hemispherical emissive power $E_{\lambda}$ is the rate at which radiation of wavelength $\lambda$ is emitted in all directions from a surface per unit wavelength $d\lambda$ and per unit surface area $dA_1$. The spectral, hemispherical emissive power is obtained by integrating the radiation intensity $I_{\lambda,e}$ over the solid angle $\sin \theta d\theta d\phi$.

$$E_{\lambda} \left[ \frac{W}{m^2 \cdot \mu m} \right] = \int_0^{\pi/2} \int_0^{2\pi} I_{\lambda,e} \cos \theta \sin \theta d\theta d\phi \quad (10.3-3)$$
\( E_{\lambda} \) is a flux based on the actual surface area \( dA_1 \), whereas the radiation intensity \( I_{\lambda,e} \) is based on the projected area \( dA_1 \cos \theta \). Therefore the \( \cos \theta \) term is included in the integrand of equation (10.3-3) so that \( \cos \theta I_{\lambda,e} \) is the radiation intensity based on the actual area \( dA_1 \).

\[
I_{\lambda,e} \cos \theta = \frac{W}{m^2 \cdot sr \cdot \mu m} = \frac{dq}{(dA_1 \cos \theta)(d\omega)(d\lambda)} \cos \theta = \frac{dq}{(dA_1)(d\omega)(d\lambda)}
\]

For a diffuse emitter, \( I_{\lambda,e} \) is independent of direction, equation (10.3-3) can easily be integrated

\[
E_{\lambda} = \int_0^{2\pi} \int_0^{\pi/2} I_{\lambda,e} \cos \theta \sin \theta d\theta d\phi = I_{\lambda,e} \int_0^{\pi/2} \sin(2\theta) d\theta d\phi
\]

\[
E_{\lambda} = \pi I_{\lambda,e} \int_0^{\pi/2} \sin(2\theta) d\theta = \pi I_{\lambda,e} \left( -\frac{\cos(2\theta)}{2} \right) \bigg|_0^{\pi/2} = -\frac{1}{2} \pi I_{\lambda,e} (-1 - 1)
\]

\[
E_{\lambda} = \pi I_{\lambda,e}
\]

(10.3-4)

The total, hemispherical emissive power \( E \) is obtained by integrating the spectral, hemispherical emissive power \( E_{\lambda} \) over all wavelengths.

\[
E = \int_0^{\infty} E_{\lambda} d\lambda
\]

For a diffuse emitter

\[
E = \pi I_e
\]

(10.3-5)

**Irradiation and Radiosity**

The same concepts may be applied to radiation intensity \( I_{\lambda,i} \) incident upon a surface. Irradiation is defined as the flux at which radiant energy arrives at a surface. The spectral irradiation \( G_{\lambda} \) is related to the incident radiation intensity \( I_{\lambda,i} \) of diffuse sources by an equation similar to (10.3-4)

\[
G_{\lambda} = \pi I_{\lambda,i}
\]

(10.3-6)

The total irradiation \( G \) is related to the incident radiation intensity \( I_i \) of diffuse sources by an equation similar to (10.3-5)

\[
G = \pi I_i
\]

(10.3-7)

Radiosity accounts for all the radiant energy leaving a surface. The spectral radiaosity \( J_{\lambda} \) is related to the radiant intensity by an equation similar to (10.3-4)

\[
J_{\lambda} = \pi I_{\lambda,e+r}
\]

(10.3-8)
The total radiosity $J$ is related to the radiation intensity of diffuse sources by an equation similar to (10.3-5)

$$J = \pi I_{e+r} \quad (10.3-9)$$

The distinctions between radiant energy of emission, irradiation, and radiosity are shown in Figure 10.3-1.

![Figure 10.3-1 Distinctions between emission, irradiation, and radiosity.](image)

**Example 10.3-1**

Consider a small surface of area $A_1 = 10^{-4}$ m$^2$, which emits diffusively with a total, hemispherical emissive power of $E_1 = 5 \times 10^4$ W/m$^2$.

1) Determine the rate of radiant energy incident upon a surface of area $A_2 = 2 \times 10^{-4}$ m$^2$ oriented as shown in Figure 10.3-2.

![Figure 10.3-2 Orientation of two small surfaces.](image)

2) Determine the irradiation $G_2$ on $A_2$.

**Solution**

---

1) Rate of radiant energy emitted from surface $A_1$ incident upon surface $A_2$

$$q_{1 \rightarrow 2} = I_{e1}(\theta, \phi) A_1 \cos \theta_1 \, d\omega_{2,1}$$

Since surface $A_1$ is diffuse $I_{e1}(\theta, \phi) = I_{e1} = \frac{E_1}{\pi}$. The solid angle subtended by $A_2$ with respect to $A_1$ is

$$d\omega_{2,1} = \frac{A_2 \cos \theta_2}{r^2}$$

$$q_{1 \rightarrow 2} = \frac{E_1}{\pi} A_1 \cos \theta_1 \frac{A_2 \cos \theta_2}{r^2}$$

$$q_{1 \rightarrow 2} = \frac{5 \times 10^4}{\pi} (10^{-4} \cos 40^\circ) \frac{2 \times 10^{-4} \cos 20^\circ}{0.6^2} = 6.36 \times 10^{-4} \text{ W}$$

2) The irradiation $G_2$ on $A_2$.

The irradiation $G_2$ is simply the flux of energy arrives at surface $A_2$, therefore

$$G_2 = q_{1 \rightarrow 2}/A_2 = 6.36 \times 10^{-4}/2 \times 10^{-4} = 3.18 \text{ W/m}^2.$$  

Example 10.3-2

A furnace with an aperture of 20-mm diameter and emissive power of $3.72 \times 10^5 \text{ W/m}^2$ is used to calibrate a heat flux gage having a sensitive area of $1.6 \times 10^{-5} \text{ m}^2$.

(a) At what distance, measured along a normal from the aperture, should the gage be positioned to receive irradiation of $1000 \text{ W/m}^2$.

(b) If the gage is tilted off normal by $20^\circ$, what will be its irradiation?^2

Solution

a) The heat transfer rate from the surface to the detector is given by

---

\[ q_{dl} = I_e A_f \cos \theta_i \omega_{dl-f} \text{ where } G = 1000 \, \text{W/m}^2 = \frac{q_{dl}}{A_d} \]

\[ \omega_{dl-f} = \text{solid angle subtended by surface } A_d \text{ with respect to } A_f = \frac{A_d \cos \theta_d}{L^2} \]

Since \( I_e = \frac{E}{\pi} \) and \( \cos \theta_i = \cos \theta_d = 1 \)

\[
G = \frac{E A_f A_d}{\pi A_d L^2} \Rightarrow L = \left( \frac{E A_f}{\pi G} \right)^{1/2} 
\]

\[
L = \left( \frac{3.72 \times 10^5 \pi \times 0.01^2}{1000} \right)^{1/2} = 0.193 \, \text{m} 
\]

b) The irradiation is given by

\[ G_{20^\circ} = G \cos \theta_i = (1000) \cos(\pi/9) = 939.7 \, \text{W/m}^2 \]

---

**Example 10.3-3**

According to its directional distribution, solar radiation incident on the earth’s surface may be divided into two components. The direct component consists of parallel rays incident at a fixed zenith angle \( \theta \), while the diffuse component consists of radiation that may be approximated as being diffusely distributed with \( \theta \). Consider clear sky conditions for which the direct radiation is incident at \( \theta = 30^\circ \), with a total flux (based on an area that is normal to the rays) of \( q''_{rad} = 100 \, \text{W/m}^2 \), and the total intensity of the diffuse radiation is \( I_{diff} = 70 \, \text{W/m}^2 \cdot \text{sr} \). What is the total solar irradiation at the earth’s surface?³

\[ q''_{dir} = 1000 \, \text{W/m}^2 \]

\[ \theta = 30^\circ \]

\[ I_{diff} = 70 \, \text{W/m}^2 \cdot \text{sr} \]

---

The direct irradiation at the earth’s surface is given by

\[ G_{\text{dir}} = q''_{\text{dir}} \cos \theta \]

The diffuse irradiation at the earth’s surface is given by

\[ G_{\text{dif}} = \pi I_{\text{dif}} \]

The total irradiation at the earth’s surface is then

\[ G = G_{\text{dir}} + G_{\text{dif}} = (1000) \cos(\pi/2) + (70)(\pi) = 1086 \, \text{W/m}^2 \]

Example 10.3-4

Solar radiation incident on the earth’s surface may be divided into two components. The direct component consists of parallel rays incident at a fixed zenith angle \( \theta \), while the diffuse component consists of radiation that may be approximated as being diffusely distributed with \( \theta \). Consider conditions for a day in which the intensity of the direct solar radiation is \( I_{\text{dir}} = 2.10 \times 10^7 \, \text{W/m}^2\cdot\text{sr} \) in the solid angle subtended by the sun with respect to the earth, \( \Delta \omega_s = 6.74 \times 10^{-5} \, \text{sr} \). The intensity of the diffuse radiation is \( I_{\text{dif}} = 70 \, \text{W/m}^2\cdot\text{sr} \).

(a) What is the total solar irradiation at the earth’s surface when the direct radiation is incident at \( \theta = 30^\circ \)?

(b) Verify the prescribed value for \( \Delta \omega_s \), recognizing that the diameter of the sun is \( 1.39 \times 10^9 \, \text{m} \) and the distance between the sun and the earth is \( 1.496 \times 10^{11} \, \text{m} \) (1 astronomical unit).\(^4\)

Solution

(a) \[ G_{\text{dir}} = I_{\text{dir}} \cos \theta \Delta \omega_s \]

\[ G_{\text{dir}} = (2.10 \times 10^7) \cos(30/6)(6.74 \times 10^{-5}) = 1225.8 \, \text{W/m}^2 \]

\[ G_{\text{dif}} = I_{\text{dif}} \pi = (70) \pi = 219.9 \, \text{W/m}^2 \]

The total solar irradiation at the earth’s surface is

\[ G = 1225.8 + 219.9 = 1446 \text{ W/m}^2 \]

(b) Verify the prescribed value for \( \Delta \omega_s \)

\[
\Delta \omega_s = \frac{dA_s}{r^2} = \frac{\pi D^2_s}{4} = \frac{\pi (1.39 \times 10^5)^2}{4(1.496 \times 10^{11})} = 6.78 \times 10^{-5} \text{ sr}
\]

--------------------------------------------------------------------------------------------

Example 10.3-5

On an overcast day the directional distribution of the solar radiation incident on the earth’s surface may be approximated by an expression of the form \( I_i = I_n \cos \theta \), where \( I_n = 80 \text{ W/m}^2 \cdot \text{sr} \) is the total intensity of radiation directed normal to the surface and \( \theta \) is the zenith angle. What is the solar irradiation at the earth’s surface?\(^5\)

\[ I_n = 80 \text{ W/m}^2 \cdot \text{sr} \]

\[ I_i = I_n \cos \theta \]

Solution

The solar irradiation at the earth’s surface is given by

\[
G = \int_0^{2\pi} \int_0^{\pi/2} I_i(\theta) \cos \theta \sin \theta d\theta d\phi
\]

\[
G = 2\pi I_n \int_0^{\pi/2} \cos^2 \theta \sin \theta d\theta = 2\pi(80) \left( -\frac{\cos^3 \theta}{3} \right)_0^{\pi/2}
\]

Therefore \[ G = 2\pi(80) \frac{1}{3} = 167.6 \text{ W/m}^2 \]

Example 10.3-6

The energy flux associated with solar radiation incident on the outer surface of the earth’s atmosphere has been accurately measured and is known to be 1353 W/m$^2$. The diameters of the sun and the earth are 1.39×10$^9$ m and 1.29×10$^7$ m, respectively, and the distance between the sun and the earth is 1.5×10$^{11}$ m.

(a) What is the emissive power of the sun?
(b) Approximating the sun’s surface as black, what is its temperature?
(c) At what wavelength is the spectral emissive power of the sun a maximum?
(d) Assuming the earth’s surface to be black and the sun to be the only source of energy for the earth, estimate the earth’s surface temperature.$^6$

Solution

(a) Emissive power of the sun: $E_s(\pi D_s^2) = 4\pi(d_{s-e} - D_e/2)q_s''$

$$E_s = \frac{4(1.5 \times 10^{11} - 1.29 \times 10^7 / 2)^2 (1353)}{(1.39 \times 10^9)^2} = 6.302 \times 10^7 \text{ W/m}^2$$

(b) Approximating the sun’s surface as black, what is its temperature?

$$E_s = \sigma T^4 \Rightarrow T = \left(\frac{E_s}{\sigma}\right)^{1/4} = \left(\frac{6.302 \times 10^7}{5.67 \times 10^{-8}}\right)^{1/4} = 5774 \text{ }^0\text{K}$$

(c) At what wavelength is the spectral emissive power of the sun a maximum?

$$\lambda_{\text{max}} = C_3/T = 2897.4/5774 = 0.50 \text{ }\mu\text{m}$$

(d) Estimate the earth’s surface temperature.

$$E_e(\pi D_e^2) = \pi(D_e/2)^2q_s'' \Rightarrow E_e = q_s''/4 = \sigma T_e^4$$

$$T_e = \left(\frac{q_s''}{4\sigma}\right)^{1/4} = \left(\frac{1353}{4 \times 5.67 \times 10^{-8}}\right)^{1/4} = 278 \text{ }^0\text{K}$$

Appendix A

Previous Exams

CHE 312 (Winter 2008) ________________

Quiz #1

Note: Your answers must be correct to 3 significant figures and have the appropriate units.

I. A thin flat plate of length \( L = 0.8 \text{ m} \), thickness \( t = 5 \text{ mm} \), and width \( w \gg L \) is thermally joined to two large heat sinks that are maintained at a temperature of \( 60^\circ\text{C} \) and \( 0^\circ\text{C} \), respectively.

The bottom of the plate is well insulated, while the net heat flux \( q'' \) to the top surface is known to have a uniform value. The thermal conductivity of the plate is \( 100 \text{ W/m}^\circ\text{K} \).

(1) The steady state temperature \( T(x) \) profile of the plate may be obtained from solving

a. \(-kwt \frac{dT}{dx}\big|_x + kwt \frac{dT}{dx}\big|_{x+dx} + q'' dx = 0\)

b. \(kwt \frac{dT}{dx}\big|_x - kwt \frac{dT}{dx}\big|_{x+dx} + q'' dx = 0\)

c. \(kwt \frac{dT}{dx}\big|_x - kwt \frac{dT}{dx}\big|_{x+dx} + q'' dx = 0\)

d. None of the above (Ans.)

(2) If the temperature profile is given as \( T = -2000x^2 + 2320x + 60 \), where \( T \) is in \( ^\circ\text{C} \) and \( x \) is in \( \text{m} \). The highest temperature of the plate is

\[
\frac{dT}{dx} = -4000x + 2320 \Rightarrow x = 0.580
\]

\[
T = -2000(0.58)^2 + 2320(0.58) + 60 = 732.8^\circ\text{C}
\]

II. Consider the heat conduction equation: \( \rho c_p \frac{\partial T}{\partial t} = \nabla(k\nabla T) + q'' \)

A. This equation is only valid for rectangular coordinate system.

B. \( q'' \) represents heat generation in \( \text{W/m}^2 \) for SI units.

a. A and B are true   b. Only A is true   c. Only B is true   d. A and B are false (Ans.)
III. Consider a cylindrical rod with radius $R$ and length $L$ with temperature at the base ($x = 0$) equal to $T_b > T_{\text{inf}}$. Temperature is a function of $x$ and $r$. A shell balance can be performed to obtain $T(x,r)$. The control volume for the shell balance is

IV. A nitrogen meat freezer uses nitrogen gas from a pressurized liquid nitrogen tank to freeze meat patties as they move carried by a conveyor belt. The nitrogen flows inside a chamber in direct contact with the meat patties, which move in the opposite direction. The heat transfer mechanism between the nitrogen gas and the meat patties is surface convection. Meat patties are to be cooled down from their processing (initial) temperature of $T_i = 10^\circ C$ to the storage (final) temperature of $T_o = -15^\circ C$. Each meat patty has a mass $M = 80$ g, diameter $D = 10$ cm, and thickness $l = 1$ cm. Assume for the meat the thermophysical properties of water, i.e., specific heat in the solid state $C_{p,s} = 1,930$ J/kg-K, specific heat in the liquid state $C_{p,l} = 4,200$ J/kg-K, heat of solidification $\Delta h_{ls} = -3.34 \times 10^5$ J/kg, and freezing temperature $T_{ls} = 0^\circ C$. The average surface-convection heat transfer between the nitrogen and the meat patties is estimated as 4,000 W/m$^2$ and the conveyor belt moves with a speed of $u_c = 0.01$ m/s. You can assume that the temperature is uniform within the meat parties and neglect the heat transfer between the conveyor belt and the meat parties. Do not neglect the heat transfer from the edge of the meat parties.

(5) Determine the time for the (liquid) meat parties to cool from $10^\circ C$ to $0^\circ C$. 

$$A = \pi(10 \times 1 + 25) \times 10^{-4} = 0.011 \text{ m}^2$$

$$\frac{(4,000)(0.011)t}{(0.08)(4,200)(10)} = t = 76.4 \text{ s}$$

(6) Determine the time for the solidification process.

$$\frac{(4,000)(0.011)t}{(0.08)(3.34 \times 10^5)} = t = 607.5 \text{ s}$$

(7) Determine the time for the (solid) meat parties to cool from $0^\circ C$ to $-15^\circ C$. 

$$A = \pi(10 \times 1 + 25) \times 10^{-4} = 0.011 \text{ m}^2$$

$$\frac{(4,000)(0.011)t}{(0.08)(1,930)(15)} = t = 52.6 \text{ s}$$
V. An experiment to determine the convection coefficient associated with air flow over the surface of a thick steel casting involves insertion of thermocouples in the casting at distances of 10 and 20 mm from the surface along a hypothetical line normal to the surface. The steel has a thermal conductivity of 10 W/m·°K. The thermocouples measure temperatures of 52 and 40°C in the steel when the air temperature is 120°C.

The convection coefficient is

\[ h(120 - 64) = k \frac{\Delta T}{\Delta x} = 10 \frac{12}{0.01} \Rightarrow h = (10)(1200)/56 = 214.3 \text{ W/m}^2 \cdot \text{oK} \]

VI. In the two-dimensional body illustrated, the gradient at surface \( A \) is found to be \( \partial T/\partial y = 25 \) K/m.

At surface B, \( \partial T/\partial x = \)

At surface B, \( \partial T/\partial x = \)

\[ (1) \partial T/\partial y|_A = (0.6) \partial T/\partial x|_B \Rightarrow \partial T/\partial x|_B = 25/0.6 = 41.67 \text{ K/m} \]

VII. Determine the work required for a heat pump to deliver 2000 kJ to a water heater if the COP for the heat pump is 3 and the compressor efficiency is 75%.

\[ \text{COP} = 3 = 2000/(0.75W_c) \Rightarrow W_c = 2000/(3 \times 0.75) = 889 \text{ kJ} \]
I. A spherical aluminum tank, inside radius $R_1 = 3\, \text{m}$, and wall thickness $l_1 = 4\, \text{mm}$, contains liquid-vapor oxygen at 1 atm pressure and $90.18\,\text{°K}$. Heat of evaporation of oxygen is $2.123 \times 10^5\,\text{J/kg}$. Under steady state, at the liquid gas surface, the heat flowing into the tank causes boil off at a rate $\dot{M}_g$. In order to prevent the pressure of the tank from rising, the gas resulting from boil off is vented through a safety valve as shown in Figure 1. An evacuated air gap, extending to location $r = R_2 = 3.1\, \text{m}$, is placed where the combined conduction-radiation effect for this gap is represented by a conductivity $k_a = 0.005\, \text{W/m·K}$. A layer of insulation with $k_i = 0.032\, \text{W/m·K}$ and thickness $l_2 = 10\, \text{cm}$ is added. The outside surface temperature is kept constant at $T_2 = 283.15\,\text{°K}$. Neglect the heat resistance through the aluminum.

![Figure 1. Liquid oxygen in a spherical container.](image)

(1) Determine the rate of heat leak $Q_{k,2-1}$ in W

$$R_{l,\text{air}} = \frac{.096}{(0.005)(4\pi)(3.004\times3.1)} = 0.1641\, \text{K/W}$$

$$R_{l,\text{insulation}} = \frac{.10}{(0.032)(4\pi)(3.1\times3.2)} = 0.0251\, \text{K/W}$$

$$Q_{k,2-1} = \frac{283.15 - 90.18}{.1641 + .0251} = 1020\, \text{W}$$
(2) If $Q_{k,2-1} = 1200$ W, determine the amount of boil off $\dot{M}_g$ in kg/s.

\[
\dot{M}_g = \frac{1200}{2.123 \times 10^3} = 5.65 \times 10^{-3} \text{ kg/s}
\]

(3) If $Q_{k,2-1} = 1200$ W, determine the temperature at the inner surface ($r = R_2$) of the insulation using the thermal resistance concept through the insulation layer.

\[
R_{t,\text{insulation}} = \frac{10}{(0.032)(4\pi)(3.1 \times 3.2)} = 0.0251 \text{ K/m}
\]

\[
1200 = \frac{T_2 - T}{R_{t,\text{insulation}}} \Rightarrow T = 283.15 - (1200)(0.0251) = 253.07 \degree \text{K}
\]

II. The air inside a chamber at $T_{\infty,i} = 50 \degree \text{C}$ is heated convectively with $h_i = 25 \text{ W/m}^2\text{K}$ by a 0.25-m-thick wall having a thermal conductivity of 5 W/m·K and a uniform heat generation of 2000 W/m$^3$. To prevent any heat generated within the wall from being lost to the outside of the chamber at $T_{\infty,o} = 15 \degree \text{C}$ with $h_o = 30 \text{ W/m}^2\text{K}$, a very thin electrical strip heater is placed on the outer wall to provide a uniform heat flux, $q_o$.

If no heat generated within the wall is lost to the outside of the chamber, determine the temperature at the wall boundary $T(L)$.

\[
(4) \quad T(L) = \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad
\]

\[
2000L = h_i(T_L - T_{\infty,i}) \Rightarrow T_L = \frac{(2000)(0.25)}{25} + 50 = 70 \degree \text{C}
\]
VI. A thin electrical heater is wrapped around the outer surface of a cylindrical tube whose inner surface is maintained at a temperature of 5°C. The tube wall has inner and outer radii of 25 and 75 mm, respectively, and a thermal conductivity of 10 W/m·K. The thermal contact resistance between the heater and the outer surface of the tube (per unit length of the tube) is $R_{t,c} = 0.01$ m·K/W. The outer surface of the heater is exposed to a fluid with $T_\infty = -10°C$ and a convection coefficient of $h = 100$ W/m²·K. The length of the tube is $L$.

![Diagram of a cylindrical tube with an electrical heater wrapped around it.](image)

10) The heater power per unit length of tube required to maintain the heater at $T_o = 25°C$ is

\[ A) \frac{T_o - T_i}{\ln(r_o/r_i)} + \frac{T_o - T_\infty}{2\pi k L} + R_{t,c} \]

\[ B) \frac{T_o - T_i}{h\pi D_o L} + R_{t,c} \frac{T_o - T_\infty}{2\pi k L} \]

\[ C) \frac{T_o - T_i}{\ln(r_o/r_i)} + \frac{T_o - T_\infty}{2\pi k} + R_{t,c} \frac{T_o - T_\infty}{h\pi D_o} \]

\[ D) \text{None of the above} \]
The questions on mass transfer were deleted from this quiz.

I. Consider a single stack of rectangular fins of length $L$ and thickness $t$, with convection conditions corresponding to $h$ and $T_\infty$. In a specific application, a stack that is 200 mm wide and 100 mm deep (let $D =$ depth $= 100$ mm) contains 50 fins, each of length $L = 12$ mm. The entire stack is made from aluminum, which is everywhere 1.0 mm thick. Data: $T_o = 400$ K, $T_L = 300$ K, $T_\infty = 100$ K, and $h = 80$ W/m$^2$·K. Note: The top surface of the upper plate and the bottom surface of the lower plate are not exposed to the convection conditions.

1) The differential equation that can be used to solve for the temperature profile along the fin is

A) $\frac{d^2T}{dx^2} - \frac{h(D+t)}{kD}(T - T_\infty) = 0$

B) $\frac{d^2T}{dx^2} - \frac{2h(D+t)}{kD}(T - T_\infty) = 0$ (Ans)

C) $\frac{d^2T}{dx^2} + \frac{2h(D+t)}{kD}(T - T_\infty) = 0$

D) None of the above.

2) The rate of heat transfer from the top surface of the lower plate to the air is ____________

$q = (0.1 \text{ m})(0.2 - 50 \times 0.001)(\text{m})(80 \text{ W/m}^2 \cdot \text{K})(300 - 100)(\text{K}) = \text{240 W}$

3) The rate of heat transfer from all the fins to the air is given by

A) $50kD\left[\frac{dT}{dx}\right]_o - \left[\frac{dT}{dx}\right]_L$

B) $-50kD\left[\frac{dT}{dx}\right]_o + \left[\frac{dT}{dx}\right]_L$

C) $-50kD\left[\frac{dT}{dx}\right]_o - \left[\frac{dT}{dx}\right]_L$ (Ans)

D) None of the above.
II. A solid steel sphere with radius $r_i$ is coated with a dielectric material layer with thickness $r_o - r_i$ and thermal conductivity $k_d$. The coated sphere is initially at a uniform temperature of 400°C and is suddenly quenched in a large oil bath for which $T_\infty = 30$ C, and $h = 2500$ W/m$^2$·K.

4) The steel sphere may be considered as a lumped capacitance system with an overall heat transfer coefficient $U$ given by

A) $\frac{1}{4\pi r^2} = \frac{r_o - r_i}{4\pi r_i^2 k_d} + \frac{1}{4\pi r_o^2 h}$

B) $\frac{1}{4\pi r^2} = \frac{r_o - r_i}{4\pi r_i^2 k_d} + \frac{1}{4\pi r_o^2 h}$

C) $\frac{1}{4\pi r^2} = \frac{r_o - r_i}{4\pi r_i^2 k_d} + \frac{1}{4\pi r_i^2 h}$

D) None of the above. Ans.

5) If $U = 40$ W/m$^2$ and $r_i = 0.10$ m, determine the time (in minutes) required for the coated sphere temperature to reach 100°C. Density of steel is 7832 kg/m$^3$ and heat capacity of steel is 560 J/kg·°K.

\[
\begin{align*}
(4/3)\pi r_i^3 \rho C_p \frac{dT}{dt} &= U 4\pi r_i^2 (T - T_\infty) \\
\frac{dT}{dt} &= \frac{3U}{r_i \rho C_p} (T - T_\infty) = \frac{3(40)}{(0.1)(7832)(560)} (T - 30)
\end{align*}
\]

\[
\frac{dT}{dt} = 2.736 \times 10^{-4} (T - 30) \Rightarrow t = \frac{1}{2.736 \times 10^{-4}} \ln \frac{370}{70} = 6,085 \text{ s} = 101 \text{ min}
\]
Quiz #4

Note: Your answers must be correct to 3 significant figures and have the appropriate units.

I. The steady-state temperature (°C) associated with selected nodal points of a two-dimensional system having a thermal conductivity of 2.0 W/m·°K are shown on the right. The ambient fluid is at 50°C with a heat transfer coefficient of 30 W/m²·°K. The isothermal surface is at 210°C.

(1) The temperature, $T_1$, at node 1 is

$$T_1 = \frac{2}{\Delta x^2} + \frac{2}{\Delta y^2} \left( T_1 = \frac{192.9 + 152.8}{0.3^2} + \frac{157 + 210}{0.2^2} \right) = 72.222 \Rightarrow T_1 = 180.22°C$$

(2) The temperature at node 2 is

$$72.222 \quad T_2 = \frac{149.4 + 0.09}{149.4 + 65.8} + \frac{2 \times 123.5}{0.04} = 8.5661 \times 10^3 \Rightarrow T_2 = 118.61°C$$

(3) The temperature at node 3 is

$$k \Delta y \left( \frac{123.5 - T_3}{\Delta x} + k(\Delta x/2) \frac{87 - T_3}{\Delta y} = k(\Delta x/2) \frac{T_3 - 65.8}{\Delta y} + h \Delta y (T_3 - 50) \right)$$

$$\frac{2}{3} (123.5 - T_3) + \frac{3}{4} (87 - T_3) = \frac{3}{4} (T_3 - 65.8) + \frac{30}{2} \times 0.2 (T_3 - 50)$$

$$T_3 = \frac{2 \times 123.5 + 3 \times 87 + 3 \times 65.8 + 3 \times 50}{2 + 3 + \frac{3}{4} + \frac{3}{4}} = 67.15°C$$
(4) If the temperature at node 3 is 75°C, calculate the heat transfer rate per unit thickness normal to the page from the right surface to the fluid.

\[ q' = 30 \times 0.2 \{0.5(210 - 50) + (87 - 50) + (75 - 50) + 0.5(65.8 - 50)\} = 899.4 \text{ W/m} \]

II A unsteady heat transfer analysis is used for the short cylinder shown with the bottom surface insulated.

5) At location A (center of the cylinder) : \( x^* = 0.5 \) \( r^* = 0 \)

6) At location B (at the top edge of the cylinder)

\( x^* = 1 \) \( r^* = 1 \)

III. A truncated solid cone is of circular cross section, and its diameter is related to the axial coordinate by an expression of the form \( D = ax^{3/2} \), where \( a = 1.0 \text{ m}^{-1/2} \). If the sides are well insulated, the rate of heat transfer is given by

a. \( \frac{a^2k\pi(T_1 - T_2)}{2\left(\frac{1}{x_1} - \frac{1}{x_2}\right)} \)

b. \( \frac{a^2k\pi(T_1 - T_2)}{\left(\frac{1}{x_1^2} - \frac{1}{x_2^2}\right)} \)

c. \( \frac{a^2k\pi(T_1 - T_2)}{4\left(\frac{1}{x_1^2} - \frac{1}{x_2^2}\right)} \)

d. None of the above

\[ q = -\pi D^2 k \frac{dT}{dx} = -\pi \frac{a^2 \chi^3}{4} k \frac{dT}{dx} \Rightarrow q \int_{x_1}^{x_2} \frac{dx}{\chi^2} = \pi \frac{a^2}{4} k(T_1 - T_2) \]

\[ -\frac{1}{2} \left(\frac{1}{x_2^2} - \frac{1}{x_1^2}\right) q = \pi \frac{a^2}{4} k(T_1 - T_2) \Rightarrow q = \frac{a^2k\pi(T_1 - T_2)}{2\left(\frac{1}{x_1^2} - \frac{1}{x_2^2}\right)} \]  

Ans: d
IV. The temperature of hot flue gases flowing through the large stack (diameter \( D \)) of a boiler is measured by means of a thermocouple enclosed within a cylindrical tube as shown. The stack is fabricated from sheet metal that is at a uniform temperature \( T_s \) and is exposed to ambient air at \( T_\infty \) and large surroundings at \( T_{sur} \).

1. A. If \( T_s \) is lower, the difference between the gas and thermocouple temperature will be larger.

   B. If the outside heat transfer coefficient (between stack and ambient air) is higher, the difference between the gas and thermocouple temperatures will be smaller.

   a. A and B are true    b. Only A is true    c. Only B is true    d. A and B are false

V. Asphalt pavement may achieve temperatures as high as 50°C on a hot summer day. Assume that such a temperature exists throughout the pavement, when suddenly a rainstorm reduces the surface temperature to 20°C. The total amount of energy (J/m²) that will be transferred from the asphalt over a time \( t \) period in which the surface is maintained at 20°C can be obtained from the following expression

\[
Q = \int_0^t q_x(t) \, dt = \int_0^t \frac{k(T_s - T_i)}{\sqrt{\pi \alpha}} \, dt = \frac{2k(T_s - T_i)}{\sqrt{\pi \alpha}} t^{1/2}
\]

Semi-infinite medium: Constant Surface Temperature: \( T(0, t) = T_s \)

\[
T(x, t) - T_s = \text{erf} \left( \frac{x}{2\sqrt{\alpha t}} \right) ; \quad q_x(t) = \frac{k(T_s - T_i)}{\sqrt{\pi \alpha t}}
\]
VI. In IC (internal combustion) engines, during injection of liquid fuel into the cylinder, it is possible for the injected fuel droplets to form a thin liquid film over the piston as shown in Figure VI. The heat transferred from the gas above the film and from the piston beneath the film causes surface evaporation. The liquid-gas interface is at the boiling $T_{lg}$, corresponding to the vapor pressure. The heat transfer from the piston side is by conduction through the piston and then by conduction through the thin liquid film. The surface-convection heat transfer from the gas side to the surface of the thin liquid film is 13,500W.

Data:
Heat of evaporation of fuel = $3.027 \times 10^5$ J/kg, thermal conductivity of fuel, $k_f = 0.083$ W/m·K, $T_{lg}$ = 398.9°K, liquid fuel density $\rho_l = 900$ kg/m$^3$, thermal conductivity of piston $k_s = 236$ W/m·K, temperature of piston at distance $L = 3$ mm from the surface is $T_1 = 500°K$. Piston diameter $D = 12$ cm, thickness of liquid film $L_f = 0.05$ mm.

![Liquid film formation on top of the piston](image)

**Figure VI.** An IC engine, showing liquid film formation on top of the piston.

The thermal resistance (K/W) from $T_1$ to the top of the liquid film is

$$R_t = \frac{L}{k_s A} + \frac{L_f}{k_f A} = \frac{1}{A} \left( \frac{L}{k_s} + \frac{L_f}{k_f} \right)$$

$$A = \frac{\pi D^2}{4} = \frac{\pi (0.12)^2}{4} = 1.131 \times 10^{-2} \text{ m}^2$$

$$R_t = \frac{1}{A} \left( \frac{L}{k_s} + \frac{L_f}{k_f} \right) = \frac{1}{1.131 \times 10^{-2}} \left( \frac{0.003}{236} + \frac{0.00005}{0.083} \right) = 0.0544 \text{ K/W}$$

---

I. Saturated steam at 99.63°C condenses on the outside of a 5-m long, 4-cm-diameter thin horizontal copper tube by cooling liquid water that enters the tube at 25°C at an average velocity of 3 m/s and leaves at 55°C. Liquid water density is 997 kg/m³, \( C_p \) of liquid water is 4.18 kJ/kg°C.

1) The rate of heat transfer to water is

\[
\dot{m} = \rho V A = (997)(3)(\pi \times 0.02)^2 = 3.7586 \text{ kg/s}
\]

\[
Q = \dot{m} C_p (T_o - T_i) = (3.7586)(4.18)(55 - 25) = 471.3 \text{ kW}
\]

2) If the rate of heat transfer to water is 500 kW, the rate of condensation of steam is

\[
\dot{m} = \frac{200}{2675.5 - 417.46} = 0.2214 \text{ kg/s}
\]

<table>
<thead>
<tr>
<th>T (°C)</th>
<th>( U_l ) (kJ/kg)</th>
<th>( U_g ) (kJ/kg)</th>
<th>( H_l ) (kJ/kg)</th>
<th>( H_g ) (kJ/kg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>99.63</td>
<td>417.36</td>
<td>2506.1</td>
<td>417.46</td>
<td>2675.5</td>
</tr>
</tbody>
</table>

II. A furnace with an area of 3.14×10⁻⁴ m² and emissive power of 3.72×10⁵ W/m² is used to calibrate a heat flux gage having a sensitive area of 1.6×10⁻⁵ m². Data: \( L = 1.5 \) m and \( \theta_d = 30° \).

3) The solid angle subtended by surface \( A_d \) with respect to \( A_f \) is

\[
\omega_{d-f} = \frac{A_d \cos \theta_d}{L^2} = (1.6\times10^{-5})(\cos 30)/1.5^2 = 6.1584\times10^{-5}
\]

4) If the solid angle in (3) is 1.6×10⁻⁵ sr, the irradiation (W/m²) received by the detector is

\[
G = \frac{q_{d-f}}{A_d} = \frac{I_f A_f \cos(\theta_f) \omega_{d-f}}{A_d} = \left( 3.72\times10^5 / \pi \right) \left( 3.14\times10^{-4} \right) (\cos 0) \left( 1.6\times10^{-5} \right) / 1.6\times10^{-5} = 37.2 \text{ W/m²}
\]
III. Consider a counter flow concentric tube heat exchanger shown with $\dot{m}_c c_p, c < \dot{m}_h c_p, h$. Let $U$ and $P$ be the overall heat transfer coefficient and the perimeter of the inner tube.

5) The following differential equation describes the temperature $T_c$ of the cold fluid in the heat exchanger

\[ A) C_{\text{min}} \frac{dT_c}{dx} = UP(T_c - T_h) \quad B) C_{\text{max}} \frac{dT_c}{dx} = -UP(T_c - T_h) \]
\[ C) C_{\text{min}} \frac{dT_c}{dx} = -UP(T_c - T_h) \quad \text{Ans} \quad D) \text{None of the above} \]

6) $T_c$ and $T_h$ are related by the following expression

\[ A) T_h = T_{hi} - C_r(T_c - T_{ci}) \quad B) T_h = T_{ho} + C_r(T_c - T_{ci}) \quad \text{Ans} \]
\[ C) T_h = T_{ho} + C_r(T_{co} - T_c) \quad D) \text{None of the above} \]
V. In IC (internal combustion) engines, during injection of liquid fuel into the cylinder, it is possible for the injected fuel droplets to form a thin liquid film over the piston as shown in Figure 3. The heat transferred from the gas above the film and from the piston beneath the film causes surface evaporation. The liquid-gas interface is at the boiling $T_{lg}$, corresponding to the vapor pressure. The heat transfer from the piston side is by conduction through the piston and then by conduction through the thin liquid film. The surface-convection heat transfer from the gas side to the surface of the thin liquid film is 13,500 W.

Data:
Heat of evaporation of fuel = $3.027 \times 10^5$ J/kg, thermal conductivity of fuel, $k_f = 0.083$ W/m-K, $T_{lg} = 398.9^\circ$K, liquid fuel density $\rho_l = 900$ kg/m$^3$, thermal conductivity of piston $k_s = 236$ W/m-K, temperature of piston at distance $L = 3$ mm from the surface is $T_1 = 500^\circ$K. Piston diameter $D = 12$ cm, thickness of liquid film $L_f = 0.03$ mm.

![Figure 3. An IC engine, showing liquid film formation on top of the piston.](image)

(7) The initial heat resistance in K/W from $T_1$ to the top surface of the liquid film is

$$A = \pi D^2/4 = \pi (0.12)^2/4 = 1.131 \times 10^{-2} \text{ m}^2$$

$$R_T = \frac{L}{k_s A} + \frac{L_f}{k_f A} = \frac{1}{0.01131} \left( \frac{3 \times 10^{-3}}{236} + \frac{3 \times 10^{-5}}{0.083} \right) = 0.0331 \text{ K/W}$$

(8) If the heat resistance from $T_1$ to the top surface of the liquid film remains constant at 0.02 K/W, determine the time it will take for the liquid film to evaporate completely

$$13,500 + (500 - 398.9)/0.02 = \dot{m} (3.027 \times 10^5) \Rightarrow \dot{m} = 0.0613 \text{ kg/s}$$

$$t = \frac{\rho L_f A}{\dot{m}} = \frac{(900)(3 \times 10^{-3})(1.131 \times 10^{-2})}{0.0613} = 0.00498 \text{ s}$$
Appendix B

Previous Exams

CHE 312 (Winter 2009)

Quiz #1

Note: Your answers must be correct to 3 significant figures and have the appropriate units.

I. An annealing process shown in Figure 1 uses a hot plate operating at an elevated temperature $T_h$. The wafer, initially at a temperature of $T_{w,i}$, is suddenly positioned at a gap separation $h = 0.2$ mm from the hot plate. The emissivity of both the hot plate and the wafer is 1.0. The silicon wafer has a thickness of $d = 0.78$ mm, a density of 2700 kg/m$^3$, and a specific heat of 875 J/kg·K. The thermal conductivity of the gas in the gap is 0.0436 W/m·K. The wafer is insulated at the bottom. Stefan-Boltzmann constant $\sigma = 5.67 \times 10^{-8}$ W/m$^2$·K$^4$.

For $T_h = 600^\circ$C and $T_{w,i} = 20^\circ$C, calculate

(1) The radiative heat flux across the gap

(2) The heat flux by conduction across the gap

(3) The initial time rate of change in the temperature of the wafer, $\left(\frac{dT_w}{dt}\right)_i$, if the total heat flux across the gap is 200 kW/m$^2$.
II. A small sphere of reference-grade iron with a specific heat of 447 J/kg-K and a mass of 0.515 kg is suddenly immersed in a water-ice mixture. Fine thermocouple wires suspend the sphere, and initial time rate of change in the sphere temperature is observed to be 0.1575 K/s. The experiment is repeated with a metallic sphere of the same diameter, but of unknown composition with a mass of 1.263 kg. If the initial time rate of change in the temperature of this sphere is 0.2179 K/s, what is the specific heat of the unknown material?

III. Consider the process arrangement where a wafer is in an evacuated chamber whose wall are maintained at 27°C and within which heating lamps maintain a radiant flux $q''_s$ at its upper surface. The wafer is 0.78 mm thick, has a thermal conductivity of 30 W/m-K, and an emissivity that equals its absorptivity to the radiant flux ($\varepsilon = \alpha = 0.65$). For $q''_s = 3.0 \times 10^5$ W/m², the temperature on its lower surface is measured by a radiation thermometer and found to have a value of $T_{w,l} = 997$°C.

![Silicon wafer, $T_{w,l} = 997$°C](image)

The temperature, $T_{w,u}$, at the top surface of the wafer can be obtained from:

A) $0.65 \times 3.0 \times 10^5 - 0.65 \times 5.67 \times 10^{-8}[T_{w,u} - (27 + 273)^4] - 30(T_{w,u} - 997)/0.00078 = 0$

B) $0.65 \times 3.0 \times 10^5 - 0.65[T_{w,u}^4 - (27 + 273)^4] - 30(T_{w,u} - 997)/0.00078 = 0$

C) $0.65 \times 3.0 \times 10^5 - 0.65 \times 5.67 \times 10^{-8}[T_{w,u}^4 - (27 + 273)^4] + 30(T_{w,u} - 997)/0.00078 = 0$

D) $0.65 \times 3.0 \times 10^5 - 0.65 \times 5.67 \times 10^{-8}[T_{w,u}^4 - (27 + 273)^4] - 30(T_{w,u} - 997)/0.00078 = 0$

E) None of the above

IV. The free convection heat transfer coefficient on a thin hot vertical plate suspended in still air can be determined from observations of the change in plate temperature with time as it cools. The ambient air temperature is 25°C and the plate measures 0.3 × 0.3 m with a mass of 3.75 kg and a specific heat of 2770 J/kg-K.

(6) Assuming the plate is isothermal and radiation exchange with its surroundings is negligible, evaluate the convection coefficient at the instant of time when the plate temperature is 225°C and the change in plate temperature with time ($dT/dt$) is −0.022 K/s.
(7) If the surroundings temperature is 25°C, the heat transfer coefficient is 10 W/m²·K, and the emissivity of the plate is 0.42, determine the ratio of the heat transfer by radiation to the heat transfer by convection at the instant of time when the plate temperature is 225°C.

V. The roof of a car in a parking lot absorbs a solar radiant flux of 800 W/m², while the underside is perfectly insulated. The convection coefficient between the roof and the ambient air is 20 W/m²·K.

(8) Neglecting radiation exchange with the surroundings, calculate the temperature of the roof under steady-state conditions if the ambient air temperature is 20°C.

(9) For the same ambient air temperature, calculate the temperature of the roof if its surface emissivity is 0.8.

VI. Liquid oxygen, which has a boiling point of 90°K and a latent heat of vaporization of 214 kJ/kg, is stored in a spherical container whose outer surface is of 500-mm diameter and at a temperature of 263 K. The container is housed in a laboratory whose air and walls are at 298 K. If the surface emissivity is 0.20 and the heat transfer coefficient associated with free convection at the outer surface of the container is 20 W/m²·K, what is the rate, in kg/s, at which oxygen vapor must be vented from the system?
I. A solar flux of 700 W/m$^2$ is incident on a flat-plate solar collector used to heat water. The area of the collector is 3 m$^2$, and 90% of the solar radiation passes through the cover and is absorbed by the absorber plate. Water flows through the tube passages on the back side of the absorber plate and is heated from an inlet temperature $T_i$ to an outlet temperature $T_o$. The cover glass, operating at a temperature of 30$^\circ$C, has an emissivity of 0.94 and experiences radiation exchange with the sky at −10$^\circ$C. The convection coefficient between the cover glass and the ambient air at 25$^\circ$C is 10 W/m$^2$·K. Stefan-Boltzmann constant $\sigma = 5.67 \times 10^{-8}$ W/m$^2$·K$^4$.

1) Determine the heat collected per unit area, $q''_c$, of the collector _____________

2) If $q''_c = 500$ W/m$^2$, flow rate of water = 0.01 kg/s, specific heat of water = 4179 J/kg ·K,
determine the temperature rise of water, $T_o - T_i$, ______________

3) The collector efficiency $\eta$ is defined as the ratio of the heat collected to the rate at which
solar energy is incident on the collector. If $q''_c = 500$ W/m$^2$, determine $\eta$ ______________
II. One-dimensional, steady-state conduction with uniform internal energy generation occurs in a plane wall with a thickness of 50 mm and a constant thermal conductivity of 10 W/m·K. For these conditions, the temperature distribution has the form, \( T(x) = a + bx + cx^2 \). The surface at \( x = 0 \) has a temperature of \( T(0) = T_0 = 150^\circ C \) and experiences convection with a fluid for which \( T_\infty = 10^\circ C \) and \( h = 400 \text{ W/m}^2\cdot\text{K} \). The surface at \( x = L \) is well insulated.

(4) Determine the internal energy generation rate, \( \dot{q} : \) 

(5) Determine the coefficient \( b : \) 

III. In an orbiting space station, an electronic package is housed in a compartment having a surface area \( A_s = 1 \text{ m}^2 \) which is exposed to space. Under normal operating conditions, the electronics dissipate 800 W, all of which must be transferred from the exposed surface to space. If the surface emissivity is 1.0 and the surface is not exposed to the sun, what is its steady-state temperature? If the surface is exposed to a solar flux of 750 W/m\(^2\) and its absorptivity to solar radiation is 0.35, what is its steady-state temperature?

VI. The air inside a chamber at \( T_{\infty,i} = 60^\circ C \) is heated convectively with \( h_i = 40 \text{ W/m}^2\cdot\text{K} \) by a 200-mm-thick wall having a thermal conductivity of 10 W/m·K and a uniform heat generation of 1600 W/m\(^3\). To prevent any heat generated within the wall from being lost to the outside of the chamber at \( T_{\infty,o} = 25^\circ C \) with \( h_o = 10 \text{ W/m}^2\cdot\text{K} \), a very thin electrical strip heater is placed on the outer wall to provide a uniform heat flux, \( \dot{q}_o \).
(7) Determine the temperature at the inside wall surface $T(L)$

(8) Determine $T(0) - T(L)$

(9) If $T(0) = 70^\circ C$, determine the value of $q_o$ that must be supplied by the strip heater so that all heat generated within the wall is transferred to the inside of the chamber.

(10) If the heat generation in the wall were switched off while the heat flux to the strip heater is 600 W/m$^2$, what would be the steady-state temperature, $T(0)$, of the outer wall surface?
I. A homeowner, whose water pipes have frozen during a period of cold weather, decides to melt the ice by passing an electric current \( I \) through the pipe wall. The inner and outer radii of the wall are designated as \( r_1 \) and \( r_2 \), and its electrical resistance per unit length is designated as \( R'_e \) (\( \Omega/m \)).

The pipe is well insulated on the outside. During melting the ice (and water) in the pipe remain at a constant temperature \( T_m = 0^\circ \text{C} \) associated with the melting process. Ice: \( \rho = 920 \text{ kg/m}^3 \), \( h_{fg} = 3.34 \times 10^5 \text{ J/kg} \). Data: \( I = 120 \text{ A} \), \( R'_e = 0.40 \Omega/m \), \( r_1 = 50 \text{ mm} \) and \( r_2 = 70 \text{ mm} \).

1) Determine the heat generated per unit volume in the pipe wall.

2) Determine the time \( t_m \) required to completely melt the ice.

II. A long cylindrical rod of diameter \( D_1 = 200 \text{ mm} \) with thermal conductivity \( k \) of 0.5 W/m·K experiences uniform volumetric heat generation \( \dot{q} \) of 24,000 W/m\(^3\). The rod is encapsulated by a circular sleeve having an outer diameter of \( D_2 = 400 \text{ mm} \) and a thermal conductivity \( k_s \) of 4 W/m·K. The outer surface of the sleeve is exposed to cross flow of air at \( T_\infty = 27^\circ \text{C} \) with a convection coefficient \( h \) of 25 W/m\(^2\)·K.

3) The temperature \( T_1 \) at the interface between the rod and sleeve can be determined from

A) \[ \dot{q} = \frac{T_1 - T_\infty}{\ln(r_2/r_1)} + \frac{1}{2\pi k} \frac{1}{h\pi D_1} \]

B) \[ \dot{q} = \frac{T_1 - T_\infty}{\ln(r_2/r_1)} + \frac{1}{2\pi k_s} \frac{1}{h\pi D_1} \]

C) \[ \dot{q} = \frac{T_1 - T_\infty}{\ln(r_2/r_1)} + \frac{1}{2\pi k_s} \frac{1}{h\pi D_2} \]

D) \[ \dot{q} \pi r_1^2 = \frac{T_1 - T_\infty}{\ln(r_2/r_1)} + \frac{1}{2\pi k_s} \frac{1}{h\pi D_2} \]

4) The temperature at the center of the rod can be determined from

A) \[ T = T_1 + \frac{\dot{q} r_2^2}{4k} \]

B) \[ T = T_1 + \frac{\dot{q} r_1^2}{2k} \]

C) \[ T = T_1 + \frac{\dot{q} r_1^2}{4k} \]

D) \[ T = T_\infty + \frac{\dot{q} r_1^2}{4k} \]
III.

Specify the control volume required to derive the temperature $T(r)$ in a circular fin with radius $R$ and thickness $t$.

IV. A solid steel sphere with radius $r_i$ is coated with a dielectric material layer with thickness $r_o - r_i$ and thermal conductivity $k_d$. The coated sphere is initially at a uniform temperature of 400°C and is suddenly quenched in a large oil bath for which $T_\infty = 30$ C, and $h = 2500$ W/m$^2$·K.

The steel sphere may be considered as a lumped capacitance system with an overall heat transfer coefficient $U$ given by

\[
A) \quad \frac{1}{4\pi_o^2 U} = \frac{r_o - r_i}{4\pi r_o k_d} + \frac{1}{4\pi_o^2 h} \\
B) \quad \frac{1}{4\pi_i^2 U} = \frac{r_o - r_i}{4\pi r_o k_d} + \frac{1}{4\pi_i^2 h} \\
C) \quad \frac{1}{4\pi_o^2 U} = \frac{r_o - r_i}{4\pi r_o k_d} + \frac{1}{4\pi_i^2 h} \\
D) \text{None of the above.}
\]

V. In IC (internal combustion) engines, during injection of liquid fuel into the cylinder, it is possible for the injected fuel droplets to form a thin liquid film over the piston as shown in Figure 5. The heat transferred from the gas above the film and from the piston beneath the film causes surface evaporation. The liquid-gas interface is at the boiling $T_{lg}$, corresponding to the vapor pressure. The heat transfer from the piston side is by conduction through the piston and then by conduction through the thin liquid film. The surface-convection heat transfer from the gas side to the surface of the thin liquid film is 9,500 W.

Data:
Heat of evaporation of fuel = 3.027×10$^5$ J/kg, thermal conductivity of fuel, $k_f = 0.083$ W/m·K, $T_{lg} = 398.9$°K, liquid fuel density $\rho = 900$ kg/m$^3$, thermal conductivity of piston $k_s = 236$ W/m·K, temperature of piston at distance $L = 3$ mm from the surface is $T_1 = 500$°K. Piston diameter $D = 12$ cm, thickness of liquid film $L_f = 0.05$ mm.

Show all your work and reasoning for this problem.
Figure 5. An IC engine, showing liquid film formation on top of the piston.

(a) (4 pts) Estimate the time it will take for the liquid film to evaporate completely assuming the thermal resistance to the liquid film remains constant at the initial value. Show all your work.

(b) (4 pts) Write down the equation needed to calculate the time it will take for the liquid film to evaporate completely if the thermal resistance to the liquid film is not a constant. Explain how you can solve the equation but do not solve it. Will the time be longer or shorter than that obtained from part (a)? Why? Show all your work
I. The steady-state temperature (°C) associated with selected nodal points of a two-dimensional system having a thermal conductivity of 2.0 W/m·°K are shown on the right. The ambient fluid is at 60°C with a heat transfer coefficient of 30 W/m²·°K. The isothermal surface is at 220°C.

(1) The temperature, $T_1$, at node 1 is ____________

(2) The temperature at node 2 is ____________

(3) The temperature at node 3 is ____________
(4) If the temperature at node 3 is 80°C, calculate the heat transfer rate per unit thickness normal to the page from the right surface to the fluid.

II. A truncated solid cone is of circular cross section, and its diameter is related to the axial coordinate by an expression of the form $D = ax^{3/2}$, where $a = 1.0 \text{ m}^{-1/2}$. If the sides are well insulated, the rate of heat transfer is given by

\begin{align*}
\text{a.} & \quad \frac{a^2 k \pi (T_1 - T_2)}{2 \left( \frac{1}{x_1} - \frac{1}{x_2} \right)} \\
\text{b.} & \quad \frac{a^2 k \pi (T_1 - T_2)}{2 \left( \frac{1}{x_1^2} - \frac{1}{x_2^2} \right)} \\
\text{c.} & \quad \frac{a^2 k \pi (T_1 - T_2)}{4 \left( \frac{1}{x_1^2} - \frac{1}{x_2^2} \right)} \\
\text{d.} & \quad \text{None of the above}
\end{align*}

III. Consider a hollow cylinder with radius $R_i$ and $R_o$ with heat transfer from the inside fluid at $T_i$ to the outside fluid at $T_o$. The inside and outside heat transfer coefficients are $h_i$ and $h_o$ respectively. The thermal conductivity of the cylinder is $k$. Specify the two boundary conditions necessary to solve for the temperature distribution within the cylinder.

IV. A. The unsteady state solution for temperature distribution in a cylinder $T(r, t)$ requires the evaluation of a Bessel function.

B. If the initial temperature for an infinite cylinder is not uniform, temperature distribution $T(r, t)$ cannot be obtained.

\begin{align*}
\text{a.} & \quad \text{A and B are true} \\
\text{b.} & \quad \text{Only A is true} \\
\text{c.} & \quad \text{Only B is true} \\
\text{d.} & \quad \text{A and B are false}
\end{align*}
V. Copper-coated, epoxy-filled fiberglass circuit boards are treated by heating a stack of them under high pressure as shown in the sketch. The stack, referred to as a book, is comprised of 10 boards and 11 pressing plates. Calculate the total thermal resistance (K/W) through the stack if each of the boards and plates has a thickness of 2.36 mm and the following properties: board (b) \( k_b = 0.30 \text{ W/m·K} \); plate (p) \( k_p = 12 \text{ W/m·K} \).

VI. A long bar of rectangular cross section is 60 mm by 90 mm on a side and has a thermal conductivity of 1.5 W/m·K. One surface is exposed to a convection process with air at 100°C and a convection coefficient of 100 W/m²·K, while the remaining surfaces are maintained at 50°C. Using a grid spacing of 30 mm the temperature at node 1 is related to the temperature at node 2 according to the equation:

\[ T_1 = AT_2 + B \]

(9) \( A = \) __________  \hspace{1cm} (10) \( B = \) __________
I. Saturated steam at 110°C condenses on the outside of a 5-m long, 4-cm-diameter thin horizontal copper tube by cooling liquid water that enters the tube at 25°C at an average velocity \( V \) of 2 m/s and leaves at 45°C. Liquid water density, \( \rho \), is 997 kg/m\(^3\), \( C_p \) of liquid water is 4.18 kJ/kg°C.

1) The rate of heat transfer to water is __________

2) If the rate of heat transfer to water is 250 kW, the rate of condensation of steam is __________

3) The following equation can be integrated over the length of the tube to determine the overall heat transfer coefficient \( U \) for the heat transfer from the steam to the water: (Note: \( D \) is the tube diameter and \( T_s \) is the steam temperature)

\[ D V \rho C_p \frac{dT}{dx} = U(T_s - T) \]

**A) \( D V \rho C_p \frac{dT}{dx} = U(T_s - T) \)**

**B) \( D V \rho C_p \frac{dT}{dx} = 4U(T_s - T) \)**

**C) \( D V \rho C_p \frac{dT}{dx} = -4U(T_s - T) \)**

**D) None of the above**
II. The steady-state temperature (°C) associated with selected nodal points of a two-dimensional system having a thermal conductivity of 2.0 W/m·°K are shown on the right. The ambient fluid is at 60°C with a heat transfer coefficient of 30 W/m²·°K. The isothermal surface is at 220°C.

Temperature at node 4 is related to temperatures at nodes 3 and 5 by the equation

\[ T_4 = AT_3 + BT_5 + C \]

4) \( A = \) __________ 
5) \( B = \) __________ 
6) \( C = \) __________
III. A shell-and-tube heat exchanger with one shell pass and two tube passes uses hot water on the tube side to heat oil on the shell side. The water enters at 87°C and 50.0 kg/s and leaves at 27°C. Inlet and outlet temperatures of the oil are 7 and 37°C. The heat exchanger contains 200 tubes which have inner and outer diameters of 20 and 24 mm and a length of 5.0 m. Density of water is 995 kg/m³ and $C_p$ of water is 4200 J/kg.K.

Evaluate $F$ correction factor

$$ R = \frac{T_j - T_o}{t_o - t_i} \text{ m} \quad P = \frac{t_o - t_i}{T_i - T_j} \quad F = \frac{\left(\sqrt{R^2 + 1}\right) \ln \left(\frac{1 - P}{(1 - RP)}\right)}{\ln \frac{2/P - 1 - R + \sqrt{R^2 + 1}}{2/P - 1 - R - \sqrt{R^2 + 1}}}$$

7) The $F$ correction factor for the log mean driving force is ____________

8) The (total) outside area for heat transfer is ____________

9) Water velocity is ____________

10) If $U_i$ is 100 W/m²-K then $U_o$ is ____________
### Answers to Quizzes 2009

#### Quiz 1

<table>
<thead>
<tr>
<th>1) 32,500 W/m²</th>
<th>2) 126,440 W/m²</th>
<th>3) 108.5 K/s</th>
<th>4) 132 J/kg K</th>
</tr>
</thead>
<tbody>
<tr>
<td>5) None of the above</td>
<td>6) 6.35 W/m²-K</td>
<td>7) 0.6385</td>
<td>8) 60.0°C</td>
</tr>
<tr>
<td>9) 311.6 K = 38.6°C</td>
<td>10) 2.7×10⁻³ kg/s</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

#### Quiz 2

<table>
<thead>
<tr>
<th>1) 385.4 W/m²</th>
<th>2) 35.9°C</th>
<th>3) 0.7143</th>
<th>4) 1.12×10⁶ W/m³</th>
</tr>
</thead>
<tbody>
<tr>
<td>5) 5,600 °C/m</td>
<td>6) 370 K</td>
<td>7) 68°C</td>
<td>8) 3.2°C</td>
</tr>
<tr>
<td>9) 450 W/m²</td>
<td>10) 67.76°C</td>
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<td></td>
</tr>
</tbody>
</table>

#### Quiz 3

<table>
<thead>
<tr>
<th>1) 7.64×10⁵ W/m³</th>
<th>2) 419 s</th>
<th>3) D</th>
<th>4) C</th>
</tr>
</thead>
<tbody>
<tr>
<td>5) 2πrtdr</td>
<td>6) B</td>
<td>7) 0.0136 s</td>
<td></td>
</tr>
</tbody>
</table>

\[ \int_0^{5×10^{-3}} \frac{dL_f}{3.083×10^{-3} + \frac{3.711×10^{-7}}{1.271×10^{-5} + 12.048L_f}} \]

#### Quiz 4

<table>
<thead>
<tr>
<th>1) 190.22°C</th>
<th>2) 128.61°C</th>
<th>3) 77.15°C</th>
<th>4) 869.4 W/m</th>
</tr>
</thead>
<tbody>
<tr>
<td>5) b</td>
<td>6) ( r = R_i, h_i(T_i - T) = -k \frac{\partial T}{\partial r}, r = R_o, -k \frac{\partial T}{\partial r} = h_o(T - T_o) )</td>
<td>7) b</td>
<td>8) 0.08083 K/W</td>
</tr>
<tr>
<td>9) A = 0.25</td>
<td>10) B = 62.5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

#### Quiz 2

<table>
<thead>
<tr>
<th>1) 209.5 kW</th>
<th>2) 0.1121 kg/s</th>
<th>3) B</th>
<th>4) 0.1452</th>
</tr>
</thead>
<tbody>
<tr>
<td>5) 0.1452</td>
<td>6) 52.0645 °C</td>
<td>7) 0.5322</td>
<td>8) 75.4 m²</td>
</tr>
<tr>
<td>9) 1.60 m/s</td>
<td>10) 83.3 W/m² K</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Appendix C

Previous Exams

CHE 312 (Winter 2010)

Quiz #1

Note: Your answers must be correct to 3 significant figures and have the appropriate units.

I. An annealing process shown below uses a hot plate operating at an elevated temperature \( T_h \). The wafer, initially at a temperature of \( T_{w,i} \), is suddenly positioned at a gap separation \( h = 0.5 \) mm from the hot plate. The emissivity of both the hot plate and the wafer is 1.0. The silicon wafer has a thickness of \( d = 0.80 \) mm, a density of 2700 \( \text{kg/m}^3 \), and a specific heat of 1050 \( \text{J/kg}\cdot\text{K} \). The thermal conductivity of the gas in the gap is 0.0436 \( \text{W/m}\cdot\text{K} \). The wafer is insulated at the bottom. Stefan-Boltzmann constant \( \sigma = 5.67 \times 10^{-8} \text{W/m}^2\cdot\text{K}^4 \).

For \( T_h = 700^\circ \text{C} \) and \( T_{w,i} = 25^\circ \text{C} \), calculate

(1) The radiative heat flux across the gap \( \boxed{50,370 \text{ W/m}^2} \)

(2) The heat flux by conduction across the gap \( \boxed{58,860 \text{ W/m}^2} \)

(3) The initial time rate of change in the temperature of the wafer, \( \frac{dT_w}{dt} \), if the total heat flux across the gap is 200 \( \text{kw/m}^2 \). \( \boxed{88.18 \text{ K/s}} \)

II. (4) A small sphere of reference-grade iron with a specific heat of 447 \( \text{J/kg}\cdot\text{K} \) and a mass of 0.515 kg is suddenly immersed in a water-ice mixture. Fine thermocouple wires suspend the sphere, and initial time rate of change in the sphere temperature is observed to be 0.1575 K/s. The experiment is repeated with a metallic sphere of the same diameter, but of unknown composition with a mass of 1.263 kg. Both spheres have the same initial temperature.

A. The initial rate of heat transfer to (or from) the iron sphere is the same as the initial heat transfer rate to (or from) the metallic sphere.
B. The initial time rate of change in the temperature of the iron sphere is the same as the initial time rate of change in the temperature of the metallic sphere.

\[ \text{a. A and B are true} \quad \text{b. Only A is true} \quad \text{c. Only B is true} \quad \text{d. A and B are false} \]
III. (5) Consider the process arrangement where a wafer is in an evacuated chamber whose wall are maintained at 27°C and within which heating lamps maintain a radiant flux \( q''_s \) at its upper surface. The wafer is 0.78 mm thick, has a thermal conductivity of 30 W/m·K, and an emissivity that equals its absorptivity to the radiant flux (\( \varepsilon = \alpha = 0.65 \)). For \( q''_s = 3.0 \times 10^5 \) W/m\(^2\), the temperature on its lower surface is measured by a radiation thermometer and found to have a value of \( T_{w,l} = 997^\circ \text{C} \).

![Diagram of wafer and heating lamps]

The temperature, \( T_{w,u} \), at the top surface of the wafer can be obtained from:

A) \[ 0.65 \times 3.0 \times 10^5 - 0.65 \times 5.67 \times 10^{-8} [T_{w,u}^4 - (27 + 273)^4] - 30(T_{w,u} - 1270)/0.00078 = 0 \]
B) \[ 0.65 \times 3.0 \times 10^5 - 0.65[T_{w,u}^4 - (27 + 273)^4] - 30(T_{w,u} - 1270)/0.00078 = 0 \]
C) \[ 0.65 \times 3.0 \times 10^5 - 0.65 \times 5.67 \times 10^{-8} [T_{w,u}^4 - (27 + 273)^4] + 30(T_{w,u} - 1270)/0.00078 = 0 \]
D) \[ 0.65 \times 3.0 \times 10^5 - 0.65 \times 5.67 \times 10^{-8} [T_{w,u}^4 - (27 + 273)^4] - 30(T_{w,u} - 1270)/0.00078 = 0 \] Ans
E) None of the above

IV. (6) Liquid oxygen, which has a boiling point of 90°C and a latent heat of vaporization of 214 kJ/kg, is stored in a spherical container whose outer surface is of 500-mm diameter and at a temperature of \(-10^\circ \text{C}\). The container is housed in a laboratory whose air and walls are at 25°C. If the surface emissivity is 0.50 and the heat transfer coefficient associated with free convection at the outer surface of the container is 20 W/m\(^2\)·K, what is the rate, in kg/s, at which oxygen vapor must be vented from the system?

\[ 2.89 \times 10^{-3} \text{ kg/s} \]

V. (7) A surface whose temperature is maintained at 400°C is separated from an air flow by a layer of insulation 25 mm thick for which the thermal conductivity is 0.05 W/m·K. If the air temperature is 25°C and the convection coefficient between the air and the outer surface of the insulation is 500 W/m\(^2\)·K, what is the temperature of this outer surface?

\[ 26.49^\circ \text{C} \]

VI. (8) An experiment to determine the convection coefficient associated with airflow over the surface of a thick steel casting involves insertion of thermocouples in the casting at distances of 10 and 20 mm from the surface along a hypothetical line normal to the surface. The steel has a thermal conductivity of 20 W/m·K. If the thermocouples measure temperatures of 55 and 40°C in the steel when the air temperature is 100°C, what is the convection coefficient?

\[ 1000 \text{ W/m}^2\cdot\text{K} \]
VII. The roof of a car in a parking lot absorbs a solar radiant flux of 900 W/m\(^2\), while the underside is perfectly insulated. The convection coefficient between the roof and the ambient air is 20 W/m\(^2\)-K.

(9) Neglecting radiation exchange with the surroundings, calculate the temperature of the roof under steady-state conditions if the ambient air temperature is 20\(^\circ\)C.

\[ T_s = \frac{900 - 20(T_s - 293) + 0.8 \times 5.67 \times 10^{-8} T_s^4}{h A_s} \]

\[ 65.0^\circ C \]

(10) For the same ambient air temperature, the following equation can determine the temperature of the roof if its surface emissivity is 0.8.

\[ q_{s,abs}^* A_s = h A_s (T_s - T_\infty) + \varepsilon \sigma A_s T_s^4 \]

A) \[ 900 - 20(T_s - 293) + 0.8 \times 5.67 \times 10^{-8} T_s^4 = 0 \]
B) \[ 800 - 20(T_s - 293) - 0.8 \times 5.67 \times 10^{-8} T_s^4 = 0 \]
C) \[ 900 - 20(T_s - 20) - 0.8 \times 5.67 \times 10^{-8} T_s^4 = 0 \]
D) \[ 900 - 20(T_s - 293) - 0.8 \times 5.67 \times 10^{-8} T_s^4 = 0 \textbf{Ans} \]
E) None of the above
Quiz #2

Note: Your answers must be correct to 3 significant figures and have the appropriate units.

I. A spherical aluminum tank, inside radius $R_1 = 3$ m, and wall thickness $l_1 = 4$ mm, contains liquid-vapor oxygen at 1 atm pressure and 90.18°K. Heat of evaporation of oxygen is $2.123 \times 10^5$ J/kg. Under steady state, at the liquid gas surface, the heat flowing (leak) into the tank causes boil off at a rate $M_g$. In order to prevent the pressure of the tank from rising, the gas resulting from boil off is vented through a safety valve as shown in Figure 1. An evacuated air gap, extending to location $r = R_2 = 3.1$ m, is placed where the combined conduction-radiation effect for this gap is represented by a conductivity $k_a = 0.008$ W/m·K. A layer of insulation with $k_i = 0.040$ W/m·K and thickness $l_2 = 10$ cm is added. The outside surface temperature is kept constant at $T_2 = 283.15$°K. Neglect the heat resistance through the aluminum.

(1) Determine the rate of heat leak $Q_{k,2-1}$ in W

$$Q_{k,2-1} = 1574 \text{ W}$$

(2) If $Q_{k,2-1} = 2000$ W, determine the amount of boil off $M_g$ in kg/s.

$$M_g = 9.42 \times 10^{-3} \text{ kg/s}$$

II. The air inside a chamber at $T_{\infty,i} = 50$°C is heated convectively with $h_i = 25$ W/m²·K by a 0.25-m-thick wall having a thermal conductivity of 5 W/m·K and a uniform heat generation of 1500 W/m³. To prevent any heat generated within the wall from being lost to the outside of the chamber at $T_{\infty,o} = 15$°C with $h_o = 10$ W/m²·K, a very thin electrical strip heater is placed on the outer wall to provide a uniform heat flux, $q_o$.

(3) Determine the temperature at the inside wall surface $T(L)$

$$T(L) = 65.0$$°C
(4) Determine \( T(0) - T(L) \) \( 9.375^\circ C \)

(5) If \( T(0) = 80^\circ C \), determine the value of \( q_a \) that must be supplied by the strip heater so that all heat generated within the wall is transferred to the inside of the chamber.

\[
650 \text{ W/m}^2
\]

(6) If the heat generation in the wall were switched off while the heat flux to the strip heater is \( 600 \text{ W/m}^2 \), what would be the steady-state temperature, \( T(0) \), of the outer wall surface?

\[
T_0 = 61.8^\circ C
\]

III. (7) A transistor, which may be approximated as a hemispherical heat source of radius \( r_0 \), is embedded in a large silicon substrate and dissipates heat at a rate \( q \). All boundaries of the silicon are maintained at an ambient temperature of \( T_\infty \), except for a plane surface that is well insulated.

The substrate temperature distribution is given by

A) \[
T = T_\infty - (T_\infty - T_s) \left( \frac{r_o}{r} \right)^2
\]

B) \[
T = T_\infty + (T_\infty - T_s) \left( \frac{r_o}{r} \right)
\]

C) \[
T = T_\infty - (T_\infty + T_s) \left( \frac{r_o}{r} \right)
\]

D) \[
T = T_\infty - (T_\infty - T_s) \left( \frac{r_o}{r} \right) \text{ Ans}
\]

E) None of the above

IV (8) Two identical closed beakers contain equal masses of liquid at a temperature of \( 20^\circ C \) as shown above. One beaker is filled with water and the other beaker is filled with ethanol (ethyl alcohol). The temperature of each liquid is increased from \( 20^\circ C \) to \( 40^\circ C \) using identical heaters immersed in the liquids. Each heater is set to the same power setting. It takes 2 minutes for the ethanol temperature to reach \( 40^\circ C \) and 3 minutes for the water temperature to reach \( 40^\circ C \).
Ignoring evaporation losses, to which liquid was more energy transferred during the heating process?

A) Water because it is heated longer. Ans
B) Alcohol because it heats up faster.
C) Both liquids received the same amount of energy because they started at the same initial temperature and ended at the same final temperature.
D) Can’t determine from the information given because heat transfer coefficients from the water and alcohol beaker surfaces are needed.
E) Can’t determine from the information given because heat capacities of water and ethanol are needed.
F) Water because it has a higher boiling point than ethanol.

V. (9) An engineering student walking barefoot (without shoes or socks) from a tile floor onto a carpeted floor notices that the tile feels cooler than the carpet. Which of the following explanations seems like the most plausible way to explain this observation?

A) The carpet has a slightly higher temperature because air trapped in the carpet retains energy from the room better.
B) The carpet has more surface area in contact with the student’s foot than the tile does, so the carpet is heated faster and feels hotter.
C) The tile conducts energy better than the carpet, so energy moves away from the student’s foot faster on tile than carpet. Ans.
D) The rate of heat transfer into the room by convection (air movement) is different for tile and carpet surfaces.
E) The carpet has a slightly higher temperature because air trapped in the carpet slows down the rate of energy transfer through the carpet into the floor.

VI. (10) You are in the business of melting ice at 0°C using hot blocks of metal as an energy source. One option is to use one metal block at a temperature of 200°C and a second option is to use two metal blocks each at a temperature of 100°C. All the metal blocks are made from the same material and have the same weight and surface area. If the blocks are placed in insulated cups filled with ice water at 0°C, which option will melt more ice?

A) The 100°C blocks.
B) The 200°C block.
C) Either option will melt the same amount of ice. Ans
D) Can’t tell from the information given.
I. A homeowner, whose water pipes have frozen during a period of cold weather, decides to melt the ice by passing an electric current $I$ through the pipe wall. The inner and outer radii of the wall are designates as $r_1$ and $r_2$, and its electrical resistance per unit length is designated as $R'_e$ ($\Omega/m$). The pipe is well insulated on the outside. During melting the ice (and water) in the pipe remain at a constant temperature $T_m = 0^\circ C$ associated with the melting process. Ice: $\rho = 920 \text{ kg/m}^3$, $h_{fg} = 3.34 \times 10^5 \text{ J/kg}$. Data: $I = 120 \text{ A}$, $R'_e = 0.40 \text{ } \Omega/m$, $r_1 = 50 \text{ mm}$ and $r_2 = 70 \text{ mm}$.

1) The differential equation required to solve for $T(r)$ is

A) $\frac{k}{r} \frac{d}{dr} \left( r \frac{dT}{dr} \right) - \dot{q} = 0$

B) $\frac{k}{r} \frac{d}{dr} \left( r \frac{dT}{dr} \right) + \dot{q} = 0 \text{ Ans}$

C) $\frac{1}{r} \frac{d}{dr} \left( r \frac{dT}{dr} \right) + \dot{q} = 0$

D) $\frac{k}{r^2} \frac{d}{dr} \left( r \frac{dT}{dr} \right) + \dot{q} = 0$

E) None of the above

2) The two boundary conditions required to solve for $T(r)$ are

$r = r_2, \frac{dT}{dr} = 0$ __________ $r = r_1, T = T_m = 0$ __________

3) Specify the control volume needed to obtain the differential equation in (1) _$2\pi rLdr$_

4) The temperature profile for $T(r)$ is given by

A) $T = T_m - \frac{\dot{q} r_2^2}{2k} \ln \left( \frac{r_1}{r} \right) - \frac{q^2}{4k} \left( r^2 - r_1^2 \right)$

B) $T = T_m + \frac{\dot{q} r_2^2}{2k} \ln \left( \frac{r_1}{r} \right) - \frac{q^2}{4k} \left( r^2 - r_1^2 \right)$

C) $T = T_m - \frac{\dot{q} r_2^2}{2k} \ln \left( \frac{r_1}{r} \right) - \frac{q^2}{2k} \left( r^2 - r_1^2 \right)$

D) $T = T_m - \frac{\dot{q} r_2^2}{2k} \ln \left( \frac{r_1}{r} \right) + \frac{q^2}{4k} \left( r^2 - r_1^2 \right)$

E) None of the above Ans
II) An air heater may be fabricated by coiling Nichrome wire and passing air in cross flow over the wire. Consider a heater fabricated from wire of diameter $D = 1\, \text{mm}$, length $L = 2\, \text{m}$, electrical resistivity $\rho_e = 10^{-6}\, \Omega\cdot\text{m}$, thermal conductivity $k = 25\, \text{W/m}\cdot\text{K}$, and emissivity $\varepsilon = 0.20$. The heater is designed to deliver air at a temperature of $T_\infty = 50\, ^\circ\text{C}$ under flow conditions that provide a convection coefficient of $h = 250\, \text{W/m}^2\cdot\text{K}$ for the wire. The temperature of the housing that enclosed the wire and through which the air flow is $T_{\text{sur}} = 50\, ^\circ\text{C}$. The temperature of the wire is $1200\, ^\circ\text{C}$. $\sigma = 5.67 \times 10^{-8}\, \text{W/m}^2\cdot\text{K}^4$.

5) Determine the heat transfer by radiation

$335\, \text{W}$

6) If the heat transfer by radiation is $600\, \text{W}$, determine the power delivered by the heater

$2410\, \text{W}$

7) You enter a cold room in a house and adjust a simple thermostat to heat the room to a more comfortable level. A simple thermostat is an on-off switch which is in the “on” position if the room temperature is below the desired setting and is in the “off” position otherwise. If you want the room temperature to increase quickly, should you set the thermostat setting to the desired temperature or set it much higher than the desired temperature?

A) All the way up
B) Set to desired temperature
C) Either setting will heat the room at the same rate (Ans)
D) Can’t determine from the information given
8) Your answer to Question 7 is correct because

A) A higher setting will produce hotter air in the furnace which will heat the house faster
B) Heat transfer is proportional to temperature difference so a higher setting will heat the house faster
C) The furnace heats at the same rate as long as the desired room temperature hasn’t been reached yet (Ans)
D) A higher setting will move air through the furnace at a faster rate which will heat the house faster
E) Can’t determine unless heating rate of the furnace is known
F) The furnace is designed to most efficiently heat a home if the thermostat is set to the desired temperature

9) Water flows steadily through the pipe shown above. The pipe wall is heated so that the temperature of flowing water increases from an average of \( T_1 \) at the pipe inlet to an average of \( T_2 \) at the outlet. Assume the pipe wall temperature is uniform and constant. If we consider the pipe and water contained in it as the unit of analysis (i.e. the control volume), which of the following statements is true?

A) Pipe control volume is at steady-state; water and pipe wall are in thermal equilibrium
B) Pipe control volume is not at steady-state; water and pipe wall are in thermal equilibrium
C) Pipe control volume is at steady-state; water and pipe wall are not in thermal equilibrium (Ans)
D) Pipe control volume is not at steady-state; water and pipe wall are not in thermal equilibrium

10) Your answer to question 9 is correct because:

A) control volume can never be at steady-state until \( T_2 \) equals the pipe wall temperature
B) steady-state and equilibrium occur together – you can’t have one without the other
C) average water temperature at any distance along the pipe is not changing with time but is less than the pipe wall temperature (Ans)
D) heat transfer is occurring at the water/pipe wall interface so the control volume can never come to steady-state
Quiz #4

Note: Your answers must be correct to 3 significant figures and have the appropriate units.

I. In IC (internal combustion) engines, during injection of liquid fuel into the cylinder, it is possible for the injected fuel droplets to form a thin liquid film over the piston as shown in the figure. The heat transferred from the gas above the film and from the piston beneath the film causes surface evaporation. The liquid-gas interface is at the boiling \( T_{lg} \), corresponding to the vapor pressure. The heat transfer from the piston side is by conduction through the piston and then by conduction through the thin liquid film. The surface-convection heat transfer from the gas side to the surface of the thin liquid film is 14,000 W.

Data:
- Heat of evaporation of fuel = 3.027 \times 10^5 J/kg, thermal conductivity of fuel, \( k_f = 0.083 \) W/m-K, \( T_{lg} = 398.9^oK \), liquid fuel density \( \rho_l = 900 \) kg/m\(^3\), thermal conductivity of piston \( k_s = 236 \) W/m-K, temperature of piston at distance \( L = 3 \) mm from the surface is \( T_1 = 500^oK \). Piston diameter \( D = 12 \) cm, thickness of liquid film \( L_f = 0.05 \) mm.

(1) Determine the initial rate of evaporation of the liquid fuel in kg/s. 
\[ 0.0524 \text{ kg/s} \]

(2) If the liquid evaporation rate is constant at 0.0350 kg/s, determine the time (in s) for the liquid film to evaporate completely.
\[ 0.0145 \text{ s} \]

(3) Consider the heat flux \( q_x'' \) for 1-dimensional heat transfer.
   A. For steady state \( q_x'' \) is a constant.
   B. For steady state with no heat generation \( q_x'' \) is a constant.

   a. A and B are true      b. Only A is true      c. Only B is true      d. A and B are false Ans b

II. Gaseous combustion occurs between two plates, as shown in the Figure below. The energy converted by combustion \( \dot{S}_{rc} \) in the gas flows through the upper and lower bounding plates. The upper plate is used for surface radiation heat transfer and is made of solid alumina \((k = 5.931 \) W/m-K). The lower plate is porous and is made of silica \((k = 0.373 \) W/m-K). Each plate has a length \( L \), a width \( w \), and a thickness \( l \). The outside surfaces of the two plates are at temperatures \( T_{s,1} \) and \( T_{s,2} \).

\[ \dot{S}_{rc} = 10^4 \text{ W}, \ T_{s,1} = 1,050^oC, \ T_{s,2} = 500^oC, \ L = 0.3 \text{ m}, \ w = 0.3 \text{ m}, \ l = 0.02 \text{ m}. \]
(4) If the inside surface temperature of the top plate is 1400°C, determine the fraction of heat generated flow through the top plate. \[ \frac{0.9333}{0.9333} \]

(5) If the inside surface areas of the two plate are at the gas temperature, determine the gas temperature. \[ 1370°C \]

(III) Given the temperature at three \( x \)-coordinates: \( T(x = 0.1 \text{ m}) = 75°C \), \( T(x = 0.2 \text{ m}) = 80°C \), and \( T(x = 0.3 \text{ m}) = 90°C \). Estimate \( \frac{dT}{dx^2} \) at \( x = 0.2 \text{ m} \) using central finite difference. \[ 500 °C/m^2 \]

IV. The steady-state temperature (°C) associated with selected nodal points of a two-dimensional system having a thermal conductivity of 2.0 W/m·°K are shown on the right. The ambient fluid is at 40°C with a heat transfer coefficient of 30 W/m²·°K. The isothermal surface is at 200°C.

Note: \( \Delta x \) is not equal to \( \Delta y \). Solve for the temperature at each node by making an energy balance around that node.

(7) The temperature, \( T_1 \), at node 1 is \[ T_1 = 170.22°C \]

(8) The temperature at node 3 is \[ 57.15°C \]
Two copper cylinders, each at 75°C, are allowed to cool in a room where the air temperature is 25°C. Cylinder 1 is taller than cylinder 2 but both cylinders have the same diameter. The top and bottom of each cylinder is perfectly insulated so the only heat loss is through the sides of the cylinders.

(9) Which of the following graphs would most closely approximate the average temperature of the two cylinders?

a. Graph A  

b. Graph B  

c. Graph C  

d. Graph D  

(10) Your answer to Question (9) is correct because:

a) Cylinder 2 contains less mass and less stored energy so will cool faster  
b) Cylinder 1 has more heat transfer area in contact with the atmosphere and therefore will cool faster  
c) Both cylinders will cool at the same rate because the surface area/volume ratio is the same for each  

**Ans**
Quiz #5

Note: Your answers must be correct to 3 significant figures and have the appropriate units.

I. A chemical with volume $V$, density $\rho$ and specific heat $c$ is heated in an adiabatic stirred reactor. The chemical is to be heated from room temperature $T_i$ to a process temperature $T$ by passing saturated steam at $T_h$ through a coiled, thin-walled with diameter $D$. Steam condensation within the tube maintains an interior convection coefficient $h_i$, while the highly agitated liquid in the stirred vessel maintains an outside convection coefficient $h_o$.

1) The overall heat transfer coefficient $U$ for this process is 1,667 W/m²·K

2) If $U = 1,200$ W/m²·K, determine the area of the submerged tubing required to heat the chemical from $T_i$ to $T$ in 60 minutes.

1.91 m²

II. (3) Air is flowing steadily through a horizontal, constant diameter pipe. The pipe wall is heated uniformly so that the temperature of the air increases as the air flows through the pipe. You may assume that the air temperature is constant across the cross-section at any length down the pipe. If the air pressure remains constant (pressure drop is small enough to ignore), what can you say about the average velocity of air in the pipe?

a) Velocity remains constant because pipe is rigid so air can’t expand or change density.
b) Velocity will increase in the flow direction because density decreases as air temperature increases. Ans
c) Velocity remains constant since flow is steady
d) Velocity will increase because increased temperature indicates the air molecules are moving faster and have higher kinetic energy
e) Velocity will decrease because hot pipe walls will increase friction in flowing air.
III. A plane wall of a furnace is fabricated from plain carbon steel \((k = 60 \text{ W/m-K}, \rho = 7850 \text{ kg/m}^3, c = 430 \text{ J/kg-K})\) and is of thickness \(L = 10 \text{ mm}\). To protect it from the corrosive effects of the furnace combustion gases, one surface of the wall is coated with a thin ceramic film which, for a unit surface area, has a thermal resistance of \(R''_{t,f} = 0.01 \text{ m}^2\text{-K/W}\). The opposite surface is well insulated from the surroundings. At furnace start up the wall is at an initial temperature \(T_i = 300^\circ\text{K}\), and the combustion gases at \(T_\infty = 1300^\circ\text{K}\) enter the furnace, providing a convection coefficient of \(h = 25 \text{ W/m}^2\text{-K}\) at the ceramic film. The film has negligible thermal capacitance. What is the temperature \(T_{s,o}\) of the exposed surface of the ceramic film at this time?

4) Determine the overall heat transfer coefficient \(U\) (W/m\(^2\)-K) between the steel and the combustion gas.

\[
20 \text{ W/m}^2\text{-K}
\]

5) What is the temperature \(T_{s,o}\) of the exposed surface of the ceramic film when \(T_{s,i} = 1100^\circ\text{K}\)?

\[
1140^\circ\text{K}
\]

6) If \(U = 50 \text{ W/m}^2\text{-K}\), how long will it take for the inner surface of the steel to achieve a temperature of \(T_{s,i} = 1100^\circ\text{K}\)?

\[
1086 \text{ s} = 18.1 \text{ min}
\]

IV. (7) If 25°C (77°F) air feels warm on our skin, why does 25°C water feel cool when we swim in it?

a) When water contacts human skin, it vaporizes at the surface, which causes the water to feel cooler than air.

b) Water holds energy better than air does, so air feels warmer since it is transferring energy faster.

c) The heat transfer rate from skin to water is faster than the rate from skin to air because of differences in fluid physical properties. Ans

d) Water opens pores in human skin better than air does, so the heat transfer area is larger with water.
PLANE WALL                     CYLINDER SPHERE

<table>
<thead>
<tr>
<th>Bi</th>
<th>$\zeta_1$ rad</th>
<th>$C_1$</th>
<th>$\zeta_1$ rad</th>
<th>$C_1$</th>
<th>$\zeta_1$ rad</th>
<th>$C_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\infty$</td>
<td>1.5708</td>
<td>1.2732</td>
<td>2.4048</td>
<td>1.6020</td>
<td>3.1416</td>
<td>2.0000</td>
</tr>
</tbody>
</table>

\[
Fo = \frac{at}{L^2} = \frac{at}{r_0^2}, \quad \theta^* = \frac{T - T_\infty}{T_i - T_\infty}, \quad \theta_0^* = C_1 \exp(-\frac{\zeta_1^2}{2} Fo)
\]

**Conduction in a slab**

\[
\theta^* = \theta_0^* \cos(\zeta_1 x^*); \quad \frac{Q}{Q_o} = 1 - \frac{\sin(\zeta_1)}{\zeta_1} \theta_0^*
\]

If the temperature at the surface $T_s$ is known $T_\infty$ will be replaced by $T_s$

$\zeta_1$ and $C_1$ will be obtained from table at $Bi = \infty$

**V.** In a tempering process, glass plate, which is initially at a uniform temperature $T_i$, is cooled by suddenly reducing the temperature of both surfaces to $T_s$. The plate is 20 mm thick, and the glass has a thermal diffusivity of \(6 \times 10^{-7} \text{ m}^2/\text{s}\).

8) How long will it take for the midplane temperature to achieve 50% of its maximum possible temperature reduction?

\[63.135 \text{ s}\]

9) The maximum temperature gradient in the glass at any time is given by

\[\begin{align*}
\text{a)} & \quad \theta_0^* \cos(\zeta_1)(T_i - T_s) \\
\text{b)} & \quad \theta_0^* \zeta_1 \sin(\zeta_1 x^*)(T_i - T_s) \\
\text{c)} & \quad \theta_0^* \zeta_1 \sin(\zeta_1)(T_i - T_s) \\
\text{d)} & \quad -\theta_0^* \zeta_1 \sin(\zeta_1)(T_i - T_s) \\
\text{e)} & \quad \text{None of the above} \quad \text{Ans}
\end{align*}\]

VI. (10) On a very cold winter day, a group of engineering students noticed that quickly licking the metal end of an ice scraper left outside overnight caused their tongues to freeze to the metal surface. **However, a quick lick of the plastic handle of the scraper didn’t cause any freezing to occur.** How can you explain this observation?

\[\begin{align*}
\text{a)} & \quad \text{Metal is colder than plastic because it transfers energy to the atmosphere faster.} \\
\text{b)} & \quad \text{Metal is colder than plastic because metal is more dense and therefore retains cold better.} \\
\text{c)} & \quad \text{Metal is colder than plastic because plastic stores energy better.} \\
\text{d)} & \quad \text{Metal conducts energy better than plastic, so energy moves away from the tongue faster when touching metal. \textbf{Ans}}
\end{align*}\]

C-15
Appendix D

Previous Exams

CHE 312 (Winter 2011)

Quiz #1

Note: Your answers must be correct to 3 significant figures and have the appropriate units.

I. An annealing process shown below uses a hot plate operating at an elevated temperature \( T_h \). The wafer, initially at a temperature of \( T_{w,i} \), is suddenly positioned at a gap separation \( h = 0.6 \) mm from the hot plate. The emissivity of both the hot plate and the wafer is 0.90. The silicon wafer has a thickness of \( d = 0.50 \) mm, a density of 2700 kg/m\(^3\), and a specific heat of 1050 J/kg-K. The thermal conductivity of the gas in the gap is 0.0436 W/m-K. The wafer is insulated at the bottom. Stefan-Boltzmann constant \( \sigma = 5.67 \times 10^{-8} \) W/m\(^2\)-K\(^4\).

For \( T_h = 650^\circ C \) and \( T_{w,i} = 25^\circ C \), calculate

(1) The radiative heat flux across the gap

\[ 36.634 \text{ W/m}^2 \]

(2) The heat flux by conduction across the gap

\[ 45.417 \text{ W/m}^2 \]

(3) The initial time rate of change in the temperature of the wafer, \( \left( \frac{dT_w}{dt} \right)_i \), if the total heat flux across the gap is 100 kW/m\(^2\).

\[ 70.55 \text{ K/s} \]

II. (4) Consider the process arrangement where a wafer is in an evacuated chamber whose wall are maintained at 27\(^\circ\)C and within which heating lamps maintain a radiant flux \( q_s' \) at its upper surface. The wafer is 0.78 mm thick, has a thermal conductivity of 30 W/m-K, and an emissivity that equals its absorptivity to the radiant flux (\( \varepsilon = \alpha = 0.65 \)). For \( q_s'' = 3.0 \times 10^5 \) W/m\(^2\), the temperature on its lower surface is measured by a radiation thermometer and found to have a value of \( T_{w,l} = 997^\circ C \). (Note: Use \( T(K) = T(C) + 273 \))
The temperature, $T_{w,u}$, at the top surface of the wafer is _________

A) 999.56°C  B) 1002.56°C  C) 1005.56°C  D) 998.56°C

$$0.65 \times 3.0 \times 10^5 - 0.65 \times 5.67 \times 10^{-8} [T_{w,u}^4 - (27 + 273)^4] - 30(T_{w,u} - 1270)/0.00078 = 0$$

III. (5) Liquid oxygen, which has a boiling point of 90°C and a latent heat of vaporization of 214 kJ/kg, is stored in a spherical container whose outer surface is of 500-mm diameter and at a temperature of $-10°C$. The container is housed in a laboratory whose air and walls are at 25°C. If the surface emissivity is 0.70 and the heat transfer coefficient associated with free convection at the outer surface of the container is 40 W/m²·K, what is the rate, in kg/s, at which oxygen vapor must be vented from the system?

$5.59 \times 10^{-3}$ kg/s

IV. (6) A surface whose temperature is maintained at 450°C is separated from an air flow by a layer of insulation 30 mm thick for which the thermal conductivity is 0.04 W/m·K. If the air temperature is 25°C and the convection coefficient between the air and the outer surface of the insulation is 500 W/m²·K, what is the temperature of this outer surface?

26.13°C

V. (7) An experiment to determine the convection coefficient associated with airflow over the surface of a thick steel casting involves insertion of thermocouples in the casting at distances of 10 and 20 mm from the surface along a hypothetical line normal to the surface. The steel has a thermal conductivity of 40 W/m·K. If the thermocouples measure temperatures of 55 and 40°C in the steel when the air temperature is 100°C, what is the convection coefficient?

2000 W/m²·K

VI. (8) The roof of a car in a parking lot absorbs a solar radiant flux of 900 W/m², while the underside is perfectly insulated. The convection coefficient between the roof and the ambient air is 15 W/m²·K. Neglecting radiation exchange with the surroundings, calculate the temperature of the roof under steady-state conditions if the ambient air temperature is 25°C.

85.0°C
VII. You have a glass of tea in a well-insulated cup that you would like to cool off before drinking. You also have 2 ice cubes to use in the cooling process and have access to an ice crusher.

9) Assuming no energy is lost from the tea into the room and no ice is lost in the crushing process, which form of ice (cubes or crushed ice) added to your tea will give a lower drink temperature?

A) The crushed ice
B) The ice cubes
C) Either will lower the drink temperature the same amount (Ans.)
D) Can’t tell from the information given

10) Your answer to the previous question is correct because:

A) Crushed ice has more surface area so energy transfer rate will be higher
B) Energy transfer is proportional to the mass of ice used (Ans.)
C) Crushed ice will melt faster and will transfer energy from the tea faster
D) Ice cubes contain less energy per mass that crushed ice so tea will cool more
E) Ice cubes have a higher heat capacity than crushed ice
Quiz #2

Note: Your answers must be correct to 3 significant figures and have the appropriate units.

I. (1). Two engineering students are conducting an experiment with air using a frictionless, well-insulated piston system as shown above. Initially the air is at room temperature and the piston sits in the lower position. They notice that when the air is heated and the temperature increases, the volume expands and the piston rises to the upper position.

What if the students conduct the experiment differently? That is, what will happen to the air temperature if they manually pull the piston from the lower position to the upper position and hold it there?

A) The air temperature will increase because the piston still moves between the same initial and final positions
B) The air temperature will decrease because work is done on the surroundings when the piston moves upward. (Ans.)
C) The air temperature will increase because pulling the piston upward does work on the air inside the piston system
D) The air temperature will decrease because energy will be lost as friction as the piston moves upward

(2) Two metal blocks of equal size and mass are initially at room temperature (~20°C) and are then placed in a furnace operating at 200°C. Block 1 reaches a uniform temperature of 200°C in 5 minutes while block 2 takes 10 minutes to reach 200°C.

To which block was more energy transferred during the heating process?  ___D_____

A) Block 2 because it is heated longer
B) Block 1 because it heats up faster (temperature rises faster)
C) Both blocks received the same amount of energy because they started at the same initial temperature and ended at the same final temperature
D) Can’t determine from the information given because heat transfer coefficients from the block surfaces are needed
E) Can’t determine from the information given because heat capacities of the metals used to make the blocks are needed

II The air inside a chamber at \( T_{x,i} = 50°C \) is heated convectively with \( h_i = 20 \text{ W/m}^2\text{-K} \) by a 200-mm-thick wall having a thermal conductivity of 4 W/m-K and a uniform heat generation of 1000 W/m³. To prevent any heat generated within the wall from being lost to the outside of the chamber at \( T_{x,o} = 25°C \) with \( h_o = 5 \text{ W/m}^2\text{-K} \), a very thin electrical strip heater is placed on the outer wall to provide a uniform heat flux, \( q'' \). No heat generated within the wall is lost to the outside of the chamber. The outer wall surface is at \( x = 0 \).
Which of the following sketch best represents the temperature profile in the wall:

(A) ![Sketch A](image)

(B) ![Sketch B](image)

(C) ![Sketch C](image)

(D) ![Sketch D](image)

(4) Determine \( T(L) \)

60°C

(5) Determine \( T(0) - T(L) \)

5°C

(6) If the heat generation in the wall were switched off while the heat flux to the strip heater remained constant, what would be the steady-state temperature, \( T(0) \), of the outer wall surface?

55°C

III (7). In a manufacturing process, a transparent film is being bonded to a substrate as shown in the sketch. To cure the bond at a temperature \( T_o \), a radiant source is used to provide a heat flux \( q_o \) (W/m²), all of which is absorbed at the bonded surface. The back of the substrate is maintained at \( T_1 \) while the free surface of the film is exposed to air at \( T_{∞} \) and a convection heat transfer coefficient \( h \). Assume the following conditions: \( T_{∞} = 20°C \), \( h = 50 \text{ W/m}^2\cdot\text{K} \), and \( T_1 = 30°C \).

Calculate the heat flux \( q_o \) required to maintain the bonded surface at \( T_o = 60°C \)

2833 W/m²

IV (8). A transistor, which may be approximated as a hemispherical heat source of radius \( r_o = 0.1 \text{ mm} \), is embedded in a large silicon substrate \((k = 125 \text{ W/m-K})\) and dissipates heat at a rate \( q \). All boundaries of the silicon are maintained at an ambient temperature of \( T_{∞} = 27°C \), except for a plane surface that is well insulated. The general expression for the substrate temperature distribution is given by

\[
T = T_{∞} - (T_{∞} - T_s) \frac{r}{r_o}
\]
Evaluate the surface temperature $T_s$ of the heat source for $q = 4\text{W}$

$77.9^\circ\text{C}$

9) Determine the heat loss in the cylindrical section of the tank $5704\text{ W}$

10) Determine the heat loss in the two hemispherical end sections of the tank $2958\text{ W}$
I. Electronic power devices are mounted to a heat sink having an exposed surface area of 0.045 m$^2$ and an emissivity of 0.80. The devices dissipate a total power of 40 W and the air and surroundings are at 27°C with a heat transfer coefficient of 25 W/m$^2$·K. Determine the steady state temperature of the heat sink using Newton method, one iteration and initial guess of 300 K.

329.73 K

II. A computer consists of an array of five printed circuit boards (PCBs), each dissipating $P_b = 20$ W power. Cooling of the electronic components on a board is provided by the forced flow of air, equally distributed in passages formed by adjoining boards, and the convection coefficient associated with heat transfer from the components to the air is approximately $h = 200$ W/m$^2$·K. Air enters the computer console at a temperature of $T_i = 20^\circ$C, and flow is driven by a fan whose power consumption is $P_f = 25$ W.

2) If the temperature rise of the air flow, $(T_o - T_i)$, is not to exceed 15°C, what is the minimum allowable volumetric flow rate of the air? The density and specific heat of the air may be approximate as $\rho = 1.161$ kg/m$^3$ and $C_p = 1007$ J/kg·K, respectively.

$7.13 \times 10^{-3}$ m$^3$/s

3) The component that is most susceptible to thermal failure dissipates 1 W/cm$^2$ of surface area. What is its surface temperature at the location to minimize thermal failure?

70°C

III) A spherical stainless steel ($\rho = 8055$ kg/m$^3$, $C_p = 510$ J/kg·°K) canister is used to store reacting chemicals that provide for a uniform heat flux $q_i$ to its inner surface. The canister is suddenly submerged in a liquid bath of temperature $T_\infty < T_i$ and heat transfer coefficient of 500 W/m$^2$·K. $T_i$ is the initial temperature of the canister wall. ($T_i = 500$ K, $T_\infty = 300$ K, $R_i = 0.5$ m, $R_o = 0.6$ m.

4) Assuming negligible temperature gradient in the canister wall and a constant heat flux determine the initial rate of change of the wall temperature if $q_i = 10^5$ W/m$^2$.

$-0.088$ K/s

5) What is the steady-state temperature of the wall?

439 K

IV) The steady-state temperatures associated with selected nodal points of a two-dimensional system having a thermal conductivity of 1.0 W/m·K are given. The grid spacing is 0.1 m for both directions. Data: Isothermal surface temperature $= T_0(\text{C}) = 170$, $T_{\text{int}}(\text{C}) = 21$, $h(\text{W/m}^2\cdot\text{K}) = 80$, $T(1) = 149.5$, $T(2) = 124.3$, $T(4) = 42.9$, $T(5) = 151.9$, $T(6) = 129.0$, $T(7) = 95.8$, $T(9) = 159.0$, $T(11) = 118.5$, and $T(12) = 64.1$ all in °C.
6) Temperature at node 10 is \(144.1 \, ^\circ C\)

7) Temperature at node 3 is \(89.7 \, ^\circ C\)

8) Temperature at node 8 is \(31.7^\circ C\)

9) Two copper cylinders, each at 75\(^\circ\)C, are allowed to cool in a room where the air temperature is 25\(^\circ\)C. As shown, cylinder 1 is taller than cylinder 2 but both cylinders are the same diameter. The top and bottom of each cylinder is perfectly insulated so the only heat loss is through the sides of the cylinders.

Which of the following graphs would most closely approximate the average temperature of the two cylinders? Graph D (Ans.)

10) A solid steel sphere (AISI 1010), 300 mm in diameter, is coated with a dielectric material layer of thickness 2 mm and thermal conductivity 0.04 W/m\(^{\circ}\)K. The coated sphere is initially at a uniform temperature of 500\(^\circ\)C and is suddenly quenched in a large oil bath for which \(T_\\infty = 100^\circ C\) and \(h = 3300 \, \text{W/m}^2\cdot\text{K}\). Estimate the time (in hr) required for the coated sphere temperature to reach 140\(^\circ\)C. Neglect the effect of energy storage in the dielectric material, since its thermal capacitance \((\rho cV)\) is small compared to that of the steel sphere. AISI steel: \(\rho = 7832 \, \text{kg/m}^3\), \(c = 559 \, \text{J/kg} \cdot \text{K}\), \(k = 48.8 \, \text{W/m} \cdot \text{K}\).

\[
25355 \, \text{s} = 7.04 \, \text{hr}
\]
I. Electronic power devices are mounted to a heat sink having an exposed surface area of 0.05 m$^2$ and an emissivity of 0.80. The devices dissipate a total power of 50 W and the air and surroundings are at 27°C with a heat transfer coefficient of 25 W/m$^2$K. Determine the steady state temperature of the heat sink using Newton method, one iteration and initial guess of 310 K.

\[ 332.97 \text{ K} \]

(II) Given the temperature at three x-coordinates: $T(x = 0.1 \text{ m}) = 76^\circ \text{C}$, $T(x = 0.2 \text{ m}) = 79^\circ \text{C}$, and $T(x = 0.3 \text{ m}) = 88^\circ \text{C}$. Estimate $\frac{d^2T}{dx^2}$ at $x = 0.2 \text{ m}$ using central finite difference.

\[ 600 \text{ } ^\circ \text{C/m}^2 \]

III. A. The unsteady state solution for temperature distribution in a cylinder $T(r, t)$ requires the evaluation of a Bessel function.

B. If the initial temperature for an infinite cylinder is not uniform, temperature distribution $T(r, t)$ cannot be obtained.

a. A and B are true  b. Only A is true (Ans)  c. Only B is true  d. A and B are false

IV. The steady-state temperature (°C) associated with selected nodal points of a two-dimensional system having a thermal conductivity of 2.0 W/m.°K are shown on the right. The ambient fluid is at 40°C with a heat transfer coefficient of 30 W/m$^2$.°K. The isothermal surface is at 200°C.

(4) The temperature, $T_2$, at node 2 is \( 108.61^\circ \text{C} \)

(5) The temperature at node 3 is 57.15°C, calculate the heat transfer rate per unit thickness normal to the page from the right surface to the fluid.

\[ 852.3 \text{ W/m} \]
V. The x-coordinate is assigned in the direction along a cylindrical fin with \( x = 0 \) at the base or left surface where the temperature is 100\(^\circ\)C. The end of the fin is insulated. Data: Length = 20 cm, diameter 1 cm, \( k = 240 \text{ W/m} \cdot \text{K} \), \( h = 40 \text{ W/m}^2 \cdot \text{K} \), and the ambient temperature \( T_\infty = 25\text{\,oC} \). The temperature profile for the fin is: 

\[ \theta = B_1 \sinh(cx) + B_2 \cosh(cx) \]

(6) Determine the numerical value of \( c \) 

8.1650 m\(^{-1}\)

If \( c = 5 \text{ m}^{-1} \), determine the numerical values of \( B_1 \) and \( B_2 \)

(7) \( B_1 = \) 75\(^{\circ}\)C 

(8) \( B_2 = -57.12\text{ \,oC} \)

V. A masonry slab of width \( W = 0.05 \text{ m} \), which is at an initial temperature of 25\(^\circ\)C, is heated by passing a hot gas through the two surfaces, with the gas temperature and the convection coefficient assumed to have constant values of \( T_\infty = 600\text{\,oC} \) and \( h = 100 \text{ W/m}^2 \cdot \text{s} \). Properties of masonry material are \( \rho = 1900 \text{ kg/m}^3 \), \( c = 800 \text{ J/kg} \cdot \text{K} \) and \( k = 0.625 \text{ W/m} \cdot \text{K} \).

(9) How long will it take to achieve 75\% of the maximum possible energy storage?

1245 s = 20.75 min

(10) What is the maximum temperature of the masonry at 20 min?

539.4\text{\,oC}
Quiz #5

Note: Your answers must be correct to 3 significant figures and have the appropriate units.

I. The x-coordinate is assigned in the direction along a square (1 cm by 1 cm) fin with \( x = 0 \) at the base or left surface where the temperature is 120°C. The end of the fin is not insulated. Data: Length = 20 cm, \( k = 240 \text{ W/m·K} \), \( h = 20 \text{ W/m}^2\text{·K} \), and the ambient temperature \( T_\infty = 25^\circ \text{C} \). The temperature profile for the fin is:

\[
\theta = B_1 \sinh(cx) + B_2 \cosh(cx)
\]

(1) Determine the numerical value of \( c \)

\[ c = 5 \text{ m}^{-1} \]

If \( c = 5 \text{ m}^{-1} \), determine the numerical values of \( B_1 \) and \( B_2 \)

(2) \( B_1 = -73.008^\circ \text{C} \)  
(3) \( B_2 = 95^\circ \text{C} \)

II. A long bar of rectangular cross section is 60 mm by 90 mm on a side and has a thermal conductivity of 2 W/m·K. One surface is exposed to a convection process with air at 100°C and a convection coefficient of 100 W/m²·K, while the remaining surfaces are maintained at 50°C. Using a grid spacing of 30 mm the temperature at node 1 is given by

\[ T_1 = AT_2 + B \]

(4) \( A = 0.2857 \)  
(5) \( B = 57.14 \text{ °C} \)

III. A process fluid having a specific heat of 3500 J/kg·K and flowing at 2 kg/s is to be cooled from 80°C to 50°C with chilled water (specific heat of 4180 J/kg·K), which is supplied at a temperature of 15°C and a flow rate of 2.5 kg/s. Assuming an overall heat transfer coefficient of 2000 W/m²·K, calculate the required heat transfer areas for the following exchanger configurations: (a) parallel flow, (b)

(6) Parallel flow \( 3.087 \text{ m}^2 \)  
(7) Counter flow \( 2.642 \text{ m}^2 \)

IV. A shell-and-tube heat exchanger with one shell pass and four tube passes uses hot water on the tube side to heat oil on the shell side. The water enters at 87°C and 50.0 kg/s and leaves at 27°C. Inlet and outlet temperatures of the oil are 7 and 37°C. The heat exchanger contains 200 tubes which have inner and outer diameters of 20 and 24 mm and a length of 5.0 m. Density of water is 995 kg/m³ and \( C_p \) of water is 4200 J/kg·K.

(8) Water velocity is \( 3.20 \text{ m/s} \)
V. A masonry slab of width $W = 0.05$ m, which is at an initial temperature of $25^\circ\text{C}$, is heated by passing a hot gas through the one surface while the other surface is insulated, with the gas temperature and the convection coefficient assumed to have constant values of $T_\infty = 600^\circ\text{C}$ and $h = 100$ W/m$^2$·s. Properties of masonry material are $\rho = 1900$ kg/m$^3$, $c = 800$ J/kg·K and $k = 0.625$ W/m·K.

(9) How long will it take to achieve 75% of the maximum possible energy storage? __________

\[3937 \text{ s} = 65.61 \text{ min} = 1.0935 \text{ h}\]

(10) What is the maximum temperature of the masonry at 50 min? __552.5^\circ\text{C}__
Appendix E

Previous Exams

CHE 312 (Winter 2012)

Quiz #1

Note: Your answers must be correct to 3 significant figures and have the appropriate units.

I. An annealing process shown below uses a hot plate operating at an elevated temperature $T_h$. The wafer, initially at a temperature of $T_{w,i}$, is suddenly positioned at a gap separation $h = 0.6$ mm from the hot plate. The emissivity of both the hot plate and the wafer is 0.90. The silicon wafer has a thickness of $d = 0.50$ mm, a density of 2700 kg/m$^3$, and a specific heat of 1050 J/kg$\cdot$K. The thermal conductivity of the gas in the gap is 0.04 W/m$\cdot$K. The wafer is insulated at the bottom. Stefan-Boltzmann constant $\sigma = 5.67 \times 10^{-8}$ W/m$^2$$\cdot$K$^4$.

For $T_h = 700^\circ$C and $T_{w,i} = 25^\circ$C, calculate

(1) Determine the total heat flux across the gap $90,336$ W/m$^2$

(2) Determine the initial time rate of change in the temperature of the wafer, \[ \frac{dT_w}{dt} \], if the total heat flux across the gap is 90 kW/m$^2$. $63.5$ K/s

II. (3)

A. The thermal conductivity is due to the flow of free electron, lattice vibrational waves, and molecular collisions.

B. For pure metals, the contribution to thermal conductivity due to phonon (lattice vibrational) is more important.

a) Both A and B are true  

b) Only A is true (A)  

c) Only B is true  

(4) Liquid oxygen, which has a boiling point of 90$^\circ$K and a latent heat of vaporization of 214 kJ/kg, is stored in a spherical container whose outer surface is of 500-mm diameter and at a temperature of −10$^\circ$C. The container is housed in a laboratory whose air is at 0$^\circ$C and walls are at 25$^\circ$C. If the surface emissivity is 0.60 and the heat transfer coefficient associated with free convection at the outer surface of the container is 40 W/m$^2$$\cdot$K, what is the rate, in kg/s, at which oxygen vapor must be vented from the system?

$1.855 \times 10^{-3}$ kg/s
IV. (5) An experiment to determine the convection coefficient associated with airflow over the surface of a thick steel casting involves insertion of thermocouples in the casting at distances of 10 and 20 mm from the surface along a hypothetical line normal to the surface. The steel has a thermal conductivity of 40 W/m·K. If the thermocouples measure temperatures of 52 and 40°C in the steel when the air temperature is 90°C, what is the convection coefficient?

1846 W/m²·K

V. (6) A surface whose temperature is maintained at 400°C is separated from an air flow by a layer of insulation 30 mm thick for which the thermal conductivity is 0.04 W/m·K. If the air temperature is 25°C and the convection coefficient between the air and the outer surface of the insulation is 200 W/m²·K, what is the temperature of this outer surface?

27.48°C

VI. In the two-dimensional body illustrated, the gradient at surface A is found to be \( \frac{\partial T}{\partial y} = 20 \) K/m.

7) Determine \( \frac{\partial T}{\partial y} \) at surface B ___0___

8) Determine \( \frac{\partial T}{\partial x} \) at surface B ___40 K/m

VII. A well-insulated pipe of 2.54 cm inside diameter carries air at 2 bar pressure and 366.5°C. It is connected to a 0.0283 m³ insulated bulge, as shown. The air in the bulge is initially at one bar pressure and 311°C. A and D are flow meters which accurately measure the air mass flow rate. Valves B and C control the air flow into and out of the bulge. Connected to the bulge is a 0.283 m³ rigid, adiabatic tank which is initially evacuated to a very low pressure.

At the start of the operation, valve B is opened to allow 4.54 g/s of air flow into the bulge; simultaneously, valve C is operated to transfer exactly 4.54 g/s from the bulge into the tank. These flows are maintained constant as measured by the flow meters. Air may be assumed to be an ideal gas with a specific heat ratio \( \gamma = C_p/C_v = 1.4 \), \( C_p = 29.3 \) J/mol·°K, and molecular weight = 29. Gas constant \( R = 8.314 \) m³·Pa/mol·°K.

(9&10). Determine \( \frac{dT}{dt} \) when the temperature in the bulge is 340°C.

5.8 K/s
Quiz #2

Note: Your answers must be correct to 3 significant figures and have the appropriate units.

I. A well-insulated pipe of 2.54 cm inside diameter carries air at 2 bar pressure and 366.5 °K. It is connected to a 0.0283 m³ insulated bulge, as shown. The air in the bulge is initially at 110 kPa pressure and 311 °K. A and D are flow meters which accurately measure the air mass flow rate. Valves B and C control the air flow into and out of the bulge. Connected to the bulge is a 0.283 m³ rigid, adiabatic tank which is initially evacuated to a very low pressure. At the start of the operation, valve B is opened to allow 4.54 g/s of air flow into the bulge; simultaneously, valve C is operated to transfer exactly 4.54 g/s from the bulge into the tank. These flows are maintained constant as measured by the flow meters. Air may be assumed to be an ideal gas with a specific heat ratio \( \gamma = C_p/C_v = 1.4 \), \( C_p = 29.3 \text{ J/mol} \cdot \text{oK} \), and molecular weight = 29. Gas constant \( R = 8.314 \text{ m}^3 \cdot \text{Pa}/\text{mol} \cdot \text{oK} \).

\[ \text{Air at 2 bar pressure} \]
\[ \text{Bulge} \]
\[ \text{Tank} \]

1) Determine the mass inside the bulge \( 34.9 \text{ g} \)

2) If the initial mass inside the bulge is 40 g, determine the initial \( \frac{dT}{dt} \) \( 8.82 \text{ K/s} \)

II. A) For one-dimensional, steady-state conduction in a plane wall with no heat generation and constant thermal conductivity, the temperature varies linearly with \( x \).

B) For one-dimensional, steady-state conduction in a plane wall with heat generation and constant thermal conductivity, the temperature varies linearly with \( x \).

a) Both A and B are true  

b) Only A is true \( (A) \)  

c) Only B is true  

d) Both A and B are false

III. (4) The wall of an oven used to cure plastic parts is of thickness \( L = 0.05 \text{ m} \) and is exposed to large surroundings and air at its outer surface. The air and the surroundings are at 300 K. If the temperature of the outer surface is 400 K and its convection coefficient and emissivity are \( h = 20 \text{ W/m}^2 \cdot \text{K} \) and \( \varepsilon = 0.8 \), respectively, what is the temperature of the inner surface if the wall has a thermal conductivity of \( k = 0.7 \text{ W/m} \cdot \text{K} \)? (Note: \( \sigma = 5.67 \times 10^{-8} \text{ W/m}^2 \cdot \text{K}^4 \))

\( 600 \text{ K} \)

IV (5). In a manufacturing process, a transparent film is being bonded to a substrate as shown in the sketch to cure the bond at a temperature \( T_o \), a radiant source is used to provide a heat flux \( q_o \) (W/m²), all of which is absorbed at the bonded surface. The back of the substrate is maintained at \( T_1 \) while the free surface of the film is exposed to air at \( T_\infty \) and a convection heat transfer coefficient \( h \). Assume the following conditions: \( T_\infty = 20^\circ \text{C}, h = 20 \text{ W/m}^2 \cdot \text{K}, \) and \( T_1 = 30^\circ \text{C} \).

Calculate the heat flux \( q_o \) required to maintain the bonded surface at \( T_o = 60^\circ \text{C} \)

\( 2167 \text{ W/m}^2 \)
V. The wind chill, which is experienced on a cold, windy day, is related to increased heat transfer from exposed human skin to the surrounding atmosphere. Consider a layer of fatty tissue that is 3 mm thick and whose interior surface is maintained at a temperature of 36°C. On a calm day the convection heat transfer coefficient at the outer surface is 25 W/m²·K, but with 30 km/h winds it reaches 65 W/m²·K. In both cases the ambient air temperature is –15°C.

6) What is the ratio of the heat loss per unit area from the skin for the calm day to that for the windy day? 
\[ \frac{0.553}{ } \]

7) What will be the skin outer surface temperature for the calm day? 
\[ \text{22.1°C} \]

8) What will be the skin outer surface temperature for the windy day? 
\[ \text{10.8°C} \]

VI. A storage tank consists of a cylindrical section that has a length and inner diameter of \( L = 2 \) m and \( D_i = 1 \) m, respectively, and two hemispherical end sections. The tank is constructed from 20-mm-thick glass (Pyrex) and is exposed to ambient air for which the temperature is 300 K and the convection coefficient is 20 W/m²·K. The tank is used to store heated oil, which maintains the inner surface at a temperature of 400 K. Radiation effects may be neglected, and the Pyrex may be assumed to have a thermal conductivity of 1.4 W/m·K.

9) Determine the heat loss in the cylindrical section of the tank 
\[ 10120 \text{ W} \]

10) Determine the heat loss in the two hemispherical end sections of the tank 
\[ 5239 \text{ W} \]
I. Consider the annular fin shown on the right with thickness $t$. The ambient air is at $T_\infty$ with heat transfer coefficient $h$. The base is at temperature $T_b$. The inside and outside radii of the annular fin are $r_1$ and $r_2$, respectively.

1) Specify the control volume required to solve for $T(r)$: __________

2) The differential equation required to solve for $T(r)$ is: __________

   a) $\frac{d}{dr} \left( r \frac{dT}{dr} \right) - \frac{h}{k} (T - T_\infty) = 0$
   
   b) $\frac{d}{dr} \left( r \frac{dT}{dr} \right) + \frac{h}{k} (T - T_\infty) = 0$
   
   c) $\frac{1}{r} \frac{d}{dr} \left( r \frac{dT}{dr} \right) - \frac{h}{k} (T - T_\infty) = 0$
   
   d) $\frac{1}{r} \frac{d}{dr} \left( r \frac{dT}{dr} \right) - \frac{h}{kt} (T - T_\infty) = 0$
   
   e) None of the above (A)

3) Specify the two boundary conditions required to solve for $T(r)$:

   a) $r = r_1$, $T = T_b$
   
   b) $r = r_2$, $-k \frac{dT}{dr} = h(T - T_\infty)$

II. Copper tubing is joined to a solar collector plate of thickness $t$, and the working fluid maintains the temperature of the plate above the tubes at $T_o$. There is a uniform net radiation heat flux $q''_{rad}$ to the top surface of the plate, while the bottom surface is well insulated. The top surface is also exposed to a fluid at $T_\infty$ that provides for a uniform convection coefficient $h$. $L$ is the mid-point between the copper tubing.

4) The differential equation required to solve for $T(x)$ is: __________

   a) $\frac{d^2T}{dx^2} + \frac{q''_{rad}}{k} = \frac{h}{kt} (T - T_\infty)$
   
   b) $\frac{d^2T}{dx^2} + \frac{q''_{rad}}{kt} = \frac{h}{kt} (T - T_\infty)$ (A)
   
   c) $\frac{d^2T}{dx^2} + \frac{q''_{rad}}{kt} + \frac{h}{kt} (T - T_\infty) = 0$
   
   d) $\frac{d^2T}{dx^2} + \frac{q''_{rad}}{k} = \frac{h}{k} (T - T_\infty)$
   
   e) None of the above

5) Specify the two boundary conditions required to solve for $T(x)$:

   a) $x = 0$, $T = T_o$
   
   b) $x = L$, $\frac{dT}{dx} = 0$
III. Circular copper \((k = 400 \text{ W/m-K})\) rods of diameter \(D = 2 \text{ mm}\) and length \(L = 25 \text{ mm}\) are used to enhance heat transfer from a surface that is maintained at \(T_{s,1} = 100^{\circ}\text{C}\). One end of the rod is attached to this surface (at \(x = 0\)), while the other end (\(x = 25 \text{ mm}\)) is joined to a second surface, which is maintained at \(T_{s,2} = 0^{\circ}\text{C}\). Air flowing between the surfaces (and over the rods) is also at a temperature of \(T_{\infty} = 0^{\circ}\text{C}\), and a convection coefficient of \(h = 100 \text{ W/m}^2\text{-K}\) is maintained. The temperature along the rod is given by \(\theta = \frac{\theta_s \sinh(mx) + \theta_h \sinh[m(L - x)]}{\sinh(mL)}\).

6) The rod temperature at \(x = 15 \text{ mm}\) is \(38.3^{\circ}\text{C}\).

7) The heat transfer from a single rod to the surface at \(x = 25 \text{ mm}\) is \(4.774 \text{ W}\).

8) If the heat transfer from a single rod to air is \(1 \text{ W}\) and a bundle of the rods is installed on \(8\)-mm centers, the total rate of heat transfer from a \(80 \text{ mm by 80 mm}\) section of the surface at \(100^{\circ}\text{C}\) to air is:

   a) \(100 \text{ W}\)  
   b) \(81 \text{ W}\)  
   c) \(160.86 \text{ W} \text{ (A)}\)  
   d) \(141.86 \text{ W}\)  
   e) None of the above

IV. A spherical system with radius of \(0.1 \text{ m}\) and uniform heat generation, \(\dot{q}\), is in an environment with \(h = 20 \text{ W/m}^2\text{-C}\) and \(T_{\infty} = 25^{\circ}\text{C}\). Thermal conductivity of the system is \(5 \text{ W/m}^{\circ}\text{C}\). The differential equation for \(T(r)\) is

\[
\frac{k}{r^2} \frac{d}{dr} \left( r^2 \frac{dT}{dr} \right) + \dot{q} = 0
\]

9) If the surface temperature (at \(r = 0.1 \text{ m}\)) of the system is \(100^{\circ}\text{C}\), determine the heat generation per unit volume of the system

   \(45,000 \text{ W/m}\)

10) If the surface temperature (at \(r = 0.1 \text{ m}\)) of the system is \(75^{\circ}\text{C}\) and the heat generation is \(30,000 \text{ W/m}^3\), the temperature at \(r = 0.05 \text{ m}\) is

   \(82.5^{\circ}\text{C}\)
Note: Your answers must be correct to 3 significant figures and have the appropriate units.

I. Consider the annular fin shown on the right with thickness \( t \). The ambient air is at \( T_\infty \) with heat transfer coefficient \( h \). The base is at temperature \( T_b \). The inside and outside radii of the annular fin are \( r_1 \) and \( r_2 \), respectively. Energy balance over a differential control volume is required to obtain the differential equation to solve for \( T(r) \).

1) Specify the expression for the heat leaving the differential control volume by convection. \( 4\pi r dr h (T - T_\infty) \)

2) Specify the expression for the heat entering the differential control volume at \( r \) by conduction. \( -2\pi r k \frac{dT}{dr} \)

II. The nodal points of a two-dimensional system having a thermal conductivity of 2.0 W/m-°K are shown on the right. The ambient fluid is at 40°C with a heat transfer coefficient of 30 W/m²-°K. The isothermal surface is at 110°C. If \( T_1 = 60°C, T_2 = 70°C, \) and \( T_3 = 85°C \), calculate the heat transfer rate per unit thickness normal to the page from the right surface to the fluid.

720 W/m

III. Irreversible cell damage occurs in living tissue maintained at temperature greater than 48°C for a duration greater than 10 seconds. You can assume that living tissue has a normal temperature of 37°C, is isotropic, and has thermal diffusivity of \( 1.513 \times 10^{-7} \text{ m}^2/\text{s} \). Calculate the thickness of a layer of cell damage if part of the body is in contact with a surface at 80°C for a period of 30 s.

Data:

\[
\frac{T(x,t) - T_s}{T_i - T_s} = \text{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)
\]

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2.7954×10⁻³ m

IV. The \( x \)-coordinate is assigned in the direction along a cylindrical fin with \( x = 0 \) at the base or left surface where the temperature is 120°C. The end of the fin is insulated. Data: Length = 30 cm, diameter 1.5 cm, \( k = 240 \text{ W/m-K} \), \( h = 40 \text{ W/m}^2\text{-K} \), and the ambient temperature \( T_\infty = 25°C \). The temperature profile for the fin is: \( \theta = B_1\sinh(cx) + B_2\cosh(cx) \)

(5) Determine the numerical value of \( c \) 6.67 m⁻¹
If \( c = 5 \text{ m}^{-1} \), determine the numerical values of \( B_1 \) and \( B_2 \)

(6) \( B_1 = 95^\circ\text{C} \) \hspace{1cm} (7) \( B_2 = -85.99^\circ\text{C} \)

V. A masonry slab of width \( W = 0.05 \text{ m} \), which is at an initial temperature of \( 25^\circ\text{C} \), is heated by passing a hot gas through the two surfaces, with the gas temperature and the convection coefficient assumed to have constant values of \( T_\infty = 600^\circ\text{C} \) and \( h = 100 \text{ W/m}^2\text{K} \). Properties of masonry material are \( \rho = 1900 \text{ kg/m}^3 \), \( c = 800 \text{ J/kg}\cdot\text{K} \) and \( k = 0.625 \text{ W/m}\cdot\text{K} \).

Data:

Conduction in a slab

\( L \) is defined as the distance from the center of the slab to the surface. If one surface is insulated, \( L \) is defined as the total thickness of the slab.

\[
F_0 = \frac{at}{L^2} = \frac{a}{r_0^2} \times \theta^* = \frac{T_0 - T_\infty}{T_i - T_\infty} \times \theta_0^* = C_1 \exp(-\xi_1^2 F_0)
\]

\[
\theta^* = \theta_0^* \cos(\xi_1 x^*) ; \quad \frac{Q}{Q_0} = 1 - \frac{\sin(\xi_1)}{\xi_1} \theta_0^*
\]

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(8) How long will it take to achieve 60% of the maximum possible energy storage? \( 798 \text{ s} \)

(9) What is the average temperature of the masonry at this time? \( 255^\circ\text{C} \)

VI. A long column with thermal conductivity \( k = 2 \text{ W/m}\cdot\text{K} \) is maintained at \( 500^\circ\text{K} \) on three surfaces while the remaining surface is exposed to a convective environment with \( h = 10 \text{ W/m}^2\cdot\text{K} \) and fluid temperature \( T_\infty \). The cross sectional area of the column is 1 m by 1 m. Using a grid spacing \( \Delta x = \Delta y = 0.25 \text{ m} \), the nodal points for this system are given in the following figure:

10) The temperature at node 7 is given by the expression: \( T_7 = AT_5 + BT_8 + C \), where

\[
A(\text{numerical value}) = 0.30769
\]
Note: Your answers must be correct to 3 significant figures and have the appropriate units.

I. Consider a square rod \((W \times W)\) with length \(L\). The ambient air is at \(T_\infty\) with heat transfer coefficient \(h\). The base is at temperature \(T_b\) with \(x\) as the distance along the rod. Energy balance over a differential control volume is required to obtain the differential equation to solve for \(T(x)\).

1) Specify the expression for the heat leaving the differential control volume by convection.
\[4Wdxh(T - T_\infty)\]

2) Specify the expression for the heat entering the differential control volume at \(r\) by conduction
\[-W^2k \frac{dT}{dx}\]

II. A shell-and-tube heat exchanger with one shell pass and four tube passes uses hot water on the tube side to heat oil on the shell side. The water enters at 87°C and 50.0 kg/s and leaves at 27°C. Inlet and outlet temperatures of the oil are 7 and 37°C. The heat exchanger contains 300 tubes which have inner and outer diameters of 20 and 24 mm and a length of 5.0 m. Density of water is 995 kg/m³ and \(C_p\) of water is 4200 J/kg.K.

3) Water velocity is \(2.13 \text{ m/s}\)

4) If \(h_o = 400 \text{ W/m}^2\cdot\text{K}\), determine the shell side convective heat transfer resistance (for the heat exchanger, not for one tube) \(2.21 \times 10^{-5} \text{ K/W}\)

III. Irreversible cell damage occurs in living tissue maintained at temperature greater than 48°C for a duration greater than 10 seconds. You can assume that living tissue has a normal temperature of 37°C, is isotropic, and has thermal diffusivity of 1.513 \times 10^{-7} \text{ m}^2/\text{s}. Calculate the thickness of a layer of cell damage if part of the body is in contact with a surface at 80°C for a period of 20 s. \(1.98 \times 10^{-3} \text{ m}\)

Data:
\[
\frac{T(x,t) - T_s}{T_i - T_s} = \text{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)
\]

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</table>

IV. Saturated steam at 110°C condenses on the outside of a 5-m long, 4-cm-diameter thin horizontal copper tube by cooling liquid water that enters the tube at 25°C at an average velocity \(V\) of 2 m/s and leaves at 45°C. Liquid water density, \(\rho\), is 997 kg/m³, \(C_p\) of liquid water is 4.18 kJ/kg°C.
The data for internal energy \((u)\) and enthalpy \((h)\) of saturated liquid and saturated vapor are given:

<table>
<thead>
<tr>
<th>T (°C)</th>
<th>(u_l) (kJ/kg)</th>
<th>(u_g) (kJ/kg)</th>
<th>(h_l) (kJ/kg)</th>
<th>(h_g) (kJ/kg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>110</td>
<td>461.2</td>
<td>2518.1</td>
<td>461.3</td>
<td>2691.5</td>
</tr>
</tbody>
</table>

6) The rate of heat transfer to water is 209.5 kW

7) If the rate of heat transfer to water is 250 kW, the rate of condensation of steam is 0.1121 kg/s

V. A long column with thermal conductivity \(k = 4\) W/m\(^o\)K is maintained at 500 K on three surfaces while the remaining surface is exposed to a convective environment with \(h = 10\) W/m\(^2\)\(°\)K and fluid temperature \(T_\infty = 300\) K. The cross sectional area of the column is 1 m by 1 m. Using a grid spacing \(\Delta x = \Delta y = 0.25\) m, the nodal points for this system are given in the following figure:

![Diagram of nodal points](image)

The temperature at node 7 is given by the expression: \(T_7 = AT_5 + BT_8 + C\), where

8) \(A\) (numerical value) = 0.38095

9) \(B\) (numerical value) = 0.19048

10) \(C\) (numerical value with unit) = 166.67 K