Some Applications of the Line integral:

In physics, the line integrals are used, in particular, for computations of

- mass of a wire;
- center of mass and moments of inertia of a wire;
- work done by a force on an object moving in a vector field;
- magnetic field around a conductor (Ampere's Law);
- voltage generated in a loop (Faraday's Law of magnetic induction).

Consider these applications in more details.

1. **Mass of a Wire**:

Suppose that a piece of a wire is described by a curve $C$ in three dimensions. The mass per unit length of the wire is a continuous function $\rho(x, y, z)$. Then the total mass of the wire is expressed through the line integral of scalar function as

$$m = \int_C \rho(x, y, z) \, ds.$$

If $C$ is a curve parameterized by the vector function $\mathbf{r}(t) = (x(t), y(t), z(t))$, then the mass can be computed by the formula

$$m = \int_a^b \rho(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt.$$

If $C$ is a curve in the $xy$-plane, then the mass of the wire is given by

$$m = \int_C \rho(x, y) \, ds$$

or in parametric form

$$m = \int_a^b \rho(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt.$$

2. **Center of Mass and Moments of Inertia of a Wire**:

Let a wire is described by a curve $C$ with a continuous density function $\rho(x, y, z)$. Then coordinates of the center of mass of the wire are defined by the formulas

$$\bar{x} = \frac{M_{xy}}{m}, \quad \bar{y} = \frac{M_{xz}}{m}, \quad \bar{z} = \frac{M_{yz}}{m},$$

where

$$M_{xy} = \int_C x \rho(x, y, z) \, ds, \quad M_{xz} = \int_C y \rho(x, y, z) \, ds, \quad M_{yz} = \int_C z \rho(x, y, z) \, ds$$

are so-called first moments.
The moments of inertia about the x-axis, y-axis and z-axis are given by the formulas

\[ I_x = \int_C (y^2 + z^2) \rho(x, y, z) \, ds, \]
\[ I_y = \int_C (x^2 + z^2) \rho(x, y, z) \, ds, \]
\[ I_z = \int_C (x^2 + y^2) \rho(x, y, z) \, ds. \]

3. **Work**

Work done by a force \( \vec{F} \) on an object moving along a curve \( C \) is given by the line integral

\[ W = \int_C \vec{F} \cdot d\vec{r}, \]

where \( \vec{F} \) is the vector force field acting on the object, \( d\vec{r} \) is the unit tangent vector (Figure 1). The notation \( \vec{F} \cdot d\vec{r} \) means dot product of \( \vec{F} \) and \( d\vec{r} \).

Note that the force field \( \vec{F} \) is not necessarily the cause of moving the object. It might be some other force acting to overcome the force field that is actually moving the object. In this case the work of the force \( \vec{F} \) could result in a negative value.

If a vector field is defined in the coordinate form

\[ \vec{F} = (P(x, y, z), Q(x, y, z), R(x, y, z)), \]

then the work done by the force is calculated by the formula

\[ W = \int_C \vec{F} \cdot d\vec{r} = \int_C P \, dx + Q \, dy + R \, dz. \]

If the object is moved along a curve \( C \) in the \( xy \)-plane, then the following formula is valid:

\[ W = \int_C \vec{F} \cdot d\vec{r} = \int_C P \, dx + Q \, dy, \]

where \( \vec{F} = (P(x, y), Q(x, y)) \).

If a path \( C \) is specified by a parameter \( t \) (often means time), the formula for calculating work becomes

\[ W = \int_a^b \left[ P(x(t), y(t), z(t)) \frac{dx}{dt} + Q(x(t), y(t), z(t)) \frac{dy}{dt} + R(x(t), y(t), z(t)) \frac{dz}{dt} \right] \, dt, \]

where \( t \) goes from \( a \) to \( b \).

If a vector field \( \vec{F} \) is **conservative**, then the work on an object moving from \( A \) to \( B \) can be found by the formula

\[ W = u(B) - u(A), \]

where \( u(x, y, z) \) is a scalar potential of the field.
4. **Ampere’s Law**: 

The line integral of a magnetic field $\vec{B}$ around a closed path $C$ is equal to the total current flowing through the area bounded by the contour $C$ (Figure 2). This is expressed by the formula 

$$ \oint_C \vec{B} \cdot d\vec{r} = \mu_0 I, $$

where $\mu_0$ is the *vacuum permeability constant*, equal to $1.26 \times 10^{-6}$ H/m.

5. **Faraday’s Law**:  

The *electromotive force* $\varepsilon$ induced around a closed loop $C$ is equal to the rate of the change of magnetic flux $\psi$ passing through the loop (Figure 3).

$$ \varepsilon = \oint_C \vec{B} \cdot d\vec{r} = -\frac{d\psi}{dt}. $$

6. **Finding the area of a closed region $D$**: 

If we use The Green’s Theorem in reverse we see that the area of the region $D$ can also be computed by evaluating any of the following line integrals.

$$ A = \oint_C x \, dy = -\oint_C y \, dx = \frac{1}{2} \oint_C x \, dy - y \, dx $$