Physical Applications of Surface Integrals

Surface integrals are used in multiple areas of physics and engineering. In particular, they are used for calculations of

- mass of a shell;
- center of mass and moments of inertia of a shell;
- gravitational force and pressure force;
- fluid flow and mass flow across a surface;
- electric charge distributed over a surface;
- electric fields (Gauss' Law in electrostatics).

1. **Mass of a Surface**

Let $S$ be a smooth thin shell. The mass per unit area of the shell is described by a continuous function $\mu(x,y,z)$. Then the total mass of the shell is expressed through the surface integral of scalar function by the formula

$$ m = \iint_S \mu(x,y,z) \, dS. $$

2. **Center of Mass and Moments of Inertia of a Surface**

Let a mass $m$ is distributed over a thin shell $S$ with a continuous density function $\mu(x,y,z)$. The coordinates of the center of mass of the shell are defined by the formulas

$$ x_C = \frac{M_{yz}}{m}, \quad y_C = \frac{M_{xz}}{m}, \quad z_C = \frac{M_{xy}}{m}, $$

where

$$ M_{yz} = \iint_S x \mu(x,y,z) \, dS, \quad M_{xz} = \iint_S y \mu(x,y,z) \, dS, \quad M_{xy} = \iint_S z \mu(x,y,z) \, dS $$

are so-called the first moments about the coordinate planes $x = 0$, $y = 0$, and $z = 0$, respectively.

The moments of inertia about the $x$-axis, $y$-axis, and $z$-axis are given by

$$ I_x = \iint_S (y^2 + z^2) \mu(x,y,z) \, dS, \quad I_y = \iint_S (x^2 + z^2) \mu(x,y,z) \, dS, \quad I_z = \iint_S (x^2 + y^2) \mu(x,y,z) \, dS. $$

The moments of inertia of a shell about the $xy$-, $yz$-, and $xz$-plane are defined by the formulas

$$ I_{xy} = \iint_S z^2 \mu(x,y,z) \, dS, \quad I_{yz} = \iint_S x^2 \mu(x,y,z) \, dS, \quad I_{xz} = \iint_S y^2 \mu(x,y,z) \, dS. $$
2. **Gravitational Force**

Let $m$ be a mass at a point $(x_0, y_0, z_0)$ outside the surface $S$ (Figure 1).

Then the force of attraction between the surface $S$ and the mass $m$ is given by

$$\vec{F} = Gm \int_S \mu(x, y, z) \frac{\vec{r}}{r^2} dS,$$

where $\vec{r} = (x - x_0, y - y_0, z - z_0)$, $G$ is gravitational constant, $\mu(x, y, z)$ is the density function.

3. **Pressure Force**

Suppose a surface $S$ be given by the position vector $\vec{r}$ and is stressed by a pressure force acting on it. Examples of such surfaces are dams, aircraft wings, compressed gas storage tanks, etc. The total force $\vec{F}$ created by the pressure $p(\vec{r})$ is given by the surface integral

$$\vec{F} = \int_S p(\vec{r}) \, d\vec{S}.$$

By definition, the pressure is directed in the direction of the normal of $S$ in each point. Therefore, we can write:

$$\vec{F} = \iint_S p(\vec{r}) \, d\vec{S} = \iint_S p\vec{n} \, dS,$$

where $\vec{n}$ is the unit normal vector to the surface $S$. 
4. **Fluid Flux and Mass Flux**

If the vector field is the fluid velocity \( \vec{v}(\vec{r}) \), the flux across a surface \( S \) is called the fluid flux. It is equal to the volume of the fluid passing across \( S \) per unit time and given by

\[
\Phi = \oint_S \vec{v}(\vec{r}) \cdot d\vec{S}.
\]

Similarly, the flux of the vector field \( \vec{F} = \rho \vec{v} \), where \( \rho \) is the fluid density, is called the mass flux and given by

\[
\Phi = \oint_S \rho \vec{v} \cdot d\vec{S}.
\]

It is equal to the mass passing across a surface \( S \) per unit time.

5. **Surface Charge**

Let \( \sigma(x,y) \) be the surface charge density. The total amount of charge distributed over the conducting surface \( S \) is expressed by the formula

\[
Q = \iint_S \sigma(x,y) dS.
\]

6. **Gauss' Law**

The electric flux \( \vec{D} \) through any closed surface \( S \) is proportional to the charge \( Q \) enclosed by the surface:

\[
\Phi = \int_S \vec{D} \cdot d\vec{S} = \sum_i Q_i,
\]

where \( \vec{D} = \varepsilon \varepsilon_0 \vec{E} \) is the magnitude of the electric field strength, \( \varepsilon \) is permittivity of material, and \( \varepsilon_0 = 8.85 \times 10^{-12} \frac{F}{m} \) is permittivity of free space.

For the discrete case the total charge \( Q \) is the sum over all the enclosed charges.

Gauss' Law is a general law applying to any closed surface. For geometries of sufficient symmetry, it simplifies the calculation of electric field. Gauss' Law is the first of Maxwell's equations, the four fundamental equations for electricity and magnetism.