A Notorious Problem in Silverman’s
A Friendly Introduction to Number Theory

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An introductory undergraduate text designed to entice non-math majors into learning some mathematics, while at the same time teaching them how to think mathematically.

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Chapter 1: What is Number Theory?

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Figure 1.1: Numbers That Form Interesting Shapes
The Notorious Problem

Problem

The first two numbers that are both squares and triangles are 1 and 36. Find the next one and, if possible, the one after that. Can you figure out an efficient way to find triangular-square numbers? Do you think that there are infinitely many?
Standard student strategies

- Google it!
Standard student strategies

▶ Google it!
▶ List them!
Standard student strategies

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- List them!

1, 4, 9, 16, 25, **36**, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400, 441, 484, 529, 576, 625, 676, 729, 784, 841, 900, 961, 1024, 1089, 1156, **1225**, 1296, 1369, 1444, 1521, 1600, 1681, 1764, 1849, 1936, 2025, 2116, 2209, 2304, 2401, 2500, 2601, 2704, 2809, 2916, 3025, 3136, 3249, 3364, 3481, 3600, 3721, ...

1, 3, 6, 10, 15, 21, 28, **36**, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210, 231, 253, 276, 300, 325, 351, 378, 406, 435, 465, 496, 528, 561, 595, 630, 666, 703, 741, 780, 820, 861, 903, 946, 990, 1035, 1081, 1128, 1176, **1225**, 1275, 1326, 1378, 1431, 1485, 1540, 1596, 1653, 1711, 1770, 1830, 1891, 1953, 2016, 2080, 2145, 2211, 2278, 2346, 2415, 2485, 2556, 2628, 2701, 2775, 2850, 2926, 3003, 3081, 3160, 3240, 3321, 3403, 3486, 3570, 3655, 3741, ...
A noble attempt...

\[ n^2 = \frac{n(n + 1)}{2} \]

Solve for \( n \).
... and a better one

\[ m^2 = \frac{n(n + 1)}{2} \]
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This line of reasoning naturally leads to the idea of Pell’s equation, an advanced topic in the book.
Available tools

Besides the definitions of square and triangular numbers, what is there to work with?
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Pictures! In fact, the following figure motivates the formula for $T_n$, the $n$th triangular number:

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(1 + 2 + 3 + 4 + 5 + 6) + 7 + (6 + 5 + 4 + 3 + 2 + 1) = 7^2

Figure 1.2: The Sum of the First $n$ Integers
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Figure 1.2: The Sum of the First $n$ Integers

We can also use the picture to see the relationship

$$S_n = T_n + T_{n-1}.$$
Experience in geometric manipulations leads to the idea of overlapping equal squares and triangles:

\[ S_6 = T_8 \iff 2T_2 = T_3 \]
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So whenever a triangle equals a square, we also have \( 2T_m = T_n \) for some \( m \) and \( n \).
Let’s run with this...

6, 10, 28, 36, 66, 78, 120, 136, 190, 210, 276, 300, 378, 406, 496, 528, 630, 666, 780, 820, 946, 990, 1128, 1176, 1326, 1378, 1540, 1596, 1770, 1830, 2016, 2080, 2278, 2346, 2556, 2628, 2850, 2926, 3160, 3240, 3486, 3570, 3828, 3916, 4186, 4278, 4560, 4656, 4950, 5050, 5356, 5460, 5778, 5886, 6216, 6328, 6670, 6786, 7140, 7260, 7626, 7750, 8128, 8256, 8646, 8778, 9180, 9316, 9730, 9870, 10296, ...

2, 6, 12, 20, 30, 42, 56, 72, 90, 110, 132, 156, 182, 210, 240, 272, 306, 342, 380, 420, 462, 506, 552, 600, 650, 702, 756, 812, 870, 930, 992, 1056, 1122, 1190, 1260, 1332, 1406, 1482, 1560, 1640, 1722, 1806, 1892, 1980, 2070, 2162, 2256, 2352, 2450, 2550, 2652, 2756, 2862, 2970, 3080, 3192, 3306, 3422, 3540, 3660, 3782, 3906, 4032, 4160, 4290, 4422, 4556, 4692, 4830, 4970, 5112, 5256, 5402, 5550, 5700, 5852, 6006, 6162, 6320, 6480, 6642, 6806, 6972, 7140, 7310, 7482, 7656, 7832, 8010, 8190, 8372, 8556, 8742, 8930, 9120, 9312, 9506, 9702, 9900, 10100, ...
7140 = 2T_m = T_n

where $m = 84$, $n = 119$. This means $204^2 = 41616$ is a triangular-square number.
A related idea

\[ T_8 = S_6 = T_6 + T_5 \]
A related idea
Note that the rectangle is $n \times (n + 1)$. 

A related idea
A related idea

\[ 2T_n = T_m \]
A related idea

\[ S_m = T_n \Rightarrow S_{m+k+1} = 3m+k+2, \]
\[ n_k+1 = 4m+k+3, \]
\[ m_1 = 1, \quad n_1 = 1, \quad m_2 = 6, \quad n_2 = 8, \quad m_3 = 35, \quad n_3 = 49, \]
\[ m_4 = 204, \quad n_4 = 288, \quad m_5 = 1189, \quad n_5 = 1681 \]

(1189^2 = 1413721)
A related idea

\[ S_{m_1} = T_{n_1} \Rightarrow S_{m_2} = T_{n_2} \]
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\[ m_{k+1} = 3m_k + 2n_k + 1 \]
\[ n_{k+1} = 4m_k + 3n_k + 1 \]
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\((1189^2 = 1413721)\)
Remember Pell’s equation?

\[ a^2 - 2b^2 = 1, \quad S_{ab} = T_{2b^2} \]
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\[ a^2 - 2b^2 = 1, \quad S_{ab} = T_{2b^2} \]

\[ a^2 - 2b^2 = -1, \quad S_{ab} = T_{a^2} \]