Dynamical Forms for Rigid Body Analysis

In this section, the general dynamical equations which apply to particle systems shall be specialized to apply to rigid bodies. To begin, note that Newton’s 2nd law as expressed through the alternative relations

\[ \sum_{\text{EXT}} \mathbf{F} = \sum_{P \in B} m_P \mathbf{a}_P = \mathbf{M} \mathbf{a}_G + \frac{d}{dt} \mathbf{G} \]  

is already in an acceptable form. For rigid body analysis, the second form expressed in terms of the acceleration of the bodies center of mass \( \mathbf{G} \) is typically preferred. Integration of the third form leads directly to the Impulse-Momentum Principle

\[ \sum_{\text{EXT}} (\mathbf{I}_{\text{imp}}) = \Delta \mathbf{G} ; \quad \mathbf{G} = \mathbf{M} \mathbf{v}_G \]  

\[ \mathbf{I}_{\text{imp}} = \int_{t_i}^{t_f} \mathbf{F} dt = \mathbf{F}_{\text{avg}} \Delta t \sim \text{Linear Impulse of a force over time} \]  

Consider now the angular momentum of a rigid body with respect to some fixed or moving moment point \( \mathbf{O} \). Beginning with the fundamental defining expression

\[ \mathbf{H}_O = \sum_{P \in B} (\mathbf{r}_{P\mathbf{O}} \times \mathbf{G}_P) = \sum_{P \in B} m_P (\mathbf{r}_{P\mathbf{O}} \times \mathbf{v}_P), \]  

rigid body specialization can be realized through the use of the so-called rigid body velocity equation

\[ \mathbf{\ddot{v}}_{P\mathbf{O}} = \mathbf{\ddot{v}}_P - \mathbf{\omega} \times \mathbf{\dot{r}}_{P\mathbf{O}} \Rightarrow \mathbf{\ddot{v}}_P = \mathbf{\ddot{v}}_O + \mathbf{\omega} \times \mathbf{\dot{r}}_{P\mathbf{O}} \]  

expressed in terms of the body point\(^1\) \( \mathbf{O}' \) which is instantaneously located at, and perhaps passing through, the selected moment point \( \mathbf{O} \).

\[ ^1 \text{This so-called body point can either be a point of the body itself or a point belonging to a real or imaginary rigid extension of the body. In either case, } \mathbf{O}' \text{ is a point which is } \text{now located at the moment point } \mathbf{O}, \text{ but moving with the body } \mathbf{B}. \]
With this substitution for each of the individual body point velocities, the specialized form

\[
\mathbf{H}_O = \sum_{P \in B} m_P (\mathbf{r}_{PO} \times \mathbf{v}_P) \\
= \sum_{P \in B} m_P [\mathbf{r}_{PO} \times (\mathbf{v}_O + \mathbf{\omega} \times \mathbf{r}_{PO})] \\
= \sum_{P \in B} m_P [\mathbf{r}_{PO} \times (\mathbf{\omega} \times \mathbf{r}_{PO})] + \sum_{P \in B} m_P (\mathbf{r}_{PO} \times \mathbf{v}_O) \\
= \sum_{P \in B} m_P [\mathbf{r}_{PO} \times (\mathbf{\omega} \times \mathbf{r}_{PO})] + \left[ \sum_{P \in B} (m_P \mathbf{r}_{PO}) \times \mathbf{v}_O \right] \\
= \sum_{P \in B} m_P [\mathbf{r}_{PO} \times (\mathbf{\omega} \times \mathbf{r}_{PO})] + (M \mathbf{r}_{GO}) \times \mathbf{v}_O \\
\mathbf{H}_O = \sum_{P \in B} m_P (\mathbf{r}_{PO} \times \mathbf{\omega} \times \mathbf{r}_{PO}) + M (\mathbf{r}_{GO} \times \mathbf{v}_O)
\]

is readily obtained. Now, the first term involving the body sum appears to be quite complex and it is by no means apparent as to how (or even if) it can be simplified. As we shall soon discover, terms of this exact form appear in a number of places in the equations governing rigid body motion. As a consequence, it will prove convenient to formally recognize its importance by defining a vector-valued function having its exact mathematical form. This function, the so-called Mass Moment of Inertia Function, is defined as

\[
\mathbf{I}_O(\mathbf{u}) = \sum_{P \in B} m_P (\mathbf{r}_{PO} \times (\mathbf{u} \times \mathbf{r}_{PO})) .
\]

In view of this definition it is clear that it maps (converts) a given argument vector \( \mathbf{u} \) into an image vector \( \mathbf{w} = \mathbf{I}_O(\mathbf{u}) \). Thus, we say that it is a vector-valued function of a single vector argument. Another key observation is that this function requires the identification and precise location of a specific rigid material body \( B \) and a moment point \( O \) in order for it to have any interpretable meaning. Put differently, this function is changed whenever we select a different material body \( B \), move it, or

\footnote{In step three, observe the use of the distributive property of the vector cross product to “factor out” the body point velocity at \( O' \) which is common to each term in the sum. This technique is used repeatedly throughout this section. Also, in step four, explicit use is made of the fact that, at this instant, the position of \( P \) relative to \( O \) is the same as its position relative to \( O' \).}
select a different moment point $O$. It is also critically important to note that this function is a *linear function* in the sense that

$$L_o(a_1 \vec{u}_1 + a_2 \vec{u}_2) = a_1[L_o(\vec{u}_1)] + a_2[L_o(\vec{u}_2)]$$

for any pair of vectors and associated scalar multipliers. This is an immediate consequence (proof left as an exercise for the reader) of the linearity of the vector cross product itself. For completeness, one should also make note of this functions alternative forms

$$L_o(\vec{u}) = \sum_{P \in B} m_P \vec{r}_{P/O} \times (\vec{u} \times \vec{r}_{P/O}) = \sum_{P \in B} m_P [\vec{r}_{P/O}^2 \vec{u} - (\vec{r}_{P/O} \cdot \vec{u}) \vec{r}_{P/O}], \quad (X.8)$$

the second of which follows from the ubiquitous double-cross product identity.

Of course, it is essential that we eventually learn how to *evaluate* this function but, for now, we will be content to recognize and identify this specific functional form whenever and wherever it appears. Following this mandate, we thus observe that the above expression (X.5) for angular momentum may now be recast in the more efficient form

$$\vec{H}_o = L_o(\vec{\omega}) + \vec{M} \times \vec{v}_o.$$  \hfill (X.9)

Two special cases of this angular momentum expression are noteworthy. Clearly, if the moment point $O$ is selected as the bodies mass center $G$ ($\vec{r}_{G/O} = \vec{r}_{G/G} = \vec{0}$), or if it is selected so as to coincide with the *instantaneous velocity center* (IVC) of the body ($\vec{v}_o = \vec{v}_{IVC} = \vec{0}$), then the second term in this equation will vanish, leaving the special forms:

$$\vec{H}_G = L_G(\vec{\omega}), \quad \vec{H}_{IVC} = L_{IVC}(\vec{\omega}). \quad (X.10)$$

In view of the general particle systems relations

$$\vec{H}_o = \vec{H}_G + \vec{r}_{G/O} \times \vec{G} : \quad \vec{G} = \vec{M} \vec{v}_G$$ \hfill (X.11)

it would then also follow that

$$\vec{H}_o = (\vec{H}_G) + \vec{r}_{G/O} \times \{\vec{G}\} = (L_G(\vec{\omega})) + \vec{r}_{G/O} \times (\vec{M} \vec{v}_G)$$

$$\vec{H}_o = L_G(\vec{\omega}) + \vec{M}(\vec{r}_{G/O} \times \vec{v}_G). \quad (X.12)$$

leading to the compilation of alternative, but completely equivalent, forms.
\[ \mathbf{H}_O = \begin{cases} \mathbf{H}_G + \mathbf{r}_{G/O} \times \mathbf{G} & \text{(generally true)} \\ \int_O (\mathbf{\omega}) + M(\mathbf{r}_{G/O} \times \mathbf{v}_O) \\ \int_G (\mathbf{\omega}) + M(\mathbf{r}_{G/O} \times \mathbf{v}_G) \end{cases} \]  

These relations are most useful for computing the angular momentum change appearing on the right-hand side of the so-called *Angular Impulse-Momentum Principle*:

\[ \sum_{\text{EXT}} (\Delta \mathbf{Imp}_O) = \Delta \mathbf{H}_O \]

\[ \Delta \mathbf{Imp}_O = \int_{t_i}^{t_f} \mathbf{M}_O \cdot dt = [\mathbf{M}_O]_{\text{avg}} \Delta t \sim 4 \text{ Impulse of a force over time} \]

\[ \text{; provided that } \mathbf{v}_O \times \mathbf{v}_G = \mathbf{0}; \text{ for each } t \text{ such that } t_i \leq t \leq t_f \]  

derived by direct integration of the previously derived general moment form

\[ \sum_{\text{EXT}} (\mathbf{M}_O) = \dot{\mathbf{H}}_O + \mathbf{v}_O \times \mathbf{G}; \mathbf{G} = M \mathbf{v}_G. \]  

The second and third of the above expressions for the angular momentum about \( O \), together with (X.4), can now be used to establish the following interesting relationship involving the mass moment function itself:

\[ \begin{align*} 
\mathbf{H}_O &= \int_O (\mathbf{\omega}) + M(\mathbf{r}_{G/O} \times \mathbf{v}_O) = \int_G (\mathbf{\omega}) + M(\mathbf{r}_{G/O} \times \mathbf{v}_G) \\
\int_O (\mathbf{\omega}) - \int_G (\mathbf{\omega}) &= M(\mathbf{r}_{G/O} \times \mathbf{v}_G) - M(\mathbf{r}_{G/O} \times \mathbf{v}_O) \\
\int_O (\mathbf{\omega}) - \int_G (\mathbf{\omega}) &= M[\mathbf{r}_{G/O} \times (\mathbf{v}_G - \mathbf{v}_O)] \\
\int_O (\mathbf{\omega}) - \int_G (\mathbf{\omega}) &= M[\mathbf{r}_{G/O} \times (\mathbf{v}_G \times \mathbf{r}_{G/O})]. 
\end{align*} \]  

As has already been observed, the mass moment function *changes* whenever the moment point is changed. The above expression specifies the *difference* between the respective mass moment functions \( \int_O \) and \( \int_G \) when they operate on (map) a specified angular velocity vector \( \mathbf{\omega} \). Since this expression would necessarily hold for any possible value of the bodies angular velocity vector, it is appropriate to infer the more general relationship

\[ \int_O (\mathbf{u}) - \int_G (\mathbf{u}) = M[\mathbf{r}_{G/O} \times (\mathbf{u} \times \mathbf{r}_{G/O})] \]

for any argument vector \( \mathbf{u} \). This important relation will eventually be recognized as corresponding to the (perhaps familiar) *Parallel Axis Theorem*. Also note its alternative forms

\[ \begin{cases} 
M[\mathbf{r}_{G/O} \times (\mathbf{u} \times \mathbf{r}_{G/O})] \\
M[\mathbf{r}_{G/O}^2 \mathbf{u} - (\mathbf{r}_{G/O} \times \mathbf{u}) \mathbf{r}_{G/O}] 
\end{cases} \]  

obtained via application of the double cross product identity.

Next, we desire to specialize the general moment equation of dynamics for use in rigid body analysis. For this, we recall the previously derived general moment equation
and make use of the *rigid body acceleration* equation

\[
\ddot{\mathbf{a}}_{\text{PO}} = \ddot{\mathbf{a}}_p - \ddot{\mathbf{a}}_O = \ddot{\mathbf{a}} \times \dot{\mathbf{r}}_{\text{PO}} + \dot{\omega} \times (\ddot{\omega} \times \dot{\mathbf{r}}_{\text{PO}})
\]

\[
\Rightarrow \ddot{\mathbf{a}}_p = \ddot{\mathbf{a}}_O + \ddot{\mathbf{a}} \times \dot{\mathbf{r}}_{\text{PO}} + \dot{\omega} \times (\ddot{\omega} \times \dot{\mathbf{r}}_{\text{PO}}) \quad (\ddot{\mathbf{r}}_{\text{PO}} = \dot{\mathbf{r}}_{\text{PO}})
\]

This moment equation expands into the following form:

\[
\sum_{\text{EXT}} (\mathbf{M}_O) = \sum_{p} m_p \{ \dot{\mathbf{r}}_{\text{PO}} \times [\ddot{\mathbf{a}}_O + \ddot{\mathbf{a}} \times \dot{\mathbf{r}}_{\text{PO}} + \dot{\omega} \times (\ddot{\omega} \times \dot{\mathbf{r}}_{\text{PO}})] \}
\]

\[
= \sum_{p} m_p \{ \dot{\mathbf{r}}_{\text{PO}} \times (\ddot{\mathbf{a}} \times \dot{\mathbf{r}}_{\text{PO}}) \} + \sum_{p} m_p \dot{\mathbf{r}}_{\text{PO}} \times (\ddot{\omega} \times (\ddot{\omega} \times \dot{\mathbf{r}}_{\text{PO}})) + \sum_{p} m_p \dot{\mathbf{r}}_{\text{PO}} \times \ddot{\mathbf{a}}_O
\]

\[
= \sum_{\text{EXT}} (\ddot{\mathbf{a}}_O) + \dot{\omega} \times \{ \sum_{p} m_p \{ \dot{\mathbf{r}}_{\text{PO}} \times (\ddot{\omega} \times \dot{\mathbf{r}}_{\text{PO}}) \} \} + (\dot{\mathbf{r}}_{\text{GO}} \times \ddot{\mathbf{a}}_O)
\]

\[
\sum_{\text{EXT}} (\mathbf{M}_O) = \sum_{\text{EXT}} (\ddot{\mathbf{a}}_O) + \dot{\omega} \times \sum_{\text{EXT}} (\dot{\omega}) + \dot{\mathbf{r}}_{\text{GO}} \times \dot{\mathbf{G}}
\]

As with the angular momentum equation (X.9), observe the specific result

\[
\sum_{\text{EXT}} (\mathbf{M}_G) = \sum_{\text{EXT}} (\ddot{\mathbf{a}}_O) + \dot{\omega} \times \sum_{\text{EXT}} (\dot{\omega})
\]

for the choice of moment point as the bodies own center of mass G. In view of this, and the general particle systems relations

\[
\sum_{\text{EXT}} (\mathbf{M}_O) = \sum_{\text{EXT}} (\ddot{\mathbf{a}}_O) + \dot{\omega} \times \sum_{\text{EXT}} (\dot{\omega})
\]

\[
\sum_{\text{EXT}} (\mathbf{M}_O) = \sum_{\text{EXT}} (\ddot{\mathbf{a}}_O) + \dot{\omega} \times \sum_{\text{EXT}} (\dot{\omega}) + \dot{\mathbf{r}}_{\text{GO}} \times \ddot{\mathbf{a}}_G
\]

Collecting results, the general moment equation of rigid body dynamics is expressed in the alternative but equivalent forms:

\[
\sum_{\text{EXT}} (\mathbf{M}_O) = \sum_{\text{EXT}} (\ddot{\mathbf{a}}_O) + \dot{\omega} \times \sum_{\text{EXT}} (\dot{\omega}) + \dot{\mathbf{r}}_{\text{GO}} \times \ddot{\mathbf{a}}_G
\]

Note that, once again, we make use of the body point O’ which is instantaneously located at the moment point O.
\[ \sum_{\text{ext}} \mathbf{M}_o = \begin{cases} \dot{\mathbf{H}}_o + \mathbf{v}_o \times \mathbf{G} & \text{(generally true)} \\ \dot{\mathbf{H}}_G + \dot{\mathbf{r}}_{GO} \times \mathbf{G} & \text{(generally true)} \\ \int_O (\alpha) + \omega \times \int_O (\alpha) + M \dot{r}_{GO} \times \ddot{a}_o \\ \int_G (\alpha) + \omega \times \int_G (\alpha) + M \dot{r}_{GO} \times \ddot{a}_G \end{cases} . \]  

(X.26)

The last two, in which specific reference to the angular momentum has been eliminated, are commonly referred to as Euler’s equations.

It is important to realize that the simplified moment equation

\[ \sum_{\text{ext}} \mathbf{M}_o = \int_O (\alpha) + \omega \times \int_O (\alpha) \]  

(X.27)

is appropriate only for special choices of the moment point O. As shown in (X.23) above, it is appropriate for the choice O=G. It is also apparent that if the body rotates about a fixed anchor point \(^4\)A, then it will also hold for the choice O=A. Another interesting case occurs for the choice O=C, where C is the point of contact of an axially symmetric wheel as it rolls along the ground. In this case it can be shown that the body point \(O'\) at C accelerates directly toward the center of mass G, thereby causing the “correction term” in the general moment equation to vanish.

Finally, it is necessary to develop an equation with which to compute the kinetic energy of a rigid body. First, consider the case where the body has a known point having zero velocity, an IVC. Using the rigid body velocity equation (X.4), and the cyclic permutation identity involving the triple scalar product, it follows that

\[ \mathbf{v}_{\text{IVC}} = \mathbf{v}_P - \mathbf{v}_{\text{IVC}} = \omega \times \mathbf{r}_{\text{IVC}} \Rightarrow \mathbf{v}_P = \mathbf{v}_{\text{IVC}} + \omega \times \mathbf{r}_{\text{IVC}} = \omega \times \mathbf{r}_{\text{IVC}} \]

\[ v_P^2 = \mathbf{v}_P \cdot \mathbf{v}_P = \omega \times \mathbf{r}_{\text{IVC}} \times \omega \times \mathbf{r}_{\text{IVC}} . \]  

(X.28)

As a consequence, the kinetic energy of the rigid body could be computed from the following compact (but specialized) equation:

\[ T = \sum_{P \in B} \frac{1}{2} m_P v_P^2 \]

\[ = \sum_{P \in B} \frac{1}{2} m_P [\omega \times (\mathbf{r}_{\text{IVC}} \times [\omega \times \mathbf{r}_{\text{IVC}}])] \]

\[ = \frac{1}{2} \omega \times \left( \sum_{P \in B} m_P [\mathbf{r}_{\text{IVC}} \times (\omega \times \mathbf{r}_{\text{IVC}})] \right) \]

\[ T = \frac{1}{2} \omega \times \int_{\text{IVC}} (\omega) . \]  

(X.29)

Unfortunately, this expression would be of little use in a circumstance where the body either has no IVC, or its position is simply not known. As an alternative, recall the general particle systems result that

\[ T = T_G = \frac{1}{2} M v_G^2 + T_{\not\exists \mathbf{G}} \]  

(X.30)

\(^4\) Such as through a pin or socket connection to the ground.
in which \( T_{\mathfrak{g} G} \) represents the kinetic energy of the body as measured (perceived) by observers in a translating frame \( \mathfrak{g} G \), which follows the body's mass center \( G \). For such observers, the body's own mass center \( G \) would appear to be its IVC, while its angular velocity would be measured exactly the same as from the ground\(^5\). As a consequence, observers in this “follower” frame could appropriately make use of (X.29) to compute the apparent kinetic energy as

\[
T_{\mathfrak{g} G} = \frac{1}{2} \ddot{\omega} \cdot J_G (\ddot{\omega}) .
\]  

(X.31)

By (X.30), it would then follow that the body's actual kinetic energy is given by

\[
T = T_{\mathfrak{g} G} = \frac{1}{2} m v^2_G + \frac{1}{2} \ddot{\omega} \cdot \dot{J}_G (\ddot{\omega}) .
\]  

(X.32)

In view of equations (X.10, 11), this can then be recast in the rather satisfying form

\[
T = \frac{1}{2} M v^2_G + \frac{1}{2} \ddot{\omega} \cdot \dot{J}_G (\ddot{\omega})
\]

\[
= \frac{1}{2} [\ddot{v}_G \cdot \ddot{G}_G] + \frac{1}{2} \ddot{\omega} \cdot \dot{J}_G (\ddot{\omega})
\]

\[
= \frac{1}{2} \left[ \ddot{v}_G \cdot (M \ddot{v}_G) + \ddot{\omega} \cdot \dot{J}_G (\ddot{\omega}) \right]
\]

\[
= \frac{1}{2} \ddot{v}_G \cdot \ddot{G}_G + \ddot{\omega} \cdot \ddot{H}_G .
\]  

(X.33)

expressed in terms of the body's linear and angular momentum.

After collecting results, we now have the following alternative forms for determining the kinetic energy of a rigid body, namely

\[
T = \begin{cases} 
\frac{1}{2} \ddot{v}_G \cdot \ddot{G}_G + \ddot{\omega} \cdot \ddot{H}_G \\
\frac{1}{2} M v^2_G + \frac{1}{2} \ddot{\omega} \cdot \dot{J}_G (\ddot{\omega}) \\
\frac{1}{2} \ddot{\omega} \cdot \dot{J}_{\text{IVC}} (\ddot{\omega}) = \frac{1}{2} \ddot{\omega} \cdot \ddot{H}_{\text{IVC}}
\end{cases}
\]

(X.34)

These formulae are most useful when applying the Work-Energy Principle to an ideal system containing one or more rigid bodies.

\[
\sum_{\text{EXT}} (U) + \sum_{\text{INT}} (U) = \Delta T
\]

\[
\sum_{\text{INT}} (U) = \emptyset \quad \text{if and only if system can be regarded as IDEAL}
\]

(X.35)

This entire set of (boxed) equations constitutes the complete set of dynamical relations for the analysis of rigid body motion. Carefully observe the prominent role played by the Inertia Function which we must now, quite obviously, learn to evaluate.

\(^5\ \ddot{\omega}_g = \ddot{\omega}_{\mathfrak{g} G} + \ddot{\omega}_{\mathfrak{g} G}; \quad [\ddot{\omega}_{\mathfrak{g} G} = \emptyset] \Rightarrow \ddot{\omega}_{\mathfrak{g} G} = \ddot{\omega}_g = \ddot{\omega} .\)