Notes on Introductory Mechanics

These notes are meant to help students understand the basic "physics" behind introductory "Newtonian" mechanics. The ideas are presented in the order in which they are taught in my first year class, and are designed to supplement the text.

The approach we take is somewhat historical. In this section we first study the conditions for static equilibrium for a rigid object: how forces combine such that a rigid object does not move or rotate. Then we will consider the basic principles of "free" motion and properties of the interactions of two-body systems. The branch of mathematics that is relevant for these topics is Vectors. We will try and understand the laws of mechanics as best we can without using calculus. Then in the next section, we will see that in order to keep the laws of motion in a simple form, we will need to use calculus. Our approach will be to first perform an experiment, and then determine what "law of physics" the experiment is demonstrating.

Mechanics before 1500

In early times the balancing of forces was important for building, and was the first "physics" to be investigated. Later on in the quarter we will analyze mechanical systems precisely. We will discuss the concepts of inertial mass, force and motion, ideas that were discovered in the 1600's. For now, we will consider a simple set of experiments with forces.

Simply speaking, a force is a push (or pull). Let’s consider forces due to an objects weight, i.e. the gravitational force due to the attraction of the earth. For this discussion, we limit ourselves to objects that are made of pure elements, i.e. copper, iron, etc. We will make the assumption that for a pure homogeneous substance the weight is proportional to the volume of the object. That is, if object A is twice as large object B, and both are made of the same substance, then object A has twice the weight as object B. If one object has a weight of 10 units, then an object of twice (or x times) the volume will have a weight of 20 units (or 10x units). The weights in our weight set are so labeled.

Combining forces in the same direction

In our initial experiments we will combine forces on a ring by using pulleys and strings. A string attached to the ring will pass over a pulley. The other end of the string will be attached to a weight. Suppose two weights, each of magnitude 10 units, pull on the ring to the right (the + direction). How much weight and where should it be placed to balance the two weights of 10 units. Your guess is probably one weight of magnitude 20 units pulling to the left. We will check this result in class. (Your guess
was correct). This means that two forces pulling in the same direction add like real numbers. Nothing profound here, with our assumption of weight and homogeneous material, we are just showing that volumes add. Similarly, we will show that two forces pulling in the opposite direction subtract. Thus for the case of one dimension, we can assign a (+ or -) to a force to signify the direction (right or left).

**Combining forces in different directions**

A forces can act in any direction. What role does the direction of a force play when combining forces? Consider the following experiment:

*A force of 40 units pulls on the ring to the east and a force of 30 units pulls on the ring to the north. What force on the ring will balance these two forces?*

After doing the experiment in class, we will find that the balancing force is a force that pulls on the ring with an amount of 50 units at a direction of $36.9^\circ$ S of W. Thus the force of 40 units to the east plus the force of 30 units to the north is equivalent to a single force of magnitude 50 units at an angle of $36.869^\circ$ N of E. In this case 30 plus 40 equals 50. You probably remember from trig that lengths of 30, 40, and 50 form a right triangle (3-4-5 triangle). Thus, if we represent each force by an arrow, whose length is proportional to its magnitude and whose direction is in the direction of the force, the two forces combine by placing the tail of one at the tip of the other. The resultant force is represented by the arrow along the hypotenuse.

In lab, you will carry out a number of experiments that demonstrate that any two forces combine using the "arrow" method described above. The mathematics that best describes how forces combine is the mathematics of **vector addition**. Since a number of physical quantities have the properties of mathematical vectors, we will spend some time in lecture discussing vectors.

Vectors are discussed in many texts and usually are defined (initially) as something with direction and magnitude. Quantities with direction and magnitude are not necessarily vectors. They must also have certain mathematical properties. An addition operation must be defined, and the sum of two vectors must also be a vector. Also, if $\vec{A}$ and $\vec{B}$ represent any two vectors, then it must be true that $\vec{A} + \vec{B} = \vec{B} + \vec{A}$. Vector addition is commutative. Vectors are also defined over a field, which in our class will be the real numbers. To be an "inner-product" vector space, a scalar product between two vectors must be defined with certain properties. We will talk about this later when we discuss energy.
The experimental result pertaining to adding "static" forces is summarized by the following experiment:

**Experiment:** If two forces, \( \vec{F}_1 \) and \( \vec{F}_2 \), act on an object that doesn’t move, the resulting force is the same as if the object were subject to the force \( \vec{F} \) where \( \vec{F} \) is the **vector sum** of the two forces: \( \vec{F} = \vec{F}_1 + \vec{F}_2 \). The same vector addition applies if more than two forces are acting on the object.

This is a wonderful experimental result! Forces didn’t have to "add" in such a simple way, but they do. It demonstrates that certain aspects of nature can be understood using geometry (trig). These ideas were developed over 3000 years ago, and helped in the building of magnificent structures. We will do many examples in class demonstrating this remarkable property of forces, but first we review the mathematical properties of vector addition.

**Multiplication of a vector by a scalar**

Multiplication of a vector by a scalar \( c \) changes the magnitude of the vector by the factor \(|c|\). If \( c > 0 \), then the direction remains the same, but if \( c < 0 \) the direction is reversed. Multiplication by a scalar enables one to define unit vectors, whose linear combinations span the vector space.

**Unit Vectors**

We can define "unit vectors" that have a magnitude of one unit and point along the cartesian axis. The notation that is usually used in physics texts is the following:

\( \hat{i} \) is defined as a vector of length one that points along the positive x-axis.

\( \hat{j} \) is defined as a vector of length one that points along the positive y-axis.

\( \hat{k} \) is defined as a vector of length one that points along the positive z-axis.

From the property of scalar multiplication and vector addition, any vector can be expressed in terms of these unit vectors. For example, if the arrow for the vector starts at the origin and ends at the coordinates \((A_x, A_y, A_z)\), then \( \vec{A} \) can be written as:

\[
\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}
\] (1)
since $A_x$, $A_y$, and $A_z$ are scalars. Note that $\hat{i}$, $\hat{j}$, and $\hat{k}$ are the only vectors on the right side of the equation. In terms of the coordinates, $|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$.

**Vector Addition**

We start with an example of vector addition. Let vector $\vec{A}$ have a magnitude of 150 units at an angle of 30 degrees N of E, vector $\vec{B}$ have a magnitude of 100 units at an angle of 45 degrees N of W, and vector $\vec{C}$ have a magnitude of 200 units at an angle of 80 degrees S of E. Find the vector $\vec{D} = \vec{A} + \vec{B} + \vec{C}$, which is the sum of the three vectors. The easiest way to add these vectors is to use unit vectors (i.e. components). If $\hat{i}$ points along the x-axis with magnitude 1, and $\hat{j}$ points along the y-axis with magnitude 1, then

$$\vec{A} = 150\cos(30)\hat{i} + 150\sin(30)\hat{j} = 130\hat{i} + 75\hat{j}$$

Similarly,

$$\vec{B} = -100\cos(45)\hat{i} + 100\sin(45)\hat{j} = -70.7\hat{i} + 70.7\hat{j}$$

and

$$\vec{C} = 200\cos(80)\hat{i} - 200\sin(80)\hat{j} = 34.7\hat{i} - 197\hat{j}$$

To add the three vectors, one simple adds the components:

$$\vec{D} = (130 - 70.7 + 34.7)\hat{i} + (75 + 70.7 - 197)\hat{j} = 94\hat{i} - 51.3\hat{j}$$

The sum can be expressed in terms of the unit vectors, or as a magnitude and direction: $|\vec{D}| = \sqrt{94^2 + 51.3^2} = 107$ units. The vector points in the fourth quadrant, with the angle $\tan^{-1}\theta = 51.3/94$, or $\theta = 28.6^\circ$. Since $\vec{D}$ is in the fourth quadrant, $\theta = 28.6^\circ$ S of E. The calculation is summarized in the figure.
\begin{array}{c|c|c}
\text{x-comp} & \text{y-comp} \\
\hline
A & 130 & +7.5 \\
B & -70.7 & +70.7 \\
C & +34.7 & -19.7 \\
\text{A} + \text{B} + \text{C} & 94 & -51.3 \\
\end{array}

\begin{align*}
\vec{A} &= 130 \hat{x} + 7.5 \hat{y} \\
\vec{B} &= -70.7 \hat{x} + 70.7 \hat{y} \\
\vec{C} &= 34.7 \hat{x} - 19.7 \hat{y} \\
\vec{P} &= 94 \hat{x} - 51.3 \hat{y} \\
\end{align*}

\[
\vec{P} = \sqrt{94^2 + (-51.3)^2} \approx 107
\]

\[
\theta = \tan^{-1} \frac{51.3}{94} \approx 23.6^\circ
\]

107 at (360 - 28.6)^\circ

107 at 331.4^\circ
The connection that this example has to physics is the following. If an object is subject to three forces: a force of magnitude 150 units at an angle of 30 degrees N of E, a force of magnitude 100 units at an angle of 45 degrees N of W, and a force of magnitude 200 units at an angle of 80 degrees S of E, then the net force on the object is 107 units at an angle of 28.6° S of E. If one wants to balance these three forces, then one needs a force of 107 units in the opposite direction at 28.6° N of W.

Expressing the vectors using unit (or basis) vectors is probably the most convenient way to perform operations with them. In general, if \( \vec{A} = A_x \hat{i} + A_y \hat{j} \), and \( \vec{B} = B_x \hat{i} + B_y \hat{j} \), then the sum \( \vec{A} + \vec{B} \) equals

\[
\vec{A} + \vec{B} = (A_x + B_x) \hat{i} + (A_y + B_y) \hat{j}
\]

for subtraction, replace the “+” with a “−”. Using unit (or basis) vectors is particularly useful when adding (or subtraction) more than two vectors.

It is important to remember that for a complete description of a vector in two dimensions, two numbers are needed: a magnitude plus direction, two components, etc. For example, the vector \( \vec{D} \) above can be expressed as: 94 \( \hat{i} \) − 51.3 \( \hat{j} \), or 107 units at a direction 28.6° S of E, or 107 units at an angle of 331.4° clockwise from the x-axis.

It should be pointed out that all quantities with a magnitude and direction are not necessarily vectors. To be a vector, a quantity needs to have certain properties. Also, here we have limited our applications to two and three dimensional vectors. These ideas can be generalized to any number of dimensions (including infinity). For higher dimensional vector spaces, it is not always useful to think of the direction of a vector.

The experiments we have done so far in lecture demonstrate the following principle concerning the conditions for static equilibrium:

If an object is subject to forces and is stationary, then the vector sum of the forces on the object equals \( \vec{0} \).

Next we will examine the condition required so that the object does not rotate.

As a final note, we point out that in order for a quantity to be a vector, the quantity must add (or combine) according to the rules of vector addition. Not all quantities with magnitude and direction add according to the rules of vector addition. An example of such a quantity is rotations. Any rotation in three dimensions can be
represented by an arrow with magnitude and direction: the direction is the axis of rotation, and the magnitude is the amount of rotation (e.g. in radians). Let $A$ be a rotation about the x-axis of $\pi/2$ radians, and $B$ be a rotation about the y-axis of $\pi/2$ radians. You can show by rotating your physics book that $A + B \neq B + A$. That forces add according to the laws of simple vector addition is remarkable.

There are a number of quantities in classical physics that behave as vectors: displacement, velocity, acceleration, force, the electric field and the magnetic field to name a few. You will encounter these during your first year of physics. Although they represent different physical quantities, they all add like the method described here. Vector addition pertains to many different physical quantities, it is a "universal mathematics", and its importance cannot be understated.

**Torque: twisting force**

We all have experience tightening a bolt. If we want it tight, we use a wrench. To apply a large amount of "twisting force", we hold the wrench as far as we can from the bolt and push with a force perpendicular to the wrench arm. The name we call "twisting force" is torque. In order to determine the amount of torque, we need to specify three things:

1) The axis of rotation. In the case of the bolt, this is where the bolt is.

2) The force that is doing the twisting. (The force, direction and magnitude, that you apply to the wrench).

3) The position of the force. (How far from the bolt that you apply the force.)

When we speak of torque, we usually say "the torque about the axis (or point) $O$ due to the force $F$ applied at the point $P$."

All three things matter in determining the torque. We know from experience that a larger force produces a larger torque, and the further away from the axis that the force is applied, the larger the torque, but what is the correct mathematical formula for torque? We will show in class that:

$$\text{torque} = |\vec{r}| \times \text{component of the force perpendicular to } \vec{r}$$

where $\vec{r}$ is the vector from the axis to the point where the force is applied. The component of the force perpendicular to $\vec{r}$ is the component of the force that does the twisting. The component of the force, $\vec{F}$, that is parallel to $\vec{r}$ just pushes on the bolt.
This component doesn’t cause any twisting about the axis. The symbol for torque used in most physics texts is $\tau$:

$$|\vec{r}| = |\vec{r}| F_\perp$$  \hspace{1cm} (3)

If we let $\theta$ be the angle between the vector $\vec{r}$ and the vector $\vec{F}$, the torque equation can be written as:

$$|\vec{r}| = |\vec{r}| |\vec{F}| \sin(\theta)$$  \hspace{1cm} (4)

Remember, the vector $\vec{r}$ goes from the axis of rotation to the point at which the force is applied.

In this class we will only consider torques that are caused by forces and about points that all lie in the same plane. However, it can be shown that the equations we develop here apply to forces in general. That is, even if the forces causing the twists do not lie in the same plane.

Is torque a vector, a scalar, or something else? Torque certainly has a magnitude, which is the amount of twisting force. We can also associate a direction with torque: The axis of rotation of the twist. The axis of rotation is in the direction perpendicular to both the force $\vec{F}$ and the position vector $\vec{r}$. Thus, we can assign a vector to a particular torque: The magnitude of the vector is the amount of twisting force, $|\vec{r}| |\vec{F}| \sin(\theta)$; and the direction is the direction perpendicular to the plane containing $\vec{r}$ and $\vec{F}$. There is one more decision to make: which direction along the axis of the torque corresponds to a clockwise twist, and which direction to a counter-clockwise twist? Since $\sin(-\theta) = -\sin(\theta)$, the vector for clockwise twist will be in the opposite direction to the one for counter-clockwise twist. The convention that is agreed upon around the world is the following: the direction of the torque vector is the direction that the bolt would move (for normal “right-handed” bolts). Using your right hand, if your fingers curl in the direction of the twist, your thumb points in the direction of the torque. So, clock-wise twists point into the page, and counter-clock-wise twists point out of the page.

We will show that torques add according to the rules of vector addition, so it is a true vector. In mathematics, there is an operation between vectors that yields another vector. It is the cross product and we can write the torque vector as $\vec{r}$ cross $\vec{F}$:

$$\vec{r} = \vec{r} \times \vec{F}$$  \hspace{1cm} (5)
Since here we will only solve problems in which all the location $\vec{r}$ vectors and all the force $\vec{F}$ vectors lie in the same plane, we will not need to carry out the cross product explicitly using vectors. We will only do problems in which the torques produce a clockwise or counter-clockwise twist, and consequently the torque vectors are either in or out of the page. Nonetheless, to prepare you for more advanced courses, we review some of the mathematical properties of the vector cross product.

**Vector Cross Product**

The cross product of vector $\vec{A}$ and $\vec{B}$ is a vector. The magnitude of $\vec{A} \times \vec{B}$ equals the magnitude of $\vec{A}$, $|\vec{A}|$ times the magnitude of $\vec{B}$, $|\vec{B}|$, times the sin of the angle between them:

$$|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}|\sin(\theta) \quad (6)$$

where the angle $\theta$ is the angle between vector $\vec{A}$ and vector $\vec{B}$. The angle goes from $\vec{A}$ to the vector $\vec{B}$. The direction of $\vec{A} \times \vec{B}$ is determined by the "right hand" rule described above. Note that since $\sin(-\theta) = -\sin(\theta)$,

$$\vec{B} \times \vec{A} = -\vec{A} \times \vec{B} \quad (7)$$

Note that the cross product is maximized when $\vec{A}$ is perpendicular to $\vec{B}$, and zero when $\theta$ is zero or 180°. The unit vectors have simple cross product properties:

$$\hat{i} \times \hat{j} = \hat{k}$$
$$\hat{j} \times \hat{k} = \hat{i}$$
$$\hat{k} \times \hat{i} = \hat{j}$$
$$\hat{i} \times \hat{i} = 0$$
$$\hat{j} \times \hat{j} = 0$$
$$\hat{k} \times \hat{k} = 0$$

We can express $\vec{A}$ and $\vec{B}$ in terms of the unit vectors: $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ and $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$. If we use the above formulas for the cross products of the unit vectors, after "foil"ing we have:

$$\vec{A} \times \vec{B} = (A_yB_z - A_zB_y)\hat{i} + (A_zB_x - A_xB_z)\hat{j} + (A_xB_y - A_yB_x)\hat{k}$$
This formula can be succinctly written in terms of the determinant of a matrix:

\[
\vec{A} \times \vec{B} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
A_x & A_y & A_z \\
B_x & B_y & B_z
\end{vmatrix}
\]  

(8)

**Condition for Static equilibrium for a rigid object**

In summary, there are two conditions for static equilibrium. In order for an object to stay at rest, the vector sum of the forces on the object must be zero. This is all one needs if the object is a point particle. However, if the object has size, the location of the forces on the object is important. Even though the sum of the forces is zero, the object might rotate. In order for the object to stay stationary and not rotate, the sum of the torques must be zero about any axis. These are the two conditions for an object to remain "static", i.e. not translate and not rotate:

\[
\sum_i \vec{F}_i = 0 \\
\sum_i \vec{\tau}_i = 0
\]

where the torque sum can be taken about any axis. Note that for the first requirement, \(\sum_i \vec{F}_i = 0\), the location of the forces do not enter in the equation. However, for the second requirement, the location of the forces are very important. Just like in Real Estate, the three most important things are Location, Location, and Location. With torque, it is just Location, Location: the Location of the axis and the Location of the force.

In class we will do many examples of many objects in static equilibrium. In evaluating the torque sum, a proper choice of axis can reduce the mathematical complexity.

**Measuring Speed, Constant Velocity**

Consider the simplest kind of motion of a small "point" object. Imagine that the "point" object is "floating" along in outer space feeling no forces (i.e. without interacting with anything). What will the motion be like? Will it come to rest on its own, or continue with a constant velocity? Newton was the first one to understand what would happen. He realized that the object would just float along with a constant speed without changing direction. This idea is a "law of motion", and we want to
formulate it quantitatively and precisely. To do so we first need to understand what is meant by constant speed. We need a method of measuring distance and time intervals. Understanding space and time belong to the realm of physics.

To measure **position**, we set up a coordinate system in three dimensions. Once a unit length has been decided upon, equal lengths can be marked on the axes. A particle’s position is specified by three real numbers (+ or -) indicating its location in the coordinate system. The distance between any two points can be determined using a ruler. Three systems of units will used in the class, MKS, cgs, and British. Length is measured in meters in the MKS, centimeters (cm) in the cgs, and in units of feet in the British system. Measuring the distance between two points is not an abstract concept. We use a straight ruler that has markings of equal length spacings and place it between the two points. Marking out equal distances on a ruler is a straightforward. Measuring time, however, is more abstract.

To measure **time**, we need an instrument (a clock) that can produce equal time intervals. Whereas we have a physical feeling for equal distances from our hands, arms and eyes, equal time intervals are difficult to get a feeling for. Physiologically a good movie lasting as long as a physics lecture might seem much shorter in time. Therefore we can’t rely on our senses to judge equal time intervals. One could use a pendulum or other oscillating device. However, using the motion of a physical system to determine equal time intervals and then using this clock to understand motion itself might bias our description and laws of physics. We might wonder if our method of measuring time is making our equations of physics to complicated. It would be nice to have a clock that does not rely on a physical system. We will discuss such a possibility in a moment. In order not to get caught up in circular arguments and deep philosophy, in this class we will accept that clocks can be made that give equal time intervals. As discussed in the text, the accepted definition of a second is 9192631770 oscillations of a particular transition in the Cesium atom. Although the atom is a physical system, we obtain a workable definition of time for developing physics theories. The interested student should take our modern physics course, where Einstein’s theory of special relativity addresses these problems.

Once we have established distance and time units, we can measure the speed of a particle moving in a straight line. We just place our ruler in the direction of the motion. The speed is the distance traveled divided by the time interval. Since the speed is constant and the direction is not changing, you get the same answer no matter how long the time interval is. When we use the term velocity, we refer to the direction of the particle as well. **Velocity can be expressed in vector notation.** The length of the vector is proportional to the speed of the particle, and the direction of the vector is the direction of the particles motion. We don’t know if velocities combine
according to the mathematics of vector addition, so we are not sure if it is a true vector yet. However, in any case the velocity of a particle can be expressed in vector notation as $\vec{v}$. For the case of constant velocity, the velocity vector does not change in time. No need for calculus yet.

**Newton’s First Law of Motion**

Under what situations will an object move with a constant velocity (i.e. constant speed in a straight line)? A person walking in a straight line down the street with a constant speed, or a car driving down a straight length of road with a constant speed are some examples that come to mind. However, there is a very important case related to a fundamental Law of Motion.

Consider a "reference frame" which is a box floating in space far from any other objects. If you were to go inside this box, floating in space, you would feel "at rest". You could not sense that you were moving. You set up a coordinate system to measure position and time. Suppose a particle had an initial velocity $\vec{v}_0$ with respect to your coordinate system. There are no forces on the particle. What would happen to the particle in time? Would it come to rest, or continue to move with the velocity $\vec{v}_0$ in a straight line? Newton and Galileo realized that the particle would continue to move with the velocity $\vec{v}_0$. If the particle was at rest ($\vec{v}_0 = 0$) it would remain at rest. We will demonstrate this phenomena in lecture using an air track. This property of motion is refered to as Newton’s first Law of Motion:

**If there are no forces acting on an object, an object at rest remains at rest and an object in motion continues in a state of uniform motion**

This idea might seem simple to us, but at the time it was proposed it was profound. It was believed that the natural state of an object was at rest, and that objects that were moving came to rest on their own. Newton’s first law of motion applies to reference frames that are floating in space. Only in these frames will an object that is released at rest in "mid air" stay at rest. The name we give to reference frames for which this law (Newton’s first law) of motion holds is an **inertial reference frame**.

Imagine two reference frames floating in space. Suppose someone named Bill was in one, and George in the other. Suppose that George observed that Bill was moving in the $+v$ direction with a *constant velocity* $+v$. George would feel at rest and say that
Bill was moving to the right with a constant velocity. Bill, however, would also feel at rest and say that George is moving to the left with a constant velocity \(-v\). Who is correct? Both are. Each of these frames is an inertial reference frame. Bill feels at rest, and so does George. There is something very special about reference frames floating in space with a constant velocity with respect to each other. They are all inertial reference frames and have the following properties:

1. A reference frame moving with a constant velocity with respect to an inertial frame is also an inertial reference frame.

2. In an inertial reference frame, one "feels" at rest.

3. There is no experiment that one can do in an inertial reference frame to determine the velocity of the reference frame.

4. The laws of physics take on the same form in all inertial reference frames.

5. There is no absolute reference frame.

The equivalence of inertial reference frames is a fundamental property of physics, and is the basis of Einstein’s theory of special relativity. It is a wonderful property of nature, and one can marvel at its simplicity.

A final note on Newton’s first law is that it allows one to define equal time intervals independent of a physical system. Here is how to do it: Set up your "x" axis, pick an origin, and a unit length. Use your unit length to mark on your "x" axis equal distances. Then set an object (with no forces) in motion. It will float along your "x" axis. A time interval occurs each time that it passes a mark. According to Newton’s first law the time intervals will be equal.

**Newton’s Third Law: Symmetry in interactions**

Let’s consider what happens when two objects, ”1” and ”2”, interact with each other. By interact, we mean that object ”1” causes a push or pull on object ”2” for a certain amount of time. Likewise object ”2” can cause a push or pull on object ”1”. We’ll do the following experiment in lecture. Two carts will be held together with a spring between them attached to cart ”1”. They will be released and the spring will push them apart. The carts can roll on the horizontal surface of the table. Is it possible for only one cart to move away, and the other one to stay still? As we
shall see, no it is not. If cart "1" moves off to the right, then cart "2" must move off to the left. If cart "2" feels a push due to cart "1", then cart "1" also feels a push. After they fly apart, and the spring is no longer pushing them, they will travel with a constant velocity. Let’s first only examine the final velocities of the cars, and not be concerned with what is happening during the pushing. Later we will deal with this problem.

The experiment will demonstrate that if the carts are identical, they end up moving with the same speed, but in opposite directions. This seems "fair" since the carts are identical. We will also see that more "massive" carts end up moving slower after "pushing away" a less massive cart. To get quantitative, we need a way to "quantify" mass. We’ll define mass as the amount of matter an object has. Just as we did with an object’s weight, we can consider objects that are a single substance, say iron. Then the mass of an iron object is proportional to its volume. That is, for pure iron carts, the mass equals the density times volume: For cart "1", \( m_1 = \rho V_1 \), and for cart "2", \( m_2 = \rho V_2 \), where the \( V_i \) are the volumes of the iron. Now we can carry out some experiments.

The experiments will show that if \( m_1 = 2m_2 \), the final speed of cart "2" will be twice that of cart "1". If \( m_1 = 3m_2 \), the final speed of cart "2" will be three times that of cart "1". Etc... The two carts move off in opposite directions with this simple relationship. WOW!, this is a very nice result. The mass times speed of cart "1" is always equal to the mass times speed of cart "2". There is something special about the product (mass)(speed). It is so special that is given a special name, momentum. Since velocity better describes the motion, we use velocity instead of speed and define the momentum \( \vec{p} \) of an object as

\[
\vec{p} \equiv m\vec{v}
\]

Note that we have written momentum as a vector. We haven’t yet shown that momenta combine according to the mathematics of vector addition, but we will soon check it out. Now, returning to the cart experiment. Let \( \vec{v}_1 \) be the final velocity of cart "1", and \( \vec{v}_2 \) be the final velocity of cart "2". Then we see that \( m_1|\vec{v}_1| = m_2|\vec{v}_2| \), or in vector notation:

\[
m_1\vec{v}_1 = -m_2\vec{v}_2 \\
\vec{p}_1 = -\vec{p}_2
\]

Since cart "1" started off at rest with no momentum, \( \vec{p}_1 \) is the change in the momentum of cart "1". Likewise, \( \vec{p}_2 \) is the change in the momentum of cart "2" resulting
from its interaction with cart "1". Newton postulated that this symmetry holds whenever any two particles interact with each other:

If object 1 interacts with object 2,
then the change in momentum of object 2 caused by object 1 is equal but opposite to the change in momentum of object 1 due to object 2.

\[ \vec{\Delta}p_1 = -\vec{\Delta}p_2 \quad (10) \]

This property of nature is part of Newton’s third law. In lecture we have demonstrated the validity of this law if the two carts are initially at rest. Experiments verify that \( \vec{\Delta}p_1 = -\vec{\Delta}p_2 \) whenever two particle interact with one another!! This is a spectacular result. One can rearrange the equation to be:

\[ \vec{\Delta}p_1 + \vec{\Delta}p_2 = 0 \]
\[ \Delta(\vec{p}_1 + \vec{p}_2) = 0 \]
\[ \vec{p}_1 + \vec{p}_2 = a \text{ constant} \]

From the last line, we see it is useful to add the two momenta as vectors to form a ”total momentum of the system” vector, \( \vec{P}_{tot} \):

\[ \vec{P}_{tot} \equiv \vec{p}_1 + \vec{p}_2 \quad (11) \]

The last line states that when two particles interact with each other, the total momentum of the system, \( \vec{P}_{tot} \) is the same before as after the interaction. During the interaction period, \( \vec{P}_{tot} \) also remains constant. Thus experiments demonstrate that the total momentum of a two-particle system remains constant during and after the interaction of the particles:

\[ \vec{P}_{tot} \equiv \vec{p}_1 + \vec{p}_2 = \text{constant} \quad (12) \]

Physicists like to discover combinations of quantities that remain constant. The above equation is referred to as a ”Conservation Law”, with the total momentum of the two particle system be conserved. This property of nature is called the Conservation of the Total Momentum of a system.

Some comments on the Conservation of Total Momentum of a system:

1. Total momentum is conserved if there are no influences (or forces) acting on the particles from outside the system. That is, the only interaction between the two particles is caused by the two particles themselves.
2. Although we only demonstrated the conservation of momentum for a two-particle system, it holds for a system of any number of particles. If the system has \( N \) particles, then \( \vec{P}_{\text{tot}} = m_1 \vec{p}_1 + m_2 \vec{p}_2 + \cdots + m_N \vec{p}_N \).

3. The amazing thing is that total momentum is conserved for all types of interactions (gravity, the electromagnetic interaction, etc.), and at all length scales. One might have to find the correct expression for momentum, but so far we are able to and \( \vec{P}_{\text{tot}} \) is always conserved.

4. The reason total momentum is conserved stems from the fact that any change in \( \vec{p}_1 \) caused by particle ”2” is equal and opposite to the change in \( \vec{p}_2 \). This is an example of a symmetry leading to a conservation law. A famous theorem by Emmy Noether states that Symmetries in nature yield conserved quantities.

In lecture we will discuss several cases where there are no external forces: collisions, explosions, etc. When no external forces act in a collision, the total momentum (vector sum of the momenta of the particles) of the system is conserved. If kinetic energy is also conserved, we call the collision elastic. It is remarkable that one doesn’t need to know the details of the collision. As long as there are no external forces, it doesn’t matter what kind of forces are involved: the total momentum remains constant before, during and after the collision.

In the derivation above, the conservation of total momentum comes from Newton’s third law, which is a result of a symmetry in nature. The interaction between object 1 and object 2 is symmetric, they each must experience the same force. Is this a general result, that symmetries in nature lead to conserved quantities? We find that in many cases this is true. Rotational symmetry leads to conservation of angular momentum, and momentum and energy conservation are a result of space and time symmetry. This is a grand idea, and helps in describing the physics of subatomic particles. We do experiments to identify conserved quantities, then develop mathematical descriptions that have the corresponding symmetry properties. The connection between symmetries in nature and conserved quantities is one of the more "beautiful" principles of physics.

We will do many examples in lecture and lab that demonstrate the conservation of total momentum of a system of particles. Another interesting consequence of the conservation of total momentum deals with the center of mass of a system of particles, which we discuss next.

Center of Mass of a system of particles
From the total momentum of a system of particles, we can define another special quantity for a system of particles: the “center-of-mass” velocity. The center-of-mass velocity, \( \vec{V}_{cm} \), is defined as the total momentum divided by the total mass of the system:

\[
\vec{V}_{cm} = \frac{\vec{P}_{tot}}{M_{tot}} \tag{13}
\]

where the total mass of the system, \( M_{tot} \) is defined as

\[
M_{tot} \equiv m_1 + m_2 + \ldots \tag{14}
\]

where \( m_1, m_2, \ldots m_i, \ldots \) refers to the mass of particle \( i \). The center-of-mass velocity is easily expressed in terms of the individual masses and velocities of the particles that make up the system:

\[
\vec{V}_{cm} = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2 + \ldots}{m_1 + m_2 + \ldots} \tag{15}
\]

Note that the center-of-mass velocity has direction and magnitude. From the center-of-mass velocity, it is straightforward to define a center-of-mass position:

\[
\vec{R}_{cm} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + \ldots}{m_1 + m_2 + \ldots} \tag{16}
\]

for the center-of-mass position. Here \( \vec{r}_i \) is the vector from the origin to the location of particle \( i \). \( \vec{R}_{cm} \) is usually called center-of-mass. These center-of-mass quantities have some nice properties. Since \( \vec{P}_{tot} = M_{tot} \vec{V}_{cm} \), we have

\[
M_{tot} \vec{V}_{cm} = \text{constant} \quad \frac{\vec{V}_{cm}}{\vec{V}_{cm}} = \text{constant}
\]

Wow, another nice result. If there are only internal influences (forces) for the system of particles, the center-of-mass of the system moves at a constant velocity. If it is initially at rest (in an inertial frame) it remains at rest. This result also defines a special reference frame for a system of particles: the reference frame for which the center-of-mass does not move.

If there are only internal interactions between the particles in a system of particles, there exists an inertial frame of reference in which the center-of-mass \( \vec{R}_{cm} \) remains at rest. We refer to this reference frame as the center-of-mass reference.
frame, or simply center-of-mass frame. Although the individual particles in a system might move in a complicated manner, there is one special position, the center-of-mass, which moves in a simple way. It is often easier to analyze the motion of a system of particles from the center-of-mass frame.

We have gone as far as we can in stating the fundamental laws of interaction and motion without using calculus. To progress in our understanding of mechanics, we need to use calculus. Calculus deals with instantaneous changes of one variable with another. We will see that if one expresses the laws of mechanics in terms of the instantaneous changes of position and velocity with time, that the laws of physics take on a simple form. This insight is generally attributed to Newton, and we will develop these ideas in the next section.