Force and Motion

In the last section we demonstrated and discussed two fundamental laws of motion and interaction: Newton’s first law and part of Newton’s third law. (1) If an object does not have any forces acting on it, it will float along with a constant velocity. (2) When two particles interact with each other, the total momentum $\vec{P}_{tot}$ of the two particle system remains constant before, during and after the interaction, where $\vec{P}_{tot} = m_1\vec{v}_1 + m_2\vec{v}_2$. Newton’s third law suggests that the product of mass times velocity is a special quantity in which to formulate the laws of motion. We now want to discover what the laws of motion are that describe the time dependence of a particle’s position and velocity.

We start with an particle moving in a straight line in one dimension, then generalize to three dimensional motion.

One Dimensional (straight-line) Frictionless Environment

Describing Motion: Position, Time, Velocity

Consider a particle that can only move in a straight line. Previously we discussed how one can quantify position (or distance) and time. Once we have established a coordinate system, which consists of distance and time units, an origin and a ”+” direction, we can define a position function, $x(t)$. The position function is just the position $x$ for the particle as a function of time, $t$.

A particle’s displacement between the times $t_1$ and $t_2$ is just equal to $x(t_2) - x(t_1)$, where $t_2 > t_1$. If the displacement is positive, the particle has moved to the right (+ direction) from the time $t_1$ to the time $t_2$. If the displacement is negative, the particle has moved to the left (- direction). To talk about displacement, you need to refer to two times.

Velocity is a measure of how fast an object is moving. Average velocity, $\bar{v}_{t_1\rightarrow t_2}$, is defined to be the displacement/(time interval):

$$\bar{v}_{t_1\rightarrow t_2} \equiv \frac{x(t_2) - x(t_1)}{t_2 - t_1}$$

As with displacement, to calculate the average velocity of an object one needs to specify the two times, $t_1$ AND $t_2$. Average time is not a particularly good quantity to express the laws of mechanics in a simple way. For understanding the laws of physics in a simple way, the velocity at an instant, instantaneous velocity, is a much better quantity. Instantaneous velocity at the time $t_1$, $v(t_1)$, is defined to be:
\[ v(t_1) \equiv \lim_{t_2 \to t_1} \frac{x(t_2) - x(t_1)}{t_2 - t_1} \]  \hspace{1cm} (2)

This can also be written as:

\[ v(t_1) \equiv \lim_{\Delta t \to 0} \frac{x(t_1 + \Delta t) - x(t_1)}{\Delta t} \]  \hspace{1cm} (3)

which you will recognize as the first derivative of \( x(t) \) evaluated at the time \( t_1 \): \( v(t_1) \equiv (dx/dt)_{t=t_1} \). A nice thing about instantaneous velocity is that it is defined at a single time, \( t_1 \). For this reason it is a better quantity to use than average velocity to try and describe how nature behaves. If the instantaneous velocity is positive (negative) then the particle is moving to the right (left) at that instant.

A particle’s speed is defined to be the magnitude (or absolute value) of the instantaneous velocity.

**Special Case of Constant Velocity**

In some special cases, a particle’s velocity may be constant for a long period of time. If this is so, then the position function takes on a simple form. Let \( v_0 \) be the velocity of the particle at the time when we start our clocks, \( t = 0 \). We will often use the subscript 0 to label the value of a variable at time \( t = 0 \). For a particle moving with a constant velocity, \( v_0 \) will be the velocity for all times \( t \). From the definition of velocity: \( (x(t) - x(0))/(t - 0) = v_0 \). If we define \( x(0) \equiv x_0 \), we have the simple relationship for \( x(t) \):

\[ x(t) = x_0 + v_0 t \]  \hspace{1cm} (4)

Often one writes \( x(t) \) as just \( x \), so the equation becomes \( x = x_0 + v_0 t \).

**Describing Motion: Acceleration**

Objects don’t always move with constant velocity, velocities change. The change in velocity per unit time is called acceleration. The average acceleration, \( \bar{a}_{t_1 \to t_2} \), between time \( t_1 \) and \( t_2 \) is defined to be

\[ \bar{a}_{t_1 \to t_2} \equiv \frac{v(t_2) - v(t_1)}{t_2 - t_1} \]  \hspace{1cm} (5)

As with average displacement and average velocity, one needs two times to calculate the average acceleration. Since two times are needed, the average acceleration is not a good quantity to describe the laws of physics. A more useful quantity is the
instantaneous acceleration, which is defined in the same way as instantaneous velocity. One takes the limit of the average acceleration as $t_2$ approaches $t_1$:

$$a(t_1) \equiv \lim_{t_2 \to t_1} \frac{v(t_2) - v(t_1)}{t_2 - t_1}$$

(6)

Instantaneous acceleration is also written as:

$$a(t_1) \equiv \lim_{\delta t \to 0} \frac{v(t_1 + \delta t) - v(t_1)}{\delta t}$$

(7)

The limit on the right side is just the first derivative of the velocity evaluated at the time $t_1$. So $a(t_1) \equiv \frac{dv}{dt} |_{t=t_1}$. Often the $t_1$ is replaced by $t$, and one writes $a \equiv \frac{dv}{dt}$. A nice property about instantaneous acceleration is that it is determined at one time. The acceleration is just the second derivative of position with respect to time: $a = \frac{d^2x}{dt^2}$.

One could also consider more derivatives such as $\frac{da}{dt}$, $\frac{d^2a}{dt^2}$, etc... to describe motion. We need to rely on experiments to determine the simplest way to relate interactions (forces) and motion.

If the position function, $x(t)$, is known, it is easy to find $v(t)$ and $a(t)$ by differentiation. If one knows $a(t)$, one needs to integrate with respect to $t$ to find $v(t)$. To find $x(t)$, one needs to integrate $v(t)$ with respect to $t$. A simple case that is discussed in many texts is the special case of motion with constant acceleration. Suppose the acceleration of a particle is constant, label it $a_0$. Then, $\frac{dv}{dt} = a_0$. Multiplying both sides by $dt$ and integrating we have $\int_0^t dv = \int_0^t a_0 dt$, which gives $v(t) - v_0 = a_0t$ where $v_0$ is the velocity at $t = 0$, $v(0)$. This is often written as $v = v_0 + a_0 t$, where $v$ means $v(t)$.

To solve for $x(t)$ for the case of constant acceleration requires one more integration. From $\frac{dx}{dt} = v(t) = v_0 + a_0 t$, one can multiply both sides by $dt$ and integrate: $\int_0^t dx = \int_0^t v_0 dt + \int_0^t a_0 t dt$. After integrating, one obtains: $x(t) - x(0) = v_0 t + a_0 t^2 / 2$. This equation is often written as $x = x_0 + v_0 t + a_0 t^2 / 2$, where $x$ means $x(t)$.

Summarizing for the special case of constant acceleration:

$$v = v_0 + a_0 t$$

(8)

$$x = x_0 + v_0 t + \frac{a_0 t^2}{2}$$

(9)

Eliminating $t$ from these two equations gives:

$$v^2 = v_0^2 + 2a_0(x - x_0)$$

(10)
In general, motion that has constant (non-zero) acceleration for extended periods of time is fairly rare. Under certain circumstances however, the motion of an object can be *approximately* one of constant acceleration for a period of time.

**One final and important note:** the sign’s of $x(t)$, $v(t)$, and $a(t)$ are not related to each other. The acceleration can be in the negative (positive) direction even though the velocity is positive (negative), etc. The sign of $a$ is in the direction of the change of $v$.

**Describing Motion in 2 and 3 Dimensions**

The extension of velocity and acceleration to 2 and 3 dimensions is straightforward, since displacements add like vectors. We can define a position vector, $\vec{r}(t)$, a velocity vector, $\vec{v}(t)$, and an acceleration vector, $\vec{a}(t)$ as follows

\[
\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}
\]
\[
\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = v_x(t)\hat{i} + v_y(t)\hat{j} + v_z(t)\hat{k}
\]
\[
\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = a_x(t)\hat{i} + a_y(t)\hat{j} + a_z(t)\hat{k}
\]

If we pick our coordinate system properly, sometimes we can change a three dimensional problem into three one dimensional problems. An important point to mention is that in general the three vectors $\vec{r}(t)$, $\vec{v}(t)$, and $\vec{a}(t)$ do not necessarily point in the same direction.

**Forces, Inertia and Motion**

With the mathematics of calculus, which enables us to work with instantaneous rates of change, we can formulate the connection between force and motion. We will start with the simple case of one dimensional motion, which takes place in an inertial reference frame in the absence of friction. Our perception of **force** is that it is a push or a pull. We also have experienced that it is easier to change the motion of a "smaller" object (an object with less matter) than a "larger" one (one with more matter). We call **inertia** the resistance to a change in motion. **Inertial mass** is a measure of how much inertia an object has. We have already used Newton’s Third
law in the last section to quantify mass as being equal to density times volume. We now need to see if there is a way to connect this idea of mass with force and motion, such that the laws of motion take on a simple form.

The case of Constant Velocity

We demonstrated that if an object has no forces acting on it, it will stay at rest or move with a constant velocity. Is there any other situation that an object will move with constant velocity. Yes, there is. If we push our physics book across the table, we can push it such that it will move with a constant velocity, say $\vec{v} = v_0 \hat{i}$. Does this mean that there are no forces on it? NO. We are pushing with a certain force, the table is exerting a force, and the earth is attracting the book (gravitational force). These three forces are acting on the book. The table is actually ”pushing up” on the book and exerting a frictional force as well. Newton realized that if an object is moving with a constant velocity then all the forces that it experiences must add up to zero (vector addition). This is consistent with our experiments on statics as follows: In a reference frame that is moving with the constant velocity of the book, $v_0 \hat{i}$, the book is at rest. We showed that the vector sum of the forces must add to zero if the book is at rest, so they will add to exactly zero in the frame where the book has a constant velocity.

If a particle has a constant velocity, it also has a constant momentum. Thus, if an object is moving with a constant momentum, the vector sum of the forces acting on it equals zero. If the net force on an object is not equal to zero, then the object will not move with a constant momentum (or velocity).

The Mathematics Needed

At present we have discovered only three basic interactions in our universe:

1. Gravity
2. Electro-magnetic-weak interaction
3. Strong interaction

As a physicist we are interested in discovering the Laws of Nature in their most simple form. Since particle interactions result in changes in the momentum of a particle, we are motivated to try and express the laws of motion in terms of the change in momentum of a particle: $d\vec{p}/dt = m d\vec{a}/dt$. Therefore, we believe that the highest derivative of position with time in the basic equations of motion should be the second
derivative $d^2\vec{r}(t)/dt^2$. There is another reason to expect that the laws of classical physics will not need higher derivatives. If the equations contained terms involving $d^3x/dt^3$, one would need three initial conditions to determine $x(t)$ for times in the future. If one believes that only the initial position, $x_0$, and initial velocity, $v_0$, are necessary to determine $x(t)$ for future times, then there can be at most second derivatives of $x(t)$ in the equations of motion. Under these constraints, two initial conditions, one only needs to consider the acceleration of a particle in formulating the Laws of Motion. It turns out that differential equations are very useful in describing nature because the laws of physics often take on a simple form when expressed in terms of infinitesimal changes. Thus, we will try and understand physics using second order differential equations. This is one of the important ideas that Newton demonstrated.

**Force as a definition?**

The three basic interactions (gravity, electromagnetism, weak and strong) can be understood without introducing the concept of force. For example, in Phy132, you will learn that the law of universal gravity between two "point" objects equals

$$
\frac{d\vec{p}_1}{dt} = G\frac{m_1m_2}{r^2}\hat{r}_{12}
$$

$$
m\ddot{a} = G\frac{m_1m_2}{r^2}\hat{r}_{12}
$$

Note that there is no mention of force in this equation. The same is true of the other fundamental interactions. One reason force is not needed is because these basic interactions occur without the particles actually touching each other. However, it is useful to consider "force". We can pull and push things around, and in practical engineering problems objects experience pushes and pulls from ropes, surfaces, etc. We will refer to forces that actually touch an object as **contact forces**. Since a push or pull changes a particles momentum, we are motivated to define and quantify the amount of force an object experiences by the following equation:

$$
F_{x\text{net}} \equiv \frac{dp_x}{dt}
$$

Note that the left hand side is the **net force**, or sum of the forces on the object. We can also write this equation different ways:
This seems reasonable, since we know from experiments that if an object has more mass, it takes more push (or pull) to change its velocity. We have demonstrated that "static weights" add according to the rules of vector addition. We will now experiment to see if this "dynamic" method of defining and quantifying forces yields consistent physics. That is, we need to check if a constant force produces constant acceleration, if masses combine like scalar quantities, and if "dynamic" forces add like vectors. We test each of these properties with an experiment using contact forces.

Three Experiments with Contact Forces

What kind of motion does a constant contact force cause?

One can ask: how do we apply a constant force to an object? One way might be to use a "perfect" spring and pull on the object such that the spring’s extension is doesn’t change. I think we can agree that if a spring is being stretched by a fixed amount the contact force it exerts will not change. By a "perfect" spring, we mean just that, that the spring does not weaken over time but keeps the same constant force. Let’s observe what happens when an object is subject to a constant contact force. What kind of motion will result? Most likely not constant velocity, since this is the case if there are no net forces. The object will accelerate, but will the acceleration change in time? We need to do the experiment, and in lecture I will pull a cart on an air track with a spring keeping the stretch of the string constant. If I do it correctly, we will see that:

**Experiment 1**: If an object is subject to a constant force, the motion is one of constant acceleration.
is only valid within the realm of non-relativistic mechanics. If the velocities are large and/or the measurements very very accurate, relativistic mechanics are needed to understand the data.

**Is inertial mass a scalar?**

We need another experiment to determine if "inertial" mass combines as a scalar. Take two identical objects, and apply the constant force $F_0$ to one object. The acceleration will be constant, call it $a_1$. Now connect the two identical objects together and apply the same force $F_0$ to the pair. The acceleration is constant, but how large is it? We will do a similar experiment in lecture. The result from the experiment is:

**Experiment 2:** If a constant force is applied to two identical objects which are connected to each other, the measured acceleration is it 1/2 the acceleration of one of the objects if it is subject to the same constant force.

This is also a very nice result, and nature didn’t have be so simple! It shows that if the amount of material is doubled, the acceleration is cut in half for the same constant force. Experiment will also show that $x$ times the amount of material results in an acceleration equal to $1/x$ times $a_1$.

As far as we know, inertial mass, is an intrinsic property of an object and therefore will have its own units. In the MKS system, the unit is the kilogram (Kg). If $a$ is in units of $M/s^2$, then force will have units of $KgM/s^2$, and is a derived quantity. One $KgM/s^2$ is called a Newton (N). If a constant force of one Newton is applied to an object whose inertial mass is 1 Kg, the object have a constant acceleration of 1 $M/s^2$. In the cgs system, mass has units of grams, and one $gm(cm/s^2)$ equals one dyne. For any single constant force, $F$, acting on an object of mass $m$, the acceleration is $a = F/m$ More force produces more acceleration, more mass results in less acceleration.

**Do forces add like vectors when an object accelerates?**

We need to do one more experiment to see if "dynamic" forces add like vectors. If they do, then we can quantify force using $F = ma$. Suppose an object is subject to two constant forces at once. Call them $F_1$ and $F_2$. Since we are still experimenting in one dimension, the forces can act towards the right (+) or left (-) direction. Forces
to the right will be considered positive and to the left negative. When both forces are applied at the same time, here is what the experiment shows:

**Experiment 3:** If a constant force $F_1$ produces an acceleration $a_1$ on an object and a constant force $F_2$ produces an acceleration $a_2$ on the same object, then if both forces $F_1 + F_2$ act on the object at the same time the measured acceleration is $a_1 + a_2$.

Wow, we should be grateful that nature behaves is such a simple way. The experiment indicates that if a 5 Newton force to the right and a 3 Newton force to the left are both applied to an object, the resulting motion is the same as if a 2 Newton force to the right were applied. Forces applied in one dimension add up like real numbers. (As discussed in the next section, in two and/or three dimensions experiment shows they add up like vectors). We refer to the "sum" of all the forces on an object as the net force, and give it the label $F_{\text{net}}$.

**Two and Three Dimensional Frictionless Environment**

The extension to two and three dimensions is greatly facilitated by using the mathematics of vectors. Displacement is a vector, since displacements have the properties that vectors need to have. A displacement 40 units east plus a displacement 30 units north is the same as one displacement 50 units at an angle of $36.869\ldots^\circ$ N of E. Similarly, relative velocity is a vector, since it is the time derivative of displacement. We have demonstrated that momenta combine as vectors. So the extension of force and motion to two and three dimensions follows:

$$F_{\text{net}} = \frac{d\vec{p}}{dt} = m\vec{a}$$

for a single particle. This equation is referred to as Newton's Second Law of motion. Although force is not needed to describe the fundamental interactions of nature, force is a useful quantity in statics, and when dealing with contact interactions (i.e. contact forces and friction). The three experiments have verified the consistency and usefulness of equating force with mass times acceleration.

In this mechanics class, we will deal with the contact forces of surfaces, ropes, human pushes or pulls, and the non-contact force of gravity near a planet’s surface.
The general approach is as follows: (1) determine all the forces acting on an object. (2) then find the vector addition of the forces, the net force. (3) once the net force is known, the acceleration of the object is $\vec{a} = \vec{F}_{\text{net}}/m$. From the acceleration, the position function $x(t)$ can be found by integration. Doing appropriate experiments we can test our understanding of the "physics" behind the interactions. Before we proceed with investigating different forces, let’s revisit Newton’s Third Law.

Revisiting Newton’s Third Law: symmetry in interactions

A small tack is in the vicinity of a huge strong magnet. The tack feels a strong attraction towards the magnet. Does the massive magnet feel a different attraction due to the little tack? We demonstrated before that the change in momentum is the same for each object. Now with our dynamic definition of force, we can restate Newton’s Third Law. Since the net force equals $d\vec{p}/dt$, we see that the magnitude of the force that each object experiences is the same. Let $\vec{F}_{12}$ be the force that object "1" exerts on object "2", and $\vec{F}_{21}$ be the force that object "2" exerts on object "1". Then,

\[ \vec{F}_{12} = -\vec{F}_{21} \] (12)

In the case of the tack and magnet, if the tack is attracted towards the magnet, the magnet is attracted towards the tack. These forces are on different objects and are opposite in direction: $\vec{F}_{\text{tack} \rightarrow \text{magnet}} = -\vec{F}_{\text{magnet} \rightarrow \text{tack}}$. Since acceleration = force/mass, the magnet will have a much smaller acceleration than the tack since its mass is so much greater. This equality of interacting forces is verified by experiment and is called "Newton’s Third Law".

It was a great insight of Newton to realize there was a certain symmetry in every interaction. He probably reasoned that something must be the same for each object. Interacting objects clearly can have different masses and different accelerations. The only quantity left are the forces that each object "feels". Nature is fair when it comes to interacting particles, object 2 is not preferred to object 1 when it comes to the force that each feels. Forces always come in pairs. Whenever there is a force on a particle, there must be another force acting on another particle.

Newton’s Third Law applies (in some form) to every type of interaction. It is true if the objects are moving or if they are not. It is true even in the static case. As
simple as it may seem, it is often miss-understood. Consider the example of a book resting on a table in a room. The book feels a gravitational force due to the earth, which is it’s weight. What is the paired force for the book’s weight? Most students answer is ”the force of the table on the book”. The table does exert a force on the book equal to it’s weight, but it is not the ”reaction” force to the book’s weight. The ”reaction” force to the book’s weight is the force on the earth due to the book. To determine the two forces that are ”paired”, just replace ”object 1” with one object and ”object 2” with the other in the statement above. ”If the earth exerts force on the book, then the book exerts an equal but opposite force on the earth.”

Some Simple Forces

Newton’s laws of motion give us a method for determining the resulting motion when an object is subject to forces. We proceed by identifying all the forces, and then add them up using vector addition to find the objects acceleration. Once the acceleration is known at all times (or all positions) then the motion is determined. The beauty of this approach is that the forces take on a simple form. That is, the quantity that affects the acceleration of an object (the thing we are calling a force) turns out to be a simple expression of position, velocity, etc. Here, we consider three types of forces: contact forces, frictional forces, and weight (gravity near the surface of a planet). In future courses we will consider the ”universal” gravitational force, spring forces, electric and magnetic forces, and atomic/nuclear/subatomic particle interactions.

Contact Force Example:

As previously stated, by a contact force we mean the pushing or pulling caused by the touching (or physical contact) of one object on another. Someone’s hand pushing on, or a rope pulling on an object are some examples. From Newton’s third law, object 2 feels the same contact force from object 1 that object 1 feels from the contact force from object 2.

Consider the following example shown in the figure:

Two masses in an inertial reference frame are connected by a rope. The inertial mass of the mass on the left is $M_1$, the inertial mass of the mass on the right is $M_2$, and the rope connecting them has a mass of $m$. Someone pulls on the mass on the right with a force of $F$ Newtons. Question: find the resulting motion, and all the contact forces.
\[ c_1 = M_1 a \]
\[ F - c_2 = M_2 a \]
\[ a = \frac{F}{M_1 + M_2 + m} \]
\[ c_2 = F - M_2 a = \frac{M_1 F}{M_1 + M_2 + m} \]

If \( m \approx 0 \), then
\[ c_2 \approx \frac{M_1 F}{M_1 + M_2} \]
\[ c_1 \approx \frac{M_1 F}{m + M_2} \]
The method of applying Newton’s Laws of motion to a system of particles is: first: find the forces on each object in the system, then: the acceleration of each object is just \( F_{\text{Net}}/(\text{mass}) \). For the example above, the forces on the ”right mass” are \( F \) minus the force that the rope pulls to the left, which we label as \( c_2 \). The rope feels the ”reaction” force \( c_2 \) to the right and a force \( c_1 \) from the left mass. Finally, the mass on the left feels only the ”reaction” force from the rope which is \( c_1 \) to the left. Summarizing:

<table>
<thead>
<tr>
<th>Object</th>
<th>Net Force</th>
<th>Equation of motion</th>
</tr>
</thead>
<tbody>
<tr>
<td>left mass</td>
<td>( c_1 )</td>
<td>( c_1 = M_1 a )</td>
</tr>
<tr>
<td>rope</td>
<td>( c_2 - c_1 )</td>
<td>( c_2 - c_1 = ma )</td>
</tr>
<tr>
<td>right mass</td>
<td>( F - c_2 )</td>
<td>( F - c_2 = M_2 a )</td>
</tr>
</tbody>
</table>

Adding up the equations of motion for the various masses gives \( a = F/(M_1 + M_2 + m) \). Solving for the contact forces gives: \( c_1 = M_1 F/(M_1 + M_2 + m) \), and \( c_2 = (M_1 + m)F/(M_1 + M_2 + m) \). If the mass \( m \) is very small compared to \( M_1 \) and \( M_2 \), then the contact forces are approximately equal \( c_1 \approx M_1 F/(M_1 + M_2) \), and \( c_2 \approx M_1 F/(M_1 + M_2) \), giving \( c_1 \approx c_2 \). This force is called the tension in the rope, and is the same throughout for a massless rope.

Weight

Weight is a force. In the Newtonian picture of gravity, the weight \( W \) of an object is the gravitational force on the object due to all the other matter in the universe. We will be considering the weight of objects on or near the surface of a planet (neglecting the rotation of the planet). In this case, the strongest force the object experiences is due to the planet. A remarkable property of nature is that the motion of all objects due to the gravitational force is not dependent on the objects inertial mass! This is true for objects falling in the classroom, satellites orbiting the earth, planets orbiting a star, etc. We will demonstrate this property in lecture by dropping two different objects with different inertial masses. They both fall with the same acceleration, which we label as \( g \). Since \( F = ma \), the gravitational force on an object must be \( W = mg \), where \( g \) depends only on the location of the object and \( m \) is the inertial mass of the object. An objects weight is proportional to it’s inertial mass. Since \( g = W/m \) is the same for all objects, if the mass is doubled, so is the objects weight.

To measure an objects weight on a planet, one can use a scale which keeps the object at rest. Since the object is not accelerating relative to the planet, the force the
scale exerts on the object equals its weight. An object’s mass is an intrinsic property and is the same everywhere. An object’s weight $W$ depends on its location (i.e. which planet it is on or near).

The Einstein picture of gravity is somewhat different. Being in a free falling elevator near the earth’s surface ”feels” the same as if you were floating in free space or in the space shuttle. In each case you are weightless. Thus, you can be near the surface of the earth and be ”weightless”. Likewise, if you are in a rocket ship in free space (outer space far away from any other objects) that is accelerating at $9.8 \text{m/s}^2$ you feel the same as if the rocket were at rest on the earth. Thus, if you sit on a scale in a rocket that is accelerating in free space, the scale will give you a ”weight” reading eventhough there are no ”gravitational forces”. Weight therefore is a relative quantity, and depends on the reference frame. In an inertial reference frame, everything is weightless. In a non-inertial reference frame, the force needed to keep the object at rest relative to the frame is the weight (or apparent weight) of the object. Mass (more specifically rest mass), on the other hand, is an absolute intrinsic quantity and is the same everywhere. Students interested in these philosophical topics should major in Physics. In this introductory class, we will take the Newtonian point of view for objects near the surface of a planet in which case the weight $W$ is:

$$W = mg$$

Next quarter you will discover what Newton discovered, that the gravitational force between two point objects of mass $m_1$ and $m_2$ that are separated by a distance $r$ is: $F_{\text{gravity}} = Gm_1m_2/r^2$. You will also show that the acceleration near the surface of a spherically symmetric planet is approximately $g = Gm_{\text{planet}}/R^2$, where $R$ is the planet’s radius and $G \approx 6.67 \times 10^{-11} \text{N m}^2/\text{kg}^2$.

**Projectile Motion**

Projectile motion is often used as an example in textbooks, and is the motion of an object ”flying through the air” near the surface of the earth (or any planet). The approximations that are made are 1) that the object is near enough to the surface to consider the surface as flat, 2) the acceleration due to gravity is constant (does not change with height), and 3) air friction is neglected. Usually the $+\hat{j}$ direction is taken as up, and the $\hat{i}$ direction parallel to the surface of the earth such that the object travels in the $x - y$ plane. The object’s acceleration is constant and given by $a_0 = -g\hat{j}$ for all objects. Letting $\vec{v}(t)$ represent the objects velocity vector and $\vec{r}(t)$ be the objects position vector we have:
\[ \vec{v}(t) = \vec{v}_0 - gt\hat{j} \]  

and

\[ \vec{r}(t) = -\frac{gt^2}{2}\hat{j} + \vec{v}_0 t + \vec{r}_0 \]  

where \( \vec{v}_0 \) is the initial velocity and \( \vec{r}_0 \) is the initial position. If \( \vec{r}_0 = 0 \) and \( \vec{v}_0 = v_0\cos(\theta)\hat{i} + v_0\sin(\theta)\hat{j} \) one has:

\[ \vec{v}(t) = v_0\cos(\theta)\hat{i} + (v_0\sin(\theta) - gt)\hat{j} \]  

and for the position vector, one has:

\[ \vec{r}(t) = v_0\cos(\theta)t\hat{i} + (v_0\sin(\theta)t - \frac{gt^2}{2})\hat{j} \]  

It is nice that the horizontal and vertical motions can be treated separately. This is because force is a vector and the only force acting on the particle is gravity which is in the vertical direction. The seemingly complicated two-dimensional motion is actually two simple one-dimensional motions.

In this example of projectile motion, two quantities remain constant: the acceleration \(( -g\hat{j} )\) and the x-component of the velocity. The x-component of the velocity is constant since there is no force in the x-direction and consequently no acceleration in the x-direction. Also note that vectors \( \vec{r}, \vec{v}, \) and \( \vec{a}_0 \) can (and usually) point in different directions.

On the earth, there is air friction which needs to be considered for an accurate calculation. Also, even in the absence of air friction, the parabolic solution above is not exactly correct. In the absence of friction, the path of a projectile near a spherical planet is elliptical.

**Frictional Forces**

In this class we consider ”contact” frictional forces. When two surfaces touch each other, one surface exerts a force on the other (and visa-versa by Newton’s third law). It is convenient to ”break-up” this force into a component perpendicular to the surfaces (Normal Force) and a component parallel to the surfaces (Frictional Force). We consider two cases for the frictional force: 1) the two surfaces slide across each other (kinetic friction) and 2) the two surfaces do not slide (static friction).
**Kinetic Friction**

If two surfaces slide across each other, the frictional force depends primarily on two things: how much the surfaces are pushing against each other (normal force $N$) and the type of material that makeup the surfaces. We will show in class that the kinetic frictional force is roughly proportional to the force pushing the surfaces together (normal force $N$), or $F_{\text{kinetic friction}} \propto N$. We can change the proportional sign to an equal sign by introducing a constant: $F_{\text{kinetic friction}} \approx \mu_K N$. The coefficient $\mu_K$ is called the coefficient of kinetic friction and depends on the material(s) of the surfaces. The direction of kinetic friction is opposite to the direction of the velocity.

**Static Friction**

If the surfaces do not slide across each other, the frictional force (parallel to the surfaces) is called static friction. The static frictional force will have a magnitude necessary to keep the surfaces from sliding. If the force necessary to keep the surfaces from sliding is too great for the frictional force, then the surfaces will slip. Thus, there is a maximum value $F_{\text{Max}}$ for the static friction: $F_{\text{static friction}} \leq F_{\text{Max}}$. As in the case of sliding friction, $F_{\text{Max}}$ will depend primarily on two things: the normal force $N$ and the type of materials that make-up the surfaces. We will also show in class the $F_{\text{Max}}$ is roughly proportional to the normal force, $F_{\text{Max}} \propto N$. Introducing the coefficient of static friction, $\mu_S$, we have: $F_{\text{Max}} = \mu_S N$. The static friction force will only be equal to $F_{\text{Max}}$ just before the surfaces start slipping. If the surfaces do not slip, $F_{\text{static friction}}$ will be just the right amount to keep the surfaces from slipping. Thus, one usually writes that $F_{\text{static friction}} \leq \mu_S N$.

Summarizing we have:

$$F_{\text{kinetic friction}} = \mu_K N \quad (18)$$

and for static friction

$$F_{\text{static friction}} \leq \mu_S N \quad (19)$$

We remind the reader that the above equations are not fundamental "Laws of Nature", but rather models that approximate the forces of contact friction. The fundamental forces involved in contact friction are the electromagnetic interactions between the atoms and electrons in the two surfaces. To determine the frictional forces from these fundamental forces is complicated, and we revert to the phenomenological models described above.

It is interesting to note that in the case of kinetic friction, the net force that the surface exerts on the object is always at angle equal to $\tan^{-1}(\mu_K)$ with respect to the
normal. For the case of static friction, the net force that the surface can exert on the object must be at an angle less than $\tan^{-1}(\mu_s)$ with respect to the normal.

**Uniform Circular Motion**

If an object travels with constant speed in a circle, we call the motion uniform circular motion. The uniform meaning constant speed. This motion is described by two parameters: the radius of the circle, $R$, and the speed of the object, $v$. The speed of the object is constant, but the direction of the velocity is always changing. Thus, the object does have an acceleration. We can determine the acceleration by differentiating the position vector twice with respect to time. For uniform circular motion, the position vector is given by:

$$\vec{r}(t) = R(\cos(\frac{vt}{R})\hat{i} + \sin(\frac{vt}{R})\hat{j})$$ (20)

where $\hat{i}$ points along the +x-direction and $\hat{j}$ points along the +y-direction. It is also convenient to define a unit vector $\hat{r}$ which points from the origin to the particle:

$$\hat{r}(t) = (\cos(\frac{vt}{R})\hat{i} + \sin(\frac{vt}{R})\hat{j})$$ (21)

In terms of $\hat{r}$, the position vector $\vec{r}$ can be written as:

$$\vec{r}(t) = R\hat{r}(t)$$ (22)

To find the velocity vector, we just differentiate the vector $\vec{r}(t)$ with respect to $t$:

$$\vec{v}(t) = \frac{d\vec{r}}{dt}$$ (23)

$$\vec{v}(t) = \frac{d}{dt}(R\hat{r}(t)) = R\frac{v}{R}(-\sin(\frac{vt}{R})\hat{i} + \cos(\frac{vt}{R})\hat{j})$$ (24)

$$\vec{v}(t) = v(-\sin(\frac{vt}{R})\hat{i} + \cos(\frac{vt}{R})\hat{j})$$ (25)

Note that $|\vec{v}| = v$ since $\sin^2 + \cos^2 = 1$.

To find the acceleration of an object moving in uniform circular motion one needs to differentiate the velocity vector $\vec{v}(t)$ with respect to $t$:

$$\vec{a}(t) = \frac{d\vec{v}}{dt}$$ (26)

17
\[\begin{align*}
\vec{a}(t) &= -\frac{v^2}{R} \hat{r} \\
\end{align*}\]

Thus for an object moving in uniform circular motion, the magnitude of the acceleration is \(|\vec{a}| = \frac{v^2}{R}\), and the direction of the acceleration is towards the center of the circle.

In an inertial reference frame, net force equals mass times acceleration. Thus, if an object is moving in a circle of radius \(R\) with a constant speed of \(v\), the net force on the object must point towards the center and have a magnitude of \(mv^2/R\).

Uniform circular motion is another case in which the three vectors \(\vec{r}\), \(\vec{v}\), and \(\vec{a}\) do not point in the same direction. In this case \(\vec{a}\) points in the opposite direction from \(\vec{r}\), and \(\vec{v}\) is perpendicular to both \(\vec{r}\) and \(\vec{a}\).
Mechanical Energy

The "force-acceleration" approach is helpful in understanding interactions in "classical" mechanics and electromagnetism. However, there are other approaches for analyzing interacting particles. In the next few weeks, we consider another approach which deals with the energy and momentum of objects. We will discuss how energy and momentum are related to an object's mass and velocity, and at the same time how the laws of physics can be expressed in terms of energy and momentum. The "force-acceleration" and the "energy-momentum" formalisms contain the same physics. Sometimes one approach is better than the other in analyzing a particular system.

An Example to motivate Mechanical Energy

Consider dropping an object of mass \( m \) from rest from a height \( h \) above the earth's surface. Let \( h \) be small compared to the earth's radius so we can approximate the gravitational force on the object as constant, \( m\vec{g} \). The object will fall straight down to the ground. Let \( y \) be the object's height above the surface (+ being up) and \( v \) be the object's speed. As the object falls, it's speed increases and its height decreases. That is, \( y \) decreases while \( v \) increases. Question: is there any combination of the quantities \( y \) and \( v \) that remain constant?

We can answer this question, since we know how speed \( v \) will change with height \( y \). If the gravitational force doesn't change, the object's acceleration is constant, and is equal to \(-g\). The minus sign is because we are choosing the up direction and positive. Let \( y_i \) and \( v_i \) be the height and speed at time \( t_i \), and let \( y_f \) and \( v_f \) be the height and speed at time \( t_f \). See Figure 1. From the formula for constant acceleration:

\[
v_f^2 = v_i^2 + 2(-g)(y_f - y_i) \tag{29}
\]

This equation can be written as:

\[
\frac{v_f^2}{2} + gy_f = \frac{v_i^2}{2} + gy_i \tag{30}
\]

Wow! This is a very nice result. Since \( t_i \) and \( t_f \) could be any two times, the quantity \( v^2/2 + gy \) does not change as the object falls to the earth! As the object falls, \( y \) decreases and \( v \) increases, but the combination \( v^2/2 + gy \) does not change. As we shall see, it is convenient to multiply this expression by the mass of the object \( m \). Since \( m \) also does not change during the fall, we have
\[ m \frac{v^2}{2} + mgy = \text{constant} \quad (31) \]

while an object falls straight down to earth. In physics when a quantity remains constant in time, we say that the quantity is conserved. The quantity that is conserved in this case is the mechanical energy. The first term, \( m \frac{v^2}{2} \), depends on the particle’s motion and is called the kinetic energy. The second term, \( mgy \), depends on the particle’s position and is called the potential energy. In these terms, we can say that the sum of the object’s kinetic energy plus its potential energy remains constant during the fall.

Can the energy considerations for this special case of an object falling straight down be generalized to other types of motion? Yes it can. The situation is a little more complicated in two and three dimensions since the direction of the net force is not necessarily in the same direction as the object’s motion (\( \vec{v} \)). Let’s first try another simple case: a block sliding without friction down an inclined plane. Let the plane’s surface make an angle of \( \theta \) with the horizontal, and let the block slide down the plane a distance \( d \) as shown in Figure 2. The net force down the plane is the component of gravity parallel to the plane: \( mgsin\theta \). So the block’s acceleration down the plane is \( a = \frac{(mgsin\theta)}{m} = gsin\theta \). If the initial speed is \( v_i \) and the final speed \( v_f \), we have

\[
\begin{align*}
    v_f^2 &= v_i^2 + 2ad \\
    v_f^2 &= v_i^2 + 2g(sin\theta) \, d
\end{align*} \quad (32) \quad (33)
\]

since the acceleration is constant. However, we note from Figure 2 that \( dsin\theta \) is just \( y_i - y_f \). With this substitution we have

\[
\begin{align*}
    v_f^2 &= v_i^2 + 2g(y_i - y_f) \quad (34) \\
    v_f^2 + 2gy_f &= v_i^2 + 2gy_i \quad (35)
\end{align*}
\]

Wow, this is the same equation we had before. Multiplying both sides by \( m \) and dividing by 2, we obtain

\[
\frac{mv_f^2}{2} + mgy_f = \frac{mv_i^2}{2} + mgy_i \quad (36)
\]

Thus, as the block slides without friction down the plane, the quantity \( (mv^2)/2 + mgy \) remains constant.
Figure 1

\[
\begin{align*}
\gamma_i - \gamma_f &= \begin{cases} 
u_i^2 = \nu_i^2 + 2g (\gamma_i - \gamma_f) \\
\gamma_f^2 &= \nu_f^2 + 2g \gamma_f \\
\gamma_i^2 + 2g \gamma_f &= \nu_i^2 + 2g \gamma_i \\
m \frac{\gamma_f^2}{2} + mg \gamma_f &= m \nu_i^2 + mg \gamma_i
\end{cases}
\end{align*}
\]

Figure 2

\[
a = \frac{mg \sin \theta}{m}
\]

\[
a = g \sin \theta
\]

\[
\gamma_i - \gamma_f = \begin{cases} 
\nu_i^2 = \nu_i^2 + 2a \sin \theta \cdot d \\
\nu_f^2 &= \nu_i^2 + 2g \sin \theta \cdot d \\
\gamma_i^2 + 2g \gamma_f &= \nu_i^2 + 2g (\gamma_i - \gamma_f)
\end{cases}
\]
Is this a general result for gravity near the surface of the earth for any kind of motion? A generalization of this analysis is facilitated by using a mathematical operation with vectors: the "dot" (or "scalar") product. Let’s summarize this vector operation, then see how it helps describe the physics.

Vector Scalar Product

The vector scalar product is an operation between two vectors that produces a scalar. Let \( \vec{A} \) and \( \vec{B} \) be two vectors. We denote the scalar product as \( \vec{A} \cdot \vec{B} \). There are a number of ways to derive a scalar quantity from two vectors. One can use \(|\vec{A}|\) and \(|\vec{B}|\). However, if we define the scalar product as the product of the magnitudes, then the angle does not play a role. If we call \( \theta \) the angle between the vectors, then we could use \( \cos(\theta) \) or \( \sin(\theta) \) in our definition. If we want \( \vec{A} \cdot \vec{B} \) to equal \( \vec{B} \cdot \vec{A} \) then we need a trig function that is symmetric in \( \theta \). Since \( \cos(-\theta) \) equals \( \cos(\theta) \) it is the best choice. The scalar product is defined as

\[
\vec{A} \cdot \vec{B} \equiv |\vec{A}||\vec{B}|\cos(\theta) \tag{37}
\]

where \( \theta \) is the angle between the two vectors.

The scalar product can be negative, positive, or zero. It is the magnitude of \( \vec{A} \) times the component of \( \vec{B} \) in the direction of \( \vec{A} \). If the vectors are perpendicular, then the scalar product is zero. The scalar products between the unit vectors are:

\[
\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0 \tag{38}
\]

and

\[
\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1 \tag{39}
\]

If the vectors are expressed in terms of the unit vectors (i.e. their components), \( \vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k} \) and \( \vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k} \), the distributive property of the scalar product gives

\[
\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z \tag{40}
\]

Work-Energy Theorum

Newton’s second law is a vector equation, it relates the acceleration of an object to the net force it experiences, \( \vec{F}_{net} = m\vec{a} \). It is also interesting to consider how a scalar
dynamical quantity changes in time and/or position. A useful quantity to consider is the speed of an object squared, \( v^2 = \vec{v} \cdot \vec{v} \). The time rate of change of \( v^2 \) is:

\[
\frac{dv^2}{dt} = \frac{d(\vec{v} \cdot \vec{v})}{dt} = \frac{d\vec{v}}{dt} \cdot \vec{v} + \vec{v} \cdot \frac{d\vec{v}}{dt} = \vec{a} \cdot \vec{v} + \vec{v} \cdot \vec{a}
\]

\[
\frac{dv^2}{dt} = 2\vec{v} \cdot \vec{a}
\]

The result above is strictly a mathematical formula. The last line states that the increase in the speed squared is proportional to the component of the acceleration in the direction of the velocity. This makes sense. If there is no component of \( \vec{a} \) in the direction of \( \vec{v} \) then the objects speed does not increase. For example in uniform circular motion, \( \vec{v} \) and \( \vec{a} \) are perpendicular and the speed does not change.

Now comes the physics. Newton’s second law relates the acceleration to the Net Force, \( \vec{a} = \vec{F}_{net}/m \):

\[
\frac{dv^2}{dt} = 2\vec{v} \cdot \vec{a} = 2\vec{v} \cdot \frac{\vec{F}_{net}}{m}
\]

Multiplying both sides by \( m/2 \) and rearranging terms gives

\[
\vec{F}_{net} \cdot \vec{v} = \frac{d(mv^2/2)}{dt}
\]

For an infinitesimal change in time, \( \Delta t \), we have

\[
\vec{F}_{net} \cdot (\vec{v} \Delta t) = \Delta \left( \frac{mv^2}{2} \right)
\]

However, \( \vec{v} \Delta t \) is the displacement of the object in the time \( \Delta t \), which we will call \( \Delta \vec{r} = \vec{v} \Delta t \). Substituting into the equation gives:

\[
\vec{F}_{net} \cdot \Delta \vec{r} = \Delta \left( \frac{mv^2}{2} \right)
\]
This is a very nice equation. It says that the (component of the net force in the direction of the motion) times the displacement equals the change in the quantity $mv^2/2$. The quantity on the right side, $mv^2/2$, and the quantity on the left side, $\vec{F}_{\text{net}} \cdot \Delta \vec{r}$, are special and have special names. $mv^2/2$ is called the kinetic energy of the object. $\vec{F}_{\text{net}} \cdot \Delta \vec{r}$ is called the Net Work. Both are scalar quantities, and the equation is a result of Newton’s second law of motion. The above equation is the infinitesimal version of the work-energy theorem.

One can integrate the above equation along the path that an object moves. This involves subdividing the path into a large number $N$ of segments. If $N$ is large enough, the segments are small enough such that $\Delta \vec{r}$ lies along the path. Adding up the results of the above equation for each segment gives:

$$\int \vec{F}_{\text{net}} \cdot d\vec{r} = \int d\left(\frac{mv^2}{2}\right)$$

If the initial position is $\vec{r}_i$ and the final position is $\vec{r}_f$, we have

$$\int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_{\text{net}} \cdot d\vec{r} = \frac{mv_f^2}{2} - \frac{mv_i^2}{2}$$

This equation is called the work-energy theorem. It is true for any path the particle takes and derives from Newton’s second law. Note that there is no explicit time variable in the equation. The work-energy theorem does not directly give information regarding the time it takes for the object to move from $\vec{r}_i$ to $\vec{r}_f$, nor does it give information about the direction of the object. The equation relates force and the distance through which the net force acts on the object to the change in the objects (speed) squared. It is a scalar equation which contains the physics of the vector equation $\vec{F}_{\text{net}} = m\vec{a}$. Since scalar quantities do not have any direction, it is often easier to analyze scalar quantities. From the work-energy theorem we discover two important scalar quantities, $mv^2/2$ and $\vec{F}_{\text{net}} \cdot \Delta \vec{r}$, and their relationship to each other. One is called the kinetic energy and the other the Net Work. Many situations for which this equation is applicable will be discussed in lecture.

**Work done by a particular Force**

Although the work-energy theorem applies to the Net Work done on an object, it is often useful to consider the work done by a particular force. The word “work” by itself is vague. To talk about work, one needs to specify two things:
1. The **force** (or net force) for the work you are calculating.

2. The **path** that the particle is traveling.

The work done by a particular force $\vec{F}_1$ for a particle moving along a path that starts at location $a$ and ends at location $b$ is written as:

$$W_{a\rightarrow b} \equiv \int_{a}^{b} \vec{F}_1 \cdot d\vec{r}$$

(46)

The integral on the right side of this equation is a "line integral". Generally, line integrals can be complicated. However, in this course we will only consider line integrals that are "easy" to evaluate. We will limit ourselves to simple situations: a) Forces that are constant, and b) for the case of forces that change with position: paths that are in the same direction as the force, and paths that are perpendicular to the direction of the force. In the later case, we will show that the work is zero.

Basically, work is the product of the (force in the direction of the motion) times (the distance the force acts). For a constant force $F$ acting through a distance $d$, which is in the *same direction* as the force, the work is simply $W = Fd$. The units of work are (force)(distance): Newton-meter, foot-pound, dyne-cm. The work done by a constant force $\vec{F}$ acting through a straight line displacement $\vec{d}$ is just $W = \vec{F} \cdot \vec{d}$.

Remember that **Work is a scalar quantity**.

For some forces, the work done only depends on the initial and final locations, and not on the path taken between these two locations. We will demonstrate this by an important example: the work done by the gravitational force near the earth's surface. Here we will assume that the force is constant and equal to $m\vec{g}$. Consider the curved path shown in Figure 3. The method of carrying out the integral is to divide up the curved path into small straight segments, which we label as $\Delta \vec{r}_i$. The work done by the gravitational force from location $a$ to location $b$ along the path shown is given by

$$W_{a\rightarrow b} = \lim_{|\Delta \vec{r}_i| \to 0} \sum_i m\vec{g} \cdot \Delta \vec{r}_i$$

(47)

Since the force $m\vec{g}$ is the same for each segment, it is a constant and can be removed from the sum:

$$W_{a\rightarrow b} = m\vec{g} \cdot \lim_{|\Delta \vec{r}_i| \to 0} \sum_i \Delta \vec{r}_i$$

(48)

Using the "tail-to-tip" method of adding vectors, the sum on the right side is easy. As seen in the figure, adding the $\Delta \vec{r}_i$ tail to tip just gives the vector from location a...
$\gamma_a - \gamma_b = |D| \cos \alpha$

$W_{a \rightarrow b} = m \vec{g} \cdot \Delta \vec{r}_i + m \vec{g} \cdot \Delta \vec{r}_2 + \ldots$

$= \sum_{i=1}^{\infty} m \vec{g} \cdot \Delta \vec{r}_i$

$= m \vec{g} \cdot \left( \sum_{i=1}^{\infty} \Delta \vec{r}_i \right) = m \vec{g} \cdot \vec{D}$

$= mg |D| \cos \alpha$

$W_{a \rightarrow b} = mg (\gamma_a - \gamma_b)$ for any path
to location b, which is $\vec{D}$ in the figure.

$$W_{a\rightarrow b} = m\vec{g} \cdot \vec{D}$$  \hspace{1cm} (49)

We would get this result for any path taken from a to b. From Figure 3 we can see that $m\vec{g} \cdot \vec{D}$ is just $mg|\vec{D}|\cos\alpha$. However, $|\vec{D}|\cos\alpha = y_a - y_b$. So the work done by the gravitational force from $a \rightarrow b$ is just

$$W_{a\rightarrow b} = mg(y_a - y_b)$$  \hspace{1cm} (50)

We will obtain this same result for any path the particle moves on from $a \rightarrow b$. We say that the work done by the gravitational force (near the surface) is "path-independent", and only depends on the initial and final positions of the particle. Along the path, the "component of the gravitational force" in the direction of the motion can change. However, when added up for the whole path, $W_{a\rightarrow b}$ is simply $mg(y_a - y_b)$. Forces that have this "path-independent" property are called conservative forces. Conservative forces lead to conserved quantities, which we discuss next.

**Potential Energy and Conservation of Mechanical Energy**

Often one can identify different forces that act on an object as it moves along its path, and the net force is the sum of these forces: $\vec{F}_{\text{net}} = \vec{F}_1 + \vec{F}_2 + ...$. For example the forces $\vec{F}_j$ can be the weight or gravitational force, the force a surface exerts on an object, an electric or magnetic force, etc. The net work on an object will be the sum of the work done by the different forces acting on the object:

$$\int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_{\text{net}} \cdot d\vec{r} = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_1 \cdot d\vec{r} + \int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_2 \cdot d\vec{r} + ...$$  \hspace{1cm} (51)

or letting $W_j$ represent the work done by the force $j$:

$$W_{\text{net}} = W_1 + W_2 + ...$$  \hspace{1cm} (52)

Note this equation motivates us to examine the work done by one force as it acts along a particular path from the position $\vec{r}_i$ to $\vec{r}_f$. We first consider the work done by particular forces, then we will discover an energy conservation principle.

**Work done by forces that only change direction**

Forces that always act perpendicular to the object’s velocity will do no work on the object. Since there is no component of force in the direction of the motion, these
forces can only change the direction but not increase the speed (kinetic energy) of the object. The tension in the rope of a simple pendulum is an example. If the rope is fixed at one end and doesn’t stretch, the tension is always perpendicular to the motion of the swinging ball. If an object slides along a frictionless surface that does not move, the force that the surface exerts, (normal force) is always perpendicular to the motion. If there is friction, the force the surface exerts on an object is often ”broken up” into a part normal to the surface and one tangential to the surface. The part of the force ”normal” to the surface will not do any work on the object. The tangential part (friction) will do work on the object. Note: if the surface moves, the normal force can do work.

**Work done by a constant force**

We will consider a particular constant force, the weight of an object near the earth’s surface. Our results will apply generally to any force that is constant in space and time. Also, in our discussion we will consider ”up” as the ”+y” direction. With this notation, the force of gravity is $\mathbf{W}_g = -mg\mathbf{j}$. Consider any arbitrary path from an initial position $\mathbf{r}_i = x_i\mathbf{i} + y_i\mathbf{j}$ to a final position $\mathbf{r}_f = x_f\mathbf{i} + y_f\mathbf{j}$. To calculate the work done by $\mathbf{W}_g$ from the initial to the final position, we divide up the path into a large number $N$ small segments. We label one segment as $\Delta\mathbf{r}$. In terms of the unit vectors, we can write $\Delta\mathbf{r} = \Delta x\mathbf{i} + \Delta y\mathbf{j}$. The work, $\Delta W_g$, done by the force of gravity for this segment is

$$\Delta W_g = (-mg\mathbf{j}) \cdot (\Delta x\mathbf{i} + \Delta y\mathbf{j})$$

which is

$$\Delta W_g = -mg\Delta y$$

The work done by the force of gravity for this segment does not depend on $\Delta x$. This makes sense, since the force acts only in the ”y” direction. If all the work done by the gravitational force for all the segments of the path are added up, the result will be

$$W_g = -mg \sum \Delta y = -mg(y_f - y_i)$$

$$W_g = mgy_i - mgy_f$$
This result is true for any path the object takes. That is, the work done by a constant gravitational force depends only on the initial and final heights of the path. If the work done by a force depends only on the initial and final positions of the path, and not the path itself, the force is called a conservative force. The force $\vec{F}_g = -mg\hat{j}$ is a conservative force. We see that the work done by the gravitational force $\vec{F}_g$ equals the difference in the function $U_g(r) = mgy$ of the initial and final positions:

$$\int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_g \cdot d\vec{r} = U_g(\vec{r}_i) - U_g(\vec{r}_f)$$

The function $U_g(r)$ is called the potential energy function for the constant gravitational force (i.e. gravity near a planet’s surface).

In general, whenever the work done by a force depends only on the initial and final positions, and is the same for any path between these starting and ending positions, the work can be written as the difference between the values of some function evaluated at the initial, $i$, and final, $f$, positions:

$$\int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{r} = U(\vec{r}_i) - U(\vec{r}_f)$$

This equation is the defining characteristic of conservative forces: the work done by a conservative force from the position $\vec{r}_i$ to the position $\vec{r}_f$ equals the difference in a function $U(\vec{r})$, $U(\vec{r}_i) - U(\vec{r}_f)$. The function $U(\vec{r})$ is called the potential energy function, and will depend on the force $\vec{F}$. Every force is not necessarily a conservative one. For example, the contact forces of friction, tension, and air friction are not conservative. The work done by these forces are not equal to the difference in a potential energy function.

Note that the potential energy function is not unique. An arbitrary constant can be added to $U(\vec{r})$ and the difference $U(\vec{r}_i) - U(\vec{r}_f)$ is unchanged. That is, if $U(\vec{r})$ is the potential energy function for some force, then $U''(\vec{r}) = U(\vec{r}) + C$ is also a valid potential energy function:

$$\int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_g \cdot d\vec{r} = U_g(\vec{r}_i) - U_g(\vec{r}_f)\]

$$= (U_g(\vec{r}_i) + C) - (U_g(\vec{r}_f) + C)$$

$$= U_g'(\vec{r}_i) - U_g'(\vec{r}_f)$$

The arbitrary constant $C$ is chosen so that the potential energy is zero at some reference point. For the case of the constant gravitational force, the reference point...
of zero potential energy is usually where one chooses $y = 0$, i.e. $U_g(\vec{r}) = mgy$.

To conclude this section, we mention the potential energy functions for some of the conservative forces that you will encounter during your first year of physics.

*Linear restoring force (ideal spring)*

For an ideal spring, the force that the spring exerts on an object is proportional to the displacement from equilibrium. If the spring acts along the x-axis and $x = 0$ is the equilibrium position, then the force the spring exerts if the end is displaced a distance $x$ from equilibrium is approximately

$$ F_x = -kx $$

The minus sign signifies that the force is a restoring force, i.e. the force is in the opposite direction as the displacement. If $x > 0$, then the force is in the negative direction towards $x = 0$. If $x < 0$, then the force is in the positive direction towards $x = 0$. The constant $k$ is called the spring constant. The work done by this force from $x_i$ to $x_f$ along the x-axis is

$$ W_{spring} = \int_{x_i}^{x_f} -kx \, dx = \frac{k}{2} x_i^2 - \frac{k}{2} x_f^2 $$

Thus, the potential energy function for the ideal spring is $U_{spring} = \frac{k}{2} x^2 / 2 + C$. The constant $C$ is usually chosen to be zero. In this case the potential energy of the spring is zero when the end is at $x = 0$, i.e. the spring is at its equilibrium position.

$$ U_{spring} = \frac{k}{2} x^2 $$

*Universal gravitational force*

Newton’s law of universal gravitation describes the force between two "point" objects: $\vec{F}_{12} = -(Gm_1m_2/r^2)\hat{r}_{12}$ where $m_1$ and $m_2$ are the masses of object 1 and 2, $r$ is the distance between the particles, $\vec{F}_{12}$ is the force on object two due to object one, $\hat{r}_{12}$ is a unit vector from object one to object two. $G$ is a constant equal to $6.67 \times 10^{-11} NM^2/kg^2$. The minus sign means that the force is always attractive, since $m_1$ and $m_2$ are positive.
The gravitational force is a “central” force, its direction is along the line connecting the two objects. This property makes the force conservative. Work is only done by this force when there is a change in \( r \) The force does no work if the path is circular, i.e. constant \( r \). The work done by the universal gravity force for paths starting at a separation distance of \( r_i \) to a separation distance of \( r_f \) is

\[
W_{\text{U.G.}} = \int_{r_i}^{r_f} \frac{Gm_1m_2}{r^2} dr
= Gm_1m_2 \left[ \frac{r}{r_f} - \frac{r}{r_i} \right]
= - \frac{Gm_1m_2}{r_i} - \left( - \frac{Gm_1m_2}{r_f} \right)
\]

Thus, the gravitational potential energy is \( U_{\text{U.G.}} = -\frac{Gm_1m_2}{r} + C \). The constant \( C \) is usually taken to be zero, which sets the potential energy to zero at \( r = \infty \).

\[
U_{\text{U.G.}} = -\frac{Gm_1m_2}{r}
\]

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**Electrostatic interaction**

Coulomb’s law describes the electrostatic force between two point objects that have charge: \( \vec{F}_{12} = \left( kq_1q_2/r^2 \right) \hat{r}_{12} \) where \( q_1 \) and \( q_2 \) are the charges of object 1 and 2, \( r \) is the distance between the particles, \( \vec{F}_{12} \) is the force on object two due to object one, \( \hat{r}_{12} \) is a unit vector from object one to object two. The constant \( k \) equals \( 9 \times 10^9 \text{NM}^2/\text{C}^2 \). If the objects have the same sign of charge, the product \( q_1q_2 \) is positive and the force is repulsive. If the objects have opposite charge, the product \( q_1q_2 \) is negative and the force is attractive. The force has the same form as the universal gravitational force. Charge is the source of the force instead of mass. Similar to the gravitation case, the work done by the electrostatic force for paths starting at a separation distance of \( r_i \) to a separation distance of \( r_f \) is

\[
W_{\text{electrostatic}} = \int_{r_i}^{r_f} \frac{kq_1q_2}{r^2} dr
= \frac{kq_1q_2}{r} \left[ \frac{r}{r_f} - \frac{r}{r_i} \right]
\]
Thus, the electrostatic potential energy is $U_{\text{electrostatic}} = kq_1 q_2 / r + C$. The constant $C$ is usually taken to be zero, which sets the potential energy to zero at $r = \infty$.

Conservation of Mechanical Energy

The reason for using the word "conservative" to describe these forces is the following. Suppose that the only forces that do work on an object are ones that are conservative, that is, the work done by these forces is equal to the difference in a potential energy function. Consider the case in which the only forces that do work on a particle are two conservative forces. From the work-energy theorem we have:

$$
\int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_{\text{net}} \cdot d\vec{r} = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_1 \cdot d\vec{r} + \int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_2 \cdot d\vec{r} = \frac{m}{2} v_f^2 - \frac{m}{2} v_i^2 + \left( U_1(\vec{r}_i) - U_1(\vec{r}_f) \right) + \left( U_2(\vec{r}_i) - U_2(\vec{r}_f) \right) = \frac{m}{2} v_f^2 - \frac{m}{2} v_i^2
$$

Rearranging terms we have:

$$
U_1(\vec{r}_i) + U_2(\vec{r}_i) + \frac{m}{2} v_i^2 = U_1(\vec{r}_f) + U_2(\vec{r}_f) + \frac{m}{2} v_f^2
$$

The left side of the equation contains only quantities at position $\vec{r}_i$, and the right side only quantities at position $\vec{r}_f$. Since $\vec{r}_f$ is arbitrary, the quantity $U_1(\vec{r}) + U_2(\vec{r}) + mv^2/2$ is a constant of the motion, it is a conserved quantity! The sum of the potential energy plus kinetic energy, which we call the total mechanical energy, is conserved. This result will hold for any number of forces, as long as the only work done on the system is by conservative forces. WOW!!

If non-conservative forces act on the object, such as frictional forces, the total mechanical energy as defined above is not conserved. However, frictional forces are
really electro-magnetic forces at the microscopic level. The electro-magnetic interaction is conservative if the energy of the electromagnetic field is included. At present, we believe there are only three fundamental interactions in nature: gravitational, electro-magnetic-weak, and strong, and they all are conservative. Frictional forces are non-conservative because of the limited description used in analyzing the system of particles. Since all the fundamental forces of nature (gravity, electro-magnetic, and strong) are conservative, at the microscopic level total energy is conserved. We refer to the energy of the all the atoms and molecules of a material as it’s internal energy.

Final comments on Energy and Momentum

The first 4 weeks of the quarter we discussed the physics of the interaction of particles using the "force-motion" approach: find the net force on each object, then \( \vec{a} = \vec{F}_{\text{net}}/m \). The symmetry of interaction was expressed via Newton’s third law. During the last 3 weeks, we expressed these laws using the energy and momentum of the particles. The two approaches are equivalent. Since we believe the fundamental forces are conservative, from the potential energy functions the forces can be determined and visa-versa:

\[
F_x = -\frac{\partial U}{\partial x} \\
F_y = -\frac{\partial U}{\partial y} \\
F_z = -\frac{\partial U}{\partial z}
\]

What are the advantages of using the "energy-momentum" approach verses the "force-motion" approach? In terms of solving problems, if one is interested in how an objects speed changes from one point to another then the energy approach is much simpler than solving for the acceleration. However, problem solving is not the main motivation for studying energy and momentum.

The laws of physics for "small" systems (atoms, nuclei, subatomic particles) and for relatively fast objects (special relativity) are best described in terms of momentum and energy. Energy and momentum transform from one inertial reference frame to another the same way as displacements in time and space. The interference properties of a particle is related to the particle’s momentum \((\hbar/p)\). In the Schrödinger equation, which is used to describe quantum systems, the interaction is expressed in terms of
the potential energy of the system. Advances in modern physics were guided by the principles of energy and momentum conservation.

Energy plays a key role in life on earth. If an animal needs to use more energy obtaining food than the food supplies, then the animal starves and species can go extinct. Energy is an important commodity in our daily lives: for our bodies, for our homes, and for our personal transportation. Wars have been fought over energy, and our lifestyle will be determined by how successful we are in using the sun’s energy. Energy is perhaps more important than momentum because it is a scalar quantity allowing it to be stored and sold. The importance of energy cannot be understated. Next quarter, a large part of your physics course (Phy132) will be devoted to an understanding of internal energy and energy transfer processes.