Orthogonal Curvilinear Coordinates

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Spring 2014

1 Introduction

This article reviews the basic mathematical and visualization skills you will need in ECE 302. You have already seen this material in your required math courses. The purpose of this review is to present these concepts in a form that improves your ability to use them to describe electromagnetic phenomena. My goal is to present this mathematics as a useful language for describing, visualizing, and understanding concepts such as forces, fields, currents, and waves that are central to electromagnetics.

In introductory electromagnetics, students become familiar with the basic concepts, build intuition, and develop computational skills. Thus, many examples and problems are symmetrical. In the appropriate coordinate system, symmetry reduces the dimensionality of the equations from three to one or two, eliminating much of the computational complexity. Many of the standard electromagnetics examples are symmetric in spherical or cylindrical coordinates, rather than in the Cartesian (rectangular) coordinates with which you are probably most comfortable. All of the problems and examples in this course can be solved using Cartesian coordinates. However, the resulting expressions are unnecessarily complicated. Working out such expressions exercises a student’s ability to carry out mathematical manipulations, but does little to build an understanding of the fundamental concepts. Using mathematics to model the world should clarify rather than complicate. One of my challenges is to teach you how to use vector calculus in that way.

In electromagnetics, math is used as a language. As with any language, the more familiar it becomes, the better a tool it becomes for describing and
understanding the phenomena we study. Because symmetry is such a useful approach for simplifying expressions and exposing the underlying electromagnetics concepts, it is important to be comfortable with the basic vector and vector calculus operations in the three most common coordinate systems (rectangular, cylindrical, and spherical). Orthogonal curvilinear coordinates (OLC) is a formalism that emphasizes the similarities of these three coordinate systems rather than their differences. This approach helps in choosing the appropriate coordinate system for a particular situation, expressing the resulting equations in the simplest possible form, and therefore facilitating understanding and insight. The purpose of this article is to present orthogonal curvilinear coordinates as they are used in electromagnetics.

1.1 Writing Coulomb’s law in various coordinate systems

The following examples illustrate the advantages and disadvantages of different choices of coordinate systems for writing Coulomb’s law, the formula for the force on a point charge \( q_1 \) caused by another point charge \( q_2 \) located a distance \( d \) away. The magnitude of this force (in SI units) is

\[
| \vec{F}_{12} | = \frac{q_1 q_2}{4 \pi \epsilon_0 d^2} \tag{1}
\]

and the direction (assuming that both \( q_1 \) and \( q_2 \) are positive) is repulsive (the vector points in the direction from \( q_2 \) toward \( q_1 \)). The charges and the force are shown in Figure 1.

Figure 1: Coulomb’s law for two point charges

1.1.1 Coulomb’s law in rectangular coordinates

In this example, I’ll implement equation (1), Coulomb’s law, to calculate the magnitude and direction of the force on a charge \( q_1 \) at \((x_1, y_1, z_1)\) caused by a charge \( q_2 \) at \((x_2, y_2, z_2)\). The distance between the charges is

\[
d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \tag{2}
\]
Substituting (2) into (1) gives
\[
|F_{12}| = \frac{q_1 q_2}{4\pi \epsilon \left[(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2\right]} \quad (3)
\]

Next I need to find the unit vector that points from \(q_2\) to \(q_1\).
\[
\hat{u} = \frac{\hat{x}(x_1 - x_2) + \hat{y}(y_1 - y_2) + \hat{z}(z_1 - z_2)}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}} \quad (4)
\]

Finally, I can combine the magnitude expression in (3) with the unit vector from (4) to give
\[
F_{12} = \frac{q_1 q_2 [\hat{x}(x_1 - x_2) + \hat{y}(y_1 - y_2) + \hat{z}(z_1 - z_2)]}{4\pi \epsilon \left[(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2\right]^{3/2}} \quad (5)
\]

The result in equation (5) is completely general. That is, substituting values for \(q_1, q_2, x_1, y_1, z_1, x_2, y_2,\) and \(z_2\) into the equation will give a numerical result for each of the components of the vector \(F_{12}\). For example, if \(q_2\) is at the origin then
\[
F_{12} = \frac{q_1 q_2 [\hat{x}(x_1) + \hat{y}(y_1) + \hat{z}(z_1)]}{4\pi \epsilon [(x_1)^2 + (y_1)^2 + (z_1)^2]^{3/2}} \quad (6)
\]

You should check this result for practice!

1.1.2 Coulomb’s law in spherical coordinates

Now consider calculating the force \(F_{12}\) in spherical coordinates. The goal is to make the results as simple as possible, so let’s use what we learned from doing the problem in rectangular coordinates: put \(q_2\) at the origin. We could, of course, convert equation (6) to spherical coordinates, but I hope to convince you that it is much easier to start by using spherical coordinates from the beginning rather than doing the conversion.

If \(q_2\) is at the origin, then the distance \(d\) between \(q_1\) and \(q_2\) is \(r\), the radial coordinate in a spherical system. The unit vector pointing from \(q_2\) to \(q_1\) is \(\hat{r}\), the radial unit vector in spherical coordinates, and
\[
F_{12} = \hat{r} \frac{q_1 q_2}{4\pi \epsilon r^2} \quad (7)
\]
1.2 Comparisons

Equations (5), (6), and (7) all give rules for computing the vector $\vec{F}_{12}$. Let’s compare them to see when each one might be useful. The rectangular form in (5) gives a means of finding both the magnitude and direction of $\vec{F}_{12}$ for arbitrary locations of the two charges relative to the origin of coordinates and to each other. The other two do not.

The spherical result in (7) closely resembles the original statement of Coulomb’s law in equation (1). This form makes it easy to see that all points at the same distance $r$ from the charge $q_2$ have the same magnitude of $\vec{F}_{12}$. Also, there is only one unit vector when using spherical coordinates, so the direction of the force is easy to visualize. The equivalent rectangular result in equation (6) has all three unit vectors and three distance components. The magnitude and direction aren’t obvious.

However, spherical coordinates pose a problem if I want to add the forces on $q_1$ from two or more other charges. I can find the magnitudes of the forces $\vec{F}_{12}$ and $\vec{F}_{13}$ easily from equation (7). Unfortunately, since the unit vector $\hat{r}$ points in different directions for the two forces, performing the vector addition is not simple in spherical coordinates.

The lesson to be learned from these examples is that no one coordinate system is best for all applications. With that statement as motivation, let’s look at how you have been taught to visualize coordinate systems in the past, what properties those coordinate systems have in common, and how to express the vector calculus operations used in electromagnetics in ways that aid visualization and understanding.

1.3 How coordinate systems are taught

I’ve learned from teaching this course that students want to use rectangular coordinates. The reasons for this preference are clear. In elementary school, the number line is introduced to visualize relationships between numbers. Somewhat later, graphing pairs of numbers (and, later, functions of one variable) is taught using two perpendicular number lines that cross at their zero points. One of the lines points to the right and the other points up. (Did you ever wonder why it’s done that way? More to the point, did your teacher ever give a reason for the convention?) There is an underlying theme: the coordinate axes define a coordinate system. You have been taught to rely on a property of rectangular coordinates systems that is not shared by spherical or cylindrical coordinates. In rectangular coordinates, the unit vectors don’t change direction from point to point.
When the cylindrical and spherical systems are introduced, the first exercises usually involve working out the transformations between these systems and rectangular coordinates. The unintentional message is that these new systems are secondary tools. It’s no wonder that a rectangular system seems to be better than the others. After all, it’s the one that has been used as the reference for all others since you first were taught to visualize numerical and mathematical concepts.

The Coulomb’s law example above shows that spherical coordinates have significant advantages in helping to understand and visualize a particular result. You probably had no trouble visualizing the direction of the force vector even though it was described using spherical coordinates. The key here is that the physical system being modeled has spherical symmetry. The charge $q_2$ looks the same from all directions. Matching the physical symmetry to the coordinate system symmetry by putting $q_2$ at the origin and using spherical coordinates made Coulomb’s law simple to write and simple to understand. The concept of choosing a coordinate system to simplify the form of the expressions that describe a particular physical situation has not been stressed in your previous education, but it is very important for learning electromagnetics.

Matching the symmetry of a given physical configuration with the appropriate coordinate system helps build mathematical models that clarify how stuff works. The barrier to treating rectangular, cylindrical, and spherical coordinate systems on an equal footing is that the unit vectors are not the best way to visualize all three systems. Generalized orthogonal curvilinear coordinates provides the insight that links these systems in a useful way.

2 Generalized orthogonal curvilinear coordinates

Before introducing the general concepts, let’s look at an example that demonstrates the difficulty with using unit vectors to visualize coordinate systems.

Example 1 At the point $(x, y, z) = (1, 2, 3)$ the radial unit vector in cylindrical coordinates $\hat{\rho} = (\hat{x} + 2\hat{y}) / \sqrt{5}$, but $\hat{\rho} = (\hat{x} + \hat{y}) / \sqrt{2}$ when $(x, y, z) = (2, 2, 3)$. Clearly, the direction of $\hat{\rho}$ changes from point to point, as is shown in Figure 2.

2.1 Surfaces of constant coordinate

By now you might see where I’m heading in this discussion. The difficulties you have with using spherical and cylindrical coordinates are usually because...
you have trouble visualizing the unit vectors, which change direction from point to point. Hopefully by now you are ready to consider a different approach, one that applies equally to rectangular, spherical, and cylindrical systems. That unifying concept is **surfaces of constant coordinate**.

**Example 2** The plane $z = 3$, the surface of a cylinder of radius 2, and the surface of a cone at an angle of 30° to the positive $z$ axis can all be constant-coordinate surfaces in the appropriate coordinate systems. (Can you visualize these surfaces?) Each of these surfaces can be described by the value of only one coordinate ($z$, $\rho$, or $\theta$) by choosing the appropriate system. If we were to use only the rectangular system, then two of these simple surfaces would be represented by more complex formulas involving two or three coordinates: $x^2 + y^2 = 4$ and $z/\sqrt{x^2 + y^2 + z^2} = \cos 30^\circ$.

Example 2 demonstrates that an equation of the form

$$\text{coordinate} = \text{value}$$

determines a surface. Three different types of surfaces are needed to uniquely specify a point in three-dimensional space. The rules for associating numbers (the coordinates) with these surfaces can be used to define a coordinate system. That is, the three coordinates of a point can be found by determining the values associated with the three constant-coordinate surfaces (one of each type) which intersect there.

**Example 3** In rectangular coordinates, all three types of surfaces are planes. These planes can be pictured as stacked sheets.
of paper, one stack perpendicular to each of the coordinate axes. However, in other systems the surfaces are not necessarily planar. The family of constant-\( r \) surfaces in spherical coordinates are concentric spheres, and constant-\( \rho \) surfaces are coaxial cylinders in cylindrical coordinates.

Now we need to express this new approach to defining a point using mathematical expressions instead of words and visualizations. In developing this formalism, I will continue to rely on your familiarity with rectangular coordinates to develop the ideas, but my goal is to make cylindrical and spherical coordinates as easy to use as rectangular.

Our coordinate systems will be defined by three functions, \( f_1, f_2, \) and \( f_3 \), each of which describes a family of surfaces. Setting \( f_i = q_i \) defines the surface by the coordinate (value) \( q_i \). Given the numbers \( q_i \), each of the following three equations describes a specific surface in three-dimensional space, and the three equations taken together define a point, the point where they intersect.

\[
\begin{align*}
f_1(x, y, z) &= q_1 \\
f_2(x, y, z) &= q_2 \\
f_3(x, y, z) &= q_3
\end{align*}
\]  

(8)

For example, in spherical coordinates \( f_1 \) describes all the spheres centered at the origin. (In terms of rectangular coordinates \( f_1 = \sqrt{x^2 + y^2 + z^2} \).) In both rectangular and cylindrical coordinates, the \( f_3 \) surfaces are the same family of parallel planes \( f_3 = z \).

Of course, I can’t choose three completely arbitrary functions, and some choices will be much more useful than others. First of all, useful sets of three functions \( f_i \) will describe surfaces such that, for any choice of numbers \( q_i \), the three surfaces generated by equations (8) will intersect at one and only one point, ensuring that the set of three numbers (coordinates) \((q_1, q_2, q_3)\) always locates exactly one point (not zero or more than one). This generalized coordinate system is called “curvilinear.”

The variables used to name the functions \( f_i \) in the general system (curvilinear) and in the three specific systems are all listed in Table 1. The functions for the cylindrical and spherical systems are also described there using the familiar rectangular \((x, y, z)\) description of the location of a point.

The curvilinear coordinates I have described so far are too general for our applications. They lack an important property shared by the rectangular,
Curvilinear | Rectangular | Cylindrical | Spherical
--- | --- | --- | ---
$f_1$ | $x$ | $\rho = \sqrt{x^2 + y^2}$ | $r = \sqrt{x^2 + y^2 + z^2}$
$f_2$ | $y$ | $\phi = \tan^{-1}(y/x)$ | $\theta = \cos^{-1}(z/r)$
$f_3$ | $z$ | $z$ | $\phi = \tan^{-1}(y/x)$

Table 1: The coordinate functions $f_i$

cylindrical, and spherical systems: the three constant-coordinate surfaces are always orthogonal at the point where they intersect. This property is so useful that it is built into all of the allowable functions $f_i$ for the coordinate systems we will be using. Constructing sets of functions with this property is not simple, but luckily we don’t have to do that because the rectangular, cylindrical, and spherical systems are orthogonal.

So far we have a method for locating points that is independent of the unit vectors. Now we need a way to find the three unit vectors at any point.

3 Unit vectors

Points in three-dimensional space are defined with ordered triplets of numbers $(q_1, q_2, q_3)$ by using the functions $f_i$ of equation (8) as specified in Table 1. The three functional equations define three surfaces of constant coordinate which intersect at the point. In order to use this formalism effectively, it is important that you practice until you can easily visualize the shapes of each of the three coordinate surfaces in the rectangular, cylindrical, and spherical systems. It may be helpful to build yourself physical models using objects like pencils, soup cans, ping pong balls, and cardboard. The better your understanding of these shapes, the more successful you will be in choosing the best coordinate system for modeling a given real-world problem.

If you feel as though you aren’t ready to use orthogonal curvilinear coordinates, that’s not surprising. So far there has been no mention of the axes or unit vectors that you have been trained for so long to use to describe and visualize points and vectors. Although this new approach may feel uncomfortable for a while, there is an important advantage: locating a point no longer relies on knowing which way the unit vectors point, so now all three of our coordinate systems have the same set of rules.

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In this new formalism, the unit vectors are determined \textit{locally} using the surfaces of constant coordinate at the point of intersection. Let’s start with a specific example: the surfaces of constant \( f_3 = z \) in rectangular or cylindrical coordinates. Each value of \( z \) defines a plane normal to the \( z \) axis. The unit vector \( \hat{z} \) is orthogonal (perpendicular) to all of these planes, and it points in the direction of planes with increasing values of \( z \) as shown in Figure 3. The length of the vector \( \hat{z} \) is one unit of \( z \). Since all of the surfaces of constant coordinate are planes in the rectangular system, the unit vectors \( \hat{x} \) and \( \hat{y} \) can be visualized in the similar way.

\[
\begin{align*}
  z &= 1.0 \\
  z &= 1.5 \\
  z &= 2.0 \\
  z &= 2.5
\end{align*}
\]

\textbf{Figure 3: The unit vector} \( \hat{x} \)

Generalizing, the unit vector associated with a particular \( f_i \)

\begin{itemize}
  \item is orthogonal to the surface \( f_i = q_i \) at a specific point \( (q_1, q_2, q_3) \),
  \item points in the direction of surfaces with increasing values of \( q_i \), and
  \item is one unit in length.
\end{itemize}

Some of the coordinates we will be using are defined by angles, not distances. You might think about the meaning of “one unit of length” in such a case.

\textbf{Example 4} The function \( f_3 = z \) has the identical meaning in both cylindrical and rectangular coordinates, so the unit vector is the same: \( \hat{z} \). In cylindrical coordinates, \( f_1 \) represents a family of infinitely-long coaxial cylinders, each characterized by its radius \( f_1 = \rho \). At any point on one of these cylinders, the unit vector \( \hat{\rho} \) points radially outward, orthogonal to the surface of the cylinder.
Example 5 The functions $f_2 = \phi$ in cylindrical coordinates and $f_3 = \phi$ in spherical coordinates describe a family of half-planes with the edge along the $\rho = 0$ surface (the $z$ axis of the rectangular system with the same origin). The associated unit vector is orthogonal to the half-plane, and also tangent to the cylindrical or spherical constant-coordinate surfaces, and it points in the direction of increasing $\phi$.

The names I will be using for the unit vectors in our four coordinate systems are given in Table 2. Note that these may not be the same as those used in your text, and may not be what you used in your math courses. Unfortunately, there is no one universally-accepted convention for naming either the coordinates or unit vectors. For example, you may see “$\hat{i}, \hat{j}, \hat{k}$,” “$\mathbf{a}_x, \mathbf{a}_y, \mathbf{a}_z$,” or “$\mathbf{u}_x, \mathbf{u}_y, \mathbf{u}_z$” for the unit vectors in a rectangular system. Similarly, you may find “$r, \phi, z$” used for the cylindrical system or “$R, \theta, \phi$” for the spherical system. I prefer to use the same symbols in typeset text that I use when working by hand, and will use the symbols in this document and throughout the course. Since there is no generally-accepted convention, it is useful to get accustomed to seeing different notations. Ultimately, it is important to make sure that you understand each author’s choice of notation.

<table>
<thead>
<tr>
<th>Curvilinear</th>
<th>Rectangular</th>
<th>Cylindrical</th>
<th>Spherical</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{u}$</td>
<td>$\hat{x}$</td>
<td>$\hat{\rho}$</td>
<td>$\hat{\rho}$</td>
</tr>
<tr>
<td>$\hat{v}$</td>
<td>$\hat{y}$</td>
<td>$\hat{\phi}$</td>
<td>$\hat{\theta}$</td>
</tr>
<tr>
<td>$\hat{w}$</td>
<td>$\hat{z}$</td>
<td>$\hat{z}$</td>
<td>$\hat{\phi}$</td>
</tr>
</tbody>
</table>

Table 2: Unit vectors.

4 Right-handedness

There is another property of the surfaces, coordinates, and unit vectors that we haven’t considered yet, and it has to do with the following question. Why are the functions $f_i$ used in a particular order? Specifically, why is the same function ($\phi$) used as $f_2$ in a cylindrical system and $f_3$ in a spherical system? This choice is not arbitrary. Orthogonality is not the only property our three coordinate systems have in common. They also share the convention...
that the vector product (cross product) of the first and second unit vectors must equal the third unit vector ($\hat{u} \times \hat{v} = \hat{w}$). So we have $\hat{x} \times \hat{y} = \hat{z}$ and $\hat{\rho} \times \hat{\phi} = \hat{z}$. In spherical coordinates $\hat{r} \times \hat{\phi} = -\hat{\theta}$ and $\hat{r} \times \hat{\theta} = \hat{\phi}$. Thus $f_2 = \theta$ and $f_3 = \phi$. This “right-handedness” is an arbitrary convention for our coordinate systems. However, once we have made that choice for one system, making the same choice for the rest of our systems is important for ensuring that the expressions for the laws of electromagnetism have the same form in all three coordinate systems.

The choice of the first coordinate ($f_1$) is arbitrary, and is also determined by convention. Once $f_1$ is chosen, the order of assignment of the constant-coordinate surface functions $f_2$ and $f_3$ must give a right-handed system as determined from the definition of the vector cross product. That is, in rectangular coordinates we could choose $(x, y, z)$, $(y, z, x)$, or $(z, x, y)$, which are all right-handed, but not $(x, z, y)$, which is left-handed.

The condition for right-handedness in the generalized orthogonal curvilinear coordinate system notation is

$$\hat{u} \times \hat{v} = \hat{w}$$

(9)

## 5 Scale factors and differential distances

There is another way in which rectangular coordinates are special: all three of the $f_i$ are constant-distance surfaces. In both cylindrical and spherical coordinates, at least one of the $f_i$ represent constant-angle surfaces. Constant-distance surfaces have the property that the differential change $dq_i$ in the coordinate $q_i$ is the same as the differential distance $dl_i$ between the surfaces $f_i = q_i$ and $f_i = q_i + dq_i$. However, constant-angle surfaces have the more complex relationship

$$dl_i = h_i dq_i$$

(10)

where $h_i$ is the appropriate radius.

**Example 6** Figure 4 shows two lines rotated by an angle $d\phi$. The distance between the two lines depends on the radial distance from the axis. If $d\phi$ is small enough that curvature can be neglected, the distance is $\rho d\phi$.

The scale factors $h_i$ for each of the coordinate systems we will be using are given in Table 3. Note that the rectangular system, the only system in which all of the functions represent constant-distance surfaces, is also the only one for which all $h_i = 1$. 

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\[ d\phi \text{ cm} \]

\[ 0 \quad 1 \text{ cm} \quad 3 \text{ cm} \]

Figure 4: The distance between surfaces of constant angular coordinate depends on the distance to the axis.

\[
\begin{array}{|c|c|c|c|}
\hline
\text{Curvilinear} & \text{Rectangular} & \text{Cylindrical} & \text{Spherical} \\
\hline
h_1 & 1 & 1 & 1 \\
\hline
h_2 & 1 & \rho & r \\
\hline
h_3 & 1 & 1 & r \sin \theta \\
\hline
\end{array}
\]

Table 3: The scale factors \( h_i \).

To solve electromagnetic problems you will need to be able to carry out one-, two-, and three-dimensional line, surface, and volume integrals. The differential distance given in equation (10), together with the scale factors in Table 3, help in writing down the differential distances, areas, and volumes required for these operations. The following examples demonstrate the use of these scale factors.

The arbitrary differential displacement vector is constructed by moving a differential distance in each of the unit vector directions.

\[ \vec{d\ell} = ˆu h_1 dq_1 + ˆv h_2 dq_2 + ˆw h_3 dq_3 \]

(11)

In rectangular coordinates this vector is

\[ \vec{d\ell} = \hat{x} dx + \hat{y} dy + \hat{z} dz \]

(12)

and in spherical coordinates it is

\[ \vec{d\ell} = \hat{r} dr + \hat{\theta} r d\theta + \hat{\phi} r \sin \theta d\phi \]

(13)

Example 7 The differential unit of area over a constant-\( \rho \)(constant-\( f_1 \)) surface in cylindrical coordinates is

\[ dl_2 dl_3 = (h_2 dq_2)(h_3 dq_3) = (\rho d\phi) dz \]
as shown in Figure 5. Thus, the surface area of a cylinder of radius 2 cm and height 5 cm is

\[ A = \int_{0}^{5 \text{ cm}} dA = \int_{0}^{2\pi} (2 \text{ cm}) d\phi \, dz = 20\pi \text{ cm}^2 \]

Figure 5: A differential constant-\(\rho\) surface in cylindrical coordinates.

**Example 8** The differential unit of volume in spherical coordinates, which is illustrated in Figure 6 is

\[ dl_1 \, dl_2 \, dl_3 = (h_1 dq_1)(h_2 dq_2)(h_3 dq_3) = (dr)(r d\theta)(r \sin \theta \, d\phi) \]

Thus, the volume of a sphere of radius \(R_0\) is

\[
V = \int_{\text{sphere}} d\tau = \int_{0}^{2\pi} \int_{0}^{\pi} \int_{0}^{R_0} r^2 \sin \theta \, dr \, d\theta \, d\phi
\]

\[
= \int_{0}^{2\pi} d\phi \int_{0}^{\pi} \sin \theta \, d\theta \int_{0}^{R_0} r^2 \, dr
\]

\[
= (2\pi)(2) \left( \frac{R_0^3}{3} \right) = \frac{4\pi}{3} R_0^3 \quad (14)
\]

6 Unit vector conversions

Equation (14) provides a method for expressing one set of unit vectors in terms of another. This method requires that you can transform the coordinates.
Figure 6: A differential volume in spherical coordinates

Example 9 Suppose we need to express the unit vectors of cylindrical coordinates in terms of the rectangular unit vectors. The conversion can be done by writing a differential length in both systems.

\[
\vec{d\ell} = \hat{\rho} \, d\rho + \hat{\phi} \, \rho \, d\phi + \hat{z} \, dz = \hat{x} \, dx + \hat{y} \, dy + \hat{z} \, dz
\]

\[
= \hat{x} \left( \frac{\partial x}{\partial \rho} \, d\rho + \frac{\partial x}{\partial \phi} \, d\phi \right) + \hat{y} \left( \frac{\partial y}{\partial \rho} \, d\rho + \frac{\partial y}{\partial \phi} \, d\phi \right) + \hat{z} \, dz
\]

\[
= \left( \frac{\partial x}{\partial \rho} + \frac{\partial y}{\partial \phi} \right) d\rho + \left( \frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial \phi} \right) d\phi + \hat{z} \, dz \quad (15)
\]

Using the coordinate transformations \( x = \rho \cos \phi \) and \( y = \rho \sin \phi \),

\[
\hat{\rho} = \hat{x} \cos \phi + \hat{y} \sin \phi \quad (16)
\]

\[
\hat{\phi} \rho = (-\hat{x} \sin \phi + \hat{y} \cos \phi) \rho \quad (17)
\]

Test your understanding by finding \( \hat{x} \) and \( \hat{y} \) in the cylindrical system (the inverse of the transformation done above). The result is

\[
\hat{x} = \hat{\rho} \cos \phi - \hat{\phi} \sin \phi \quad (18)
\]

\[
\hat{y} = \hat{\rho} \sin \phi + \hat{\phi} \cos \phi \quad (19)
\]

7 Vector derivatives

Now we come to a much tougher test of my claim that orthogonal curvilinear coordinates will simplify the vector calculus used in electromagnetics: the gradient, divergence, curl, and Laplacian. These operators give the vector
and scalar derivatives of scalar and vector fields (scalar and vector functions of position).

Let

\[ S = S(q_1, q_2, q_3) \]  

be an arbitrary scalar field like temperature or voltage and

\[ \vec{V} = \hat{u}V_1(q_1, q_2, q_3) + \hat{v}V_2(q_1, q_2, q_3) + \hat{w}V_3(q_1, q_2, q_3) \]

be an arbitrary vector function like force or the electric field.

### 7.1 Gradient

The gradient is one of the vector derivative operations. It is used, for example, to find the force given the potential energy. Since the electric field is force per unit charge and the voltage (electrostatic potential) is electrostatic potential energy per unit charge, the gradient is also used to find the electric field given the voltage.

To understand the gradient, consider how a change \(dS\) in the scalar function \(S\) depends on changes \(dx\), \(dy\), and \(dz\) in the coordinates.

\[ dS = \left( \frac{\partial S}{\partial x} \right) dx + \left( \frac{\partial S}{\partial y} \right) dy + \left( \frac{\partial S}{\partial z} \right) dz \]  

This expression can be written as the dot product of the differential length vector

\[ \vec{d}l = \hat{x} dx + \hat{y} dy + \hat{z} dz \]

and a second vector called the gradient \(\vec{\nabla}S\) of \(S\), where

\[ \vec{\nabla}S = \hat{x} \frac{\partial S}{\partial x} + \hat{y} \frac{\partial S}{\partial y} + \hat{z} \frac{\partial S}{\partial z} \]

That is, a differential change in the function \(S\) where the change is taken in the direction of \(\vec{d}l\) is

\[ dS = \vec{\nabla}S \cdot \vec{d}l = \left| \vec{\nabla}S \right| \left| \vec{d}l \right| \cos \alpha \Rightarrow \frac{dS}{\left| \vec{d}l \right|} = \left| \vec{\nabla}S \right| \cos \alpha \]

where \(\alpha\) is the angle between the vectors \(\vec{\nabla}S\) and \(\vec{d}l\).

Suppose I choose a constant length for \(\vec{d}l\) and rotate its direction about a point so that only the angle \(\alpha\) between \(\vec{\nabla}S\) and \(\vec{d}l\) changes. Then Equation 25 tells us that the change \(dS\) in the function \(S\) at this point in the distance
|\vec{d}l| is maximum when \( \alpha = 0 \) (\( \vec{d}l \) has the same direction as \( \vec{\nabla}S \)). That is, \( \vec{\nabla}S \) is a vector whose magnitude is the maximum value of \( dS/dl \) and whose direction is the direction in which \( dS/dl \) is maximum. Translating this concept into the generalized OLC formalism using the expression for the differential distance in the direction of a unit vector gives

\[
\vec{\nabla}S = \hat{u} \frac{1}{h_1} \frac{\partial S}{\partial q_1} + \hat{v} \frac{1}{h_2} \frac{\partial S}{\partial q_2} + \hat{w} \frac{1}{h_3} \frac{\partial S}{\partial q_3}
\]

(26)

**Example 10** The surface defined by \( h = x^2y \) is plotted in Figure 7. The gradient of \( h \) is \( \vec{\nabla}h = \hat{x} 2xy + \hat{y} x^2 \). A vector plot of the gradient is shown in Figure 8.

![Figure 7: The surface \( x^2y \).](image)

### 7.2 The “Del” Operator

Equations 24 and 26 can be written using a vector differentiation operator

\[
\vec{\nabla} = \hat{u} \frac{1}{h_1} \frac{\partial}{\partial q_1} + \hat{v} \frac{1}{h_2} \frac{\partial}{\partial q_2} + \hat{w} \frac{1}{h_3} \frac{\partial}{\partial q_3}
\]

(27)

As we have seen above, applying \( \vec{\nabla} \) to a scalar function gives a vector function: the gradient. Now I’ll explore what happens when this operator is
applied to a vector function. There are two possibilities, the scalar (dot) product and the vector (cross) product. Each of them is used in electromagnetics.

7.3 Divergence

The divergence describes the “sources” and “sinks” of a vector field. One of the uses of the divergence in electromagnetics is to find the distribution of charge that creates an electric field. Two of Maxwell’s four equations use the divergence to relate the electric and magnetic fields to sources.

Since you have a lot of everyday experience with water, I’ll use water flow to illustrate the divergence. First consider water flowing in a pipe. Assume that the velocity \( \vec{v} \) of the water is constant across the cross section of the pipe and points along the axis. If the pipe ends with a cut perpendicular to the axis as shown on the left in Figure 9, then the weight of water collected in a bucket in a time \( t \) is \( W = \eta |\vec{v}| Atg \) where \( \eta \) is the density of water, \( A \) is the area of the open end of the pipe, and \( g \) is the gravitational acceleration. That is, the rate of flow of water weight into the bucket per unit area of the pipe is \( W/(At) = \eta g|\vec{v}| \).

Now suppose the end of the pipe is cut at an angle \( \theta \) to the axis as shown on the right in Figure 9. The rate at which water is supplied to the end
of the pipe doesn’t change, so the rate at which water is collected in the bucket should be the same even though the area \( A_\theta \) of the opening is larger. Some thought should convince you that it is the projection of the area of the opening in the direction of the water velocity that matters. That is, \(|\vec{v}|\) should be replaced with \( v \cdot \hat{n} \) where \( \hat{n} \) is a vector normal to the opening of the pipe. Then \( W/t = (\eta g \vec{v}) \cdot (\hat{n} A) = \eta g |\vec{v}| A_\theta \cos \theta = \eta g |\vec{v}| A \) as before.

![Figure 9: Water flow out of a pipe.](image)

This water flow example shows that, for a vector function \( \vec{V} \), \( \vec{V} \cdot d\sigma \) is the proportion of \( |\vec{V}| \) that “flows through” the differential surface area \( d\sigma \). (In the example, \( \vec{V} = \eta g \vec{v} \) and \( d\sigma = \hat{n} dA \).)

Now consider the total “flow” of \( \vec{V} \) “through” the surface of the differential volume with sides of lengths \( dx \), \( dy \), and \( dz \) that is shown in Figure 10. On the face whose normal is \( \hat{x} \), \( d\tilde{\sigma} = \hat{x} dy dz \). On the face parallel to it, \( d\tilde{\sigma} = -\hat{x} dy dz \). Thus, for this pair of faces, the total “flow” of \( \vec{V} \) out of the volume element.
volume is
\[
\mathbf{V} \cdot d\mathbf{\sigma} \bigg|_x = \left( \mathbf{V} \bigg|_{x=dx} - \mathbf{V} \bigg|_{x=0} \right) \cdot (\mathbf{\hat{x}} dy dz) = \left( \frac{\partial V_x}{\partial x} dx \right) dy dz
\] (28)

Adding the contributions of the other two pairs of faces gives the total “flow” of \( \mathbf{V} \) out of this volume.
\[
\mathbf{V} \cdot d\mathbf{\sigma} = \left( \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \right) dx dy dz = \left( \mathbf{\nabla} \cdot \mathbf{V} \right) dx dy dz
\] (29)

Thus, \( \mathbf{\nabla} \cdot \mathbf{V} \) is a local measure (at a particular point) of the source of \( \mathbf{V} \) per unit volume. We have, as expected,
\[
\mathbf{\nabla} \cdot \mathbf{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}
\] (30)

or, for the general OLC system,
\[
\mathbf{\nabla} \cdot \mathbf{V} = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial q_1} (V_1 h_2 h_3) + \frac{\partial}{\partial q_2} (V_2 h_3 h_1) + \frac{\partial}{\partial q_3} (V_3 h_1 h_2) \right]
\] (31)

Another image for the divergence is that it measures how much \( \mathbf{V} \) spreads out from a point. A two-dimensional visualization aid is to imagine leaves floating on the surface of a pond or stream. If the leaves spread out away from a point, then that point has a positive divergence. If they move toward the point, then it has a negative divergence.

**Example 11** The vector function \( \mathbf{V} = \mathbf{\hat{r}} r^2 \) is plotted in Figure 11. The divergence \( \mathbf{\nabla} \cdot \mathbf{V} = 3r \) is plotted in Figure 12. These two images show that for larger distance away from the origin the increasing magnitude of \( \mathbf{V} \) requires larger and larger local “sources” (divergence).

### 7.4 Curl

The curl gives the local vorticity (rotation) of a vector field. Two of Maxwell’s four equations relate the electric and magnetic fields through the curl. One use of these equations is to derive the wave equation that describes electromagnetic waves.

In two dimensions, the leaves on the surface of a pond or stream can also demonstrate the curl. If a leaf rotates, then the water flow at that point has a curl. The formula for the curl in OLC is given in Equation 32.
Figure 11: The vector function $\vec{V} = \hat{\rho} \rho^2$.

Figure 12: The divergence $\nabla \cdot \vec{V} = 3\rho$ of the vector function $\vec{V} = \hat{\rho} \rho^2$. 
and Example 12 gives an opportunity to try this formula in two coordinate systems.

\[
\vec{\nabla} \times \vec{V} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix}
h_1 \dot{u} & h_2 \dot{v} & h_3 \dot{w} \\
\frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\
h_1 V_1 & h_2 V_2 & h_3 V_3 
\end{vmatrix}
\]  
(32)

**Example 12** The vector function \( \vec{V} = \hat{x} y - \hat{y} x = \hat{z}(\rho) \) is plotted in Figure 13. The curl \( \vec{\nabla} \times \vec{V} = \hat{z}(-2) \) everywhere. Take the point \((2, 0)\) as an example. If \( \vec{V} \) represents water flow on the surface of a pond, then a leaf at that point would rotate clockwise because the flow to the right is faster than the flow to the left. The vector direction of the associated torque is given by the right-hand rule, so it points to \(-\hat{z}\).

![Figure 13: The vector function \( \vec{V} = \hat{x} y - \hat{y} x = \hat{z}(\rho) \).](image-url)
8 Second derivatives

There are five possible second derivatives: the divergence and curl of the gradient of a scalar field, the gradient of the divergence of a vector field, and the divergence and curl of the curl of a vector field. Each of these operations will be examined in the following sections. Three of these expressions lead to formulas which I have not derived. These derivations are easily found in textbooks on vector calculus or mathematical physics.

8.1 Divergence of the gradient of a scalar

The divergence of the gradient of a scalar field is called the Laplacian. This operation is so widely used that it has its own symbol, \( \nabla^2 \). The Laplacian is used in electromagnetics in the relationship between voltage and charge and in the wave equation. It is also used in the diffusion equation that describes heat conduction.

The Laplacian of a scalar field is

\[
\nabla^2 S = \nabla \cdot \nabla S = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial q_1} \left( \frac{h_2 h_3}{h_1} \frac{\partial S}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left( \frac{h_3 h_1}{h_2} \frac{\partial S}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left( \frac{h_1 h_2}{h_3} \frac{\partial S}{\partial q_3} \right) \right]
\]

(33)

8.2 Curl of the gradient of a scalar

This operation gives a useful identity:

\[
\nabla \times \left( \nabla S \right) = 0
\]

(34)

8.3 Gradient of the divergence of a vector

The operation \( \nabla \left( \nabla \cdot \vec{V} \right) \) is not common in physics. However, it is important to note that this is not the Laplacian of a vector.

8.4 Divergence of the curl of a vector

This operation also leads to a useful identity:

\[
\nabla \cdot \left( \nabla \times \vec{V} \right) = 0
\]

(35)
8.5 Curl of the curl of a vector

This operation gives a definition of the Laplacian of a vector. Using the definition of $\vec{\nabla}$, it can be shown that

$$\nabla^2 \vec{V} = \vec{\nabla} \left( \vec{\nabla} \cdot \vec{V} \right) - \vec{\nabla} \times \vec{\nabla} \times \vec{V}$$  \hfill (36)

This is a tricky concept. The Laplacian of a vector is a vector. In rectangular coordinates, the derivatives of the unit vectors with respect to position are all zero, so this operation takes the simple form

$$\nabla^2 \vec{v} = \hat{x} \nabla^2 V_x + \hat{y} \nabla^2 V_y + \hat{z} \nabla^2 V_z$$  \hfill (37)

It is important to remember that this simplification does not apply to other coordinate systems.