Our development of the Lebesgue integral relied upon the set $L$ of step functions $\phi : \mathbb{R} \to \mathbb{R}$ satisfying the properties

1.) If $\phi, \psi \in L$, $a, b \in \mathbb{R}$ then
   \[ a\phi + b\psi \in L, \quad \phi \lor \psi \in L \quad \text{and} \quad \phi \land \psi \in L. \]

2.) If $\phi, \psi \in L$ and $a, b \in \mathbb{R}$ then
   \[ \int (a\phi + b\psi) \, dx = a\int \phi \, dx + b\int \psi \, dx \]

3.) If $\phi \in L$ with $\phi \geq 0$ then
   \[ \int \phi \, dx \geq 0. \]

4.) if $\{\phi_n\}_{n=1}^{\infty}$ is a decreasing sequence of nonnegative piecewise linear functions such that
   \[ \lim_{n \to \infty} \phi_n(x) = 0 \]
   for all $x$, then
   \[ \lim_{n \to \infty} \int \phi_n \, dx = 0. \]

In the following set of problems, you will show that we can develop the Lebesgue integral in a similar manner using piecewise linear functions.

**Definition 1** A function $\phi : \mathbb{R} \to \mathbb{R}$ is a piecewise linear function if there is a finite sequence of points

\[ a_0 \leq a_1 \leq \ldots \leq a_n \]

and real numbers $c_1, c_2, \ldots, c_{n-1}$ such that with $c_0 = c_n = 0$,

\[ \phi(x) = \begin{cases} 
0 & \text{for } x \leq a_0 \text{ and } x \geq a_n \\
(c_k & \text{for } x = a_k, 1 \leq k \leq n - 1 \\
\frac{(a_k - x)c_{k-1} + (x - a_{k-1})c_k}{a_k - a_{k-1}} & \text{for } a_{k-1} < x < a_k, 1 \leq k \leq n. 
\end{cases} \]
A piecewise linear function is a continuous function that is linear on the intervals \([a_k - 1, a_k]\). The sequence \((a_0, \ldots, a_n; c_1, \ldots, c_{n-1})\) is called a presentation for \(\phi\) as a piecewise linear function.

1. Show that the definition
   \[
   \int \phi \, dx = \sum_{k=1}^{n} \frac{(c_k + c_{k-1})(a_k - a_{k-1})}{2}
   \]
   of the integral of the piecewise linear function \(\phi\) is independent of the presentation \((a_0, \ldots, a_n; c_1, \ldots, c_{n-1})\) for \(\phi\).

2. Show that the set of all piecewise linear functions is a vector space and that if \(\phi\) and \(\psi\) are piecewise linear functions then \(\phi \lor \psi\) and \(\phi \land \psi\) are also. Thus Property 1.) is satisfied for piecewise linear functions.

3. Show that if \(\phi\) and \(\psi\) are piecewise linear functions and \(a, b \in \mathbb{R}\) then
   \[
   \int (a\phi + b\psi) \, dx = a \int \phi \, dx + b \int \psi \, dx
   \]
   so that Property 2.) is satisfied for piecewise linear functions.

4. Show that if \(\phi\) is a piecewise linear function with \(\phi \geq 0\) then
   \[
   \int \phi \, dx \geq 0
   \]
   so that Property 3.) is satisfied for piecewise linear functions.

5. Show that Property 4.) is satisfied by piecewise linear functions. That is, show if \(\{\phi_n\}_{n=1}^{\infty}\) is a decreasing sequence of nonnegative piecewise linear
functions such that
\[ \lim_{n \to \infty} \phi_n(x) = 0 \]
for all \( x \), then
\[ \lim_{n \to \infty} \int \phi_n \, dx = 0. \]

Hint: Let \( M_n \) be the maximum value of \( \phi_n \), prove that \( M_n \to 0 \).

6. Show that \( E \subset \mathbb{R} \) is a null set if and only if there is an increasing sequence \( \{ \phi_n \}_{n=1}^{\infty} \) of piecewise linear functions such that \( \{ \phi_n(x) \}_{n=1}^{\infty} \) diverges for each \( x \in E \) and \( \{ \int |\phi_n| \, dx \}_{n=1}^{\infty} \) converges.

Hint: Mimic the proof of the Proposition on P.67

7. Let \( S = \mathbb{R} \) and \( L \) be the set of piecewise linear functions. Verify that if we define for \( \phi \in L \) an integral as
\[
\int \phi \, d\mu = \int \phi \, dx
\]

as in Problem 1., then by Problems 2. - 5. the integral defined in (1) is a Daniell integral.

Let \( L_1 \) be the set of step functions with their integral
\[
\int \phi \, d\mu_1, \text{ for } \phi \in L_1,
\]
defined by the (elementary) Lebesgue integral. Let \( L_2 \) be the set of continuous functions, each of which vanishes off some interval, with their integral
\[
\int f \, d\mu_2, \text{ for } f \in L_2,
\]
defined as the Riemann integral. Let \( L_3 \) be the set of piecewise linear functions with their integral
\[
\int \psi \, d\mu_3, \text{ for } \psi \in L_3,
\]
defined by equation (1).

Let \( L_i^1 \) be the space of \( d\mu_i \)-integrable functions for \( i = 1, 2, 3 \).

8. Show that
\[
L_3 \subset L_2 \subset L_1^1
\]
and that
\[
\int \psi \, d\mu_3 = \int \psi \, d\mu_2 = \int \psi \, d\mu_1 \text{ for } \psi \in L_3.
\]
Hint: Examine the definition of each of these integrals. The first equality is shown in elementary calculus, explain! We showed the second equality this quarter, see P. 89.

9. Using the Comparison Theorem for the Daniell integral, conclude from Problem 8, that
\[
L_3^1 \subset L_2^1 \subset L_1^1.
\]
Further, show that
\[
\int f \, d\mu_3 = \int f \, d\mu_2 = \int f \, d\mu_1 \text{ for } f \in L_3^1,
\]
and
\[
\int f \, d\mu_2 = \int f \, d\mu_1 \text{ for } f \in L_2^1.
\]
10. Using the Comparison Theorem for Daniell integrals show that $L_1^1 \subset L_3^1$ so that

$$L_3^1 = L_2^1 = L_1^1.$$  

Hint: Let $\chi$ be the characteristic function of $[a, b]$ and let

$$\psi_n(x) = \begin{cases} 
0 & \text{for } x \leq a - \frac{1}{n} \\
na + 1 & \text{for } a - \frac{1}{n} \leq x \leq a \\
1 & \text{for } a \leq x \leq b \\
nb + 1 - nx & \text{for } b \leq x \leq b + \frac{1}{n} \\
0 & \text{for } x \geq b + \frac{1}{n}.
\end{cases}$$

Show that $\{\psi_n\}_{n=1}^\infty$ defines a sequence of piecewise linear function (hence in $L_3^1$) that decreases to $\chi$ and which satisfies

$$\int \psi_n \, d\mu_3 \geq 0.$$ 

Using the monotone convergence theorem, show that $\chi$ is $d\mu_3$-integrable and

$$\int \chi \, d\mu_3 = \lim_{n \to \infty} \int \psi_n \, d\mu_3 = \lim_{n \to \infty} \int \psi_n \, d\mu_1 = \int \chi \, d\mu_1.$$ 

Conclude from this that $L_1^1 \subset L_3^1$ and

$$\int \phi \, d\mu_1 = \int \phi \, d\mu_3 \text{ for } \phi \in L_1.$$ 

Remark: From Problem 10, we can conclude that the set of step functions $L_1$, the set of piecewise linear functions $L_3$ and the set of continuous functions $L_2$, could each serve as the basis for the theory of Lebesgue integration. Of course, the step functions are the simplest functions to use in developing this theory.