**A Simple Death Process**

Due Thursday April 16

In lecture we considered the pure birth process. In this problem, we introduce a \textit{pure death process}. In this rather macabre process, individuals persist only until they die and there are no replacements. The assumptions are similar to those in the pure birth process, but now each individual, if still alive at time \( t \), is removed in \( (t, t + \Delta t) \) with probability \( \mu \Delta t \). The transitions and rates for this process are given as:

<table>
<thead>
<tr>
<th>Transition</th>
<th>Rate</th>
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<tbody>
<tr>
<td>( i \rightarrow i + 1 )</td>
<td>( \lambda_i = 0 ) for ( i = 0, 1, 2, \ldots )</td>
</tr>
<tr>
<td>( i \rightarrow i - 1 )</td>
<td>( \mu_i = i\mu ) for ( i = 1, 2, 3, \ldots )</td>
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If we let \( X(t) \) be the number of individuals alive at time \( t \), then we obtain a system of differential equations similar to those obtained in the pure birth process. Suppose initially, there are \( n_0 \) individuals, that is \( X(0) = n_0 \).

\textbf{a}) Show that the differential equations for the time-dependent probability distribution of the pure death process are:

\[
\begin{align*}
P'_{n_0}(t) &= -n_0\mu P_{n_0}(t) \\
P'_{n_0-1}(t) &= n_0\mu P_{n_0}(t) - (n_0 - 1)\mu P_{n_0-1}(t) \\
&\vdots \\
P'_{0}(t) &= \mu P_{1}(t)
\end{align*}
\]

where

\[ P_k(t) = P[X(t) = k \mid X(0) = n_0]. \]
b) Show that the system of differential equations (1) has solution

\[ P_k(t) = \binom{n_0}{k} e^{-\mu kt} (1 - e^{-\mu t})^{n_0 - k} \text{ for } k = n_0, n_0 - 1, \ldots, 1, 0, \]

by first finding a solution for \( P_{n_0}(t) \) and then using this solution to find the subsequent \( P_k(t) \)’s.

c) The mean and variance of the pure death process. Using the mean and variance of a binomial random variable with \( P_1(t) = e^{-\mu t} \) as the probability of a success, show that the mean of the pure death process is

\[ E(X(t)) = n_0 e^{-\mu t}. \]

and that the variance is

\[ V(X(t)) = n_0 e^{-\mu t} (1 - e^{-\mu t}). \]

Compare the expression for the mean of the pure death process with that of a deterministic exponential population model when the growth rate is negative.

d) Extinction. In the pure death process the population either remains constant or it decreases. It may eventually reach zero in which case we say that the population has gone extinct. Show that the probability the population is extinct at time \( t \) is

\[ P[X(t) = 0 \mid X(0) = n_0] = (1 - e^{-\mu t})^{n_0}. \]

Use this expression to show that extinction is inevitable.