Asymptotic Behaviour of the $q$-Poisson Distributions Heine and Euler by Pointwise Convergence

Andreas Kyriakoussis and Malvina Vamvakari

1 Extended Abstract

Kemp [7, 8, 9], introduced the $q$-Poisson distributions, Heine and Euler, with probability functions given respectively by

$$f_{H}^{X}(x) = e_{q}(-\lambda) \frac{q(z)^x \lambda^x}{[x]_q !}, x = 0, 1, 2, \ldots, 0 < q < 1, \ 0 < \lambda < \infty$$  \hspace{1cm} (1)

and

$$f_{E}^{X}(x) = E_{q}(-\lambda) \frac{\lambda^x}{[x]_q !}, x = 0, 1, 2, \ldots, 0 < q < 1, \ 0 < \lambda(1 - q) < 1,$$  \hspace{1cm} (2)

where

$$e_{q}(z) := \sum_{n=0}^{\infty} \frac{(1 - q)^n z^n}{(q; q)_n} = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q !} = \frac{1}{((1 - q)z; q)_\infty}, \ |z| < 1$$ \hspace{1cm} (3)

and

$$E_{q}(z) := \sum_{n=0}^{\infty} \frac{(1 - q)^n q(z)^n}{(q; q)_n} = \sum_{n=0}^{\infty} \frac{q(z)^n z^n}{[n]_q !} = (-1 - qz; q)_{\infty}, \ |z| < 1.$$ \hspace{1cm} (4)

Both these $q$-Poisson distributions are unimodal and logconcave with Euler being infinitely divisible but Heine not. Also, Heine is underdispersed but Euler overdispersed.

Charalambides [3], reproduced Heine as direct approximation, as $n \to \infty$, of the $q$-Binomial distribution and the negative $q$-Binomial one, with probability functions given respectively by

$$f_{B}^{X}(x) = \binom{n}{x} q(z)^x \prod_{j=1}^{n} (1 + \theta q^{j-1})^{-1}, \ x = 0, 1, \ldots, n,$$ \hspace{1cm} (5)

and

$$f_{NB}^{X}(x) = \binom{n+x-1}{x} q(z)^x \prod_{j=1}^{n+x} (1 + \theta q^{j-1})^{-1}, \ x = 0, 1, \ldots,$$ \hspace{1cm} (6)

where $\theta > 0, \ 0 < q < 1$. Moreover, Charalambides [3], reproduced Euler as direct approximation of the $q$-Binomial and the

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*Harokopion University, Department of Informatics and Telematics*
negative $q$-Binomial distributions of the second kind one, as $n \to \infty$, with probability functions given respectively by

\[ f^n_{BS}(x) = \binom{n}{x}_q \theta^x \prod_{j=1}^{n-x}(1 - \theta q^{j-1}), \ x = 0, 1, \ldots, n, \quad (7) \]

and

\[ f^n_{NBS}(x) = \binom{n+x-1}{x}_q \theta^x \prod_{j=1}^{n}(1 - \theta q^{j-1}), \ x = 0, 1, \ldots, \quad (8) \]

where $0 < \theta < 1$ and $0 < q < 1$ or $1 < q < \infty$ with $\theta q^{n-1} < 1$.

Kyriakoussis and Vamvakari [11, 12], for $q$ constant, established a $q$-Stirling formula and proved limit theorems for the $q$-binomial distribution (5) and negative $q$-Binomial distribution (6), by using pointwise convergence in a $q$-analogue sense of the DeMoivre-Laplace classical limit theorem. Specifically in [11], the pointwise convergence of the $q$-binomial distribution to a deformed Stieltjes-Wigert continuous distribution was proved. In detail, transferred from the random variable $X$ of the $q$-binomial distribution (5) to the equal-distributed deformed random variable $Y = \left[X\right]_{1/q}$ and for $n \to \infty$, the $q$-binomial distribution was approximated by a deformed standardized continuous Stieltjes-Wigert distribution as follows

\[ f^B_X(x) \cong \frac{q^{1/8} \log q^{-1/2}}{(2\pi)^{1/2}} \left( q^{-3/2} (1 - q)^{1/2} \left[ x \right]_{1/q}^{1/2} \frac{\mu_q}{\sigma_q} + q^{-1} \right)^{1/2} \cdot \exp \left( \frac{1}{2 \log q} \log^2 \left( q^{-3/2} (1 - q)^{1/2} \left[ x \right]_{1/q}^{1/2} \frac{\mu_q}{\sigma_q} + q^{-1} \right) \right), \ x \geq 0, \quad (9) \]

where $\theta = \theta_n$, $n = 0, 1, 2, \ldots$ such that $\theta_n = q^{-\alpha n}$ with $0 < \alpha < 1$ constant and $\mu_q$ and $\sigma_q^2$ the mean value and variance of the random variable $Y$, respectively. Also in [12], a similar asymptotic result has been provided for the negative $q$-binomial distribution.

The aim of this work is to study the pointwise convergence of both Heine and Euler distributions as $\lambda \to \infty$. Specifically, the pointwise convergence of the Heine distribution to a deformed Stieltjes-Wigert continuous distribution and of the Euler distribution to a deformed Gauss are proved. Moreover, pointwise convergence of the $q$-binomial of the second kind and the negative $q$-binomial of the second kind, to a deformed Gauss are analogously deduced. Also, the associated $q$-orthogonal polynomials in respect of their weight functions to the above $q$-distributions and the related Stieltjes-Wigert polynomials moment problem are presented (see Andrews[1, 2], Christiansen [4, 5], Ismail[6]).

References


