## Chapter 2

# Integral Transform

## 2.1 Introduction

Consider pairs of functions related by an expression of the form:

$$\hat{F}(\alpha) = \int_a^b K(\alpha, x) f(x) dx \tag{2.1-1}$$

The function $\hat{F}(\alpha)$ is called the integral transform of $f(x)$ by the kernel $K(\alpha, x)$. The operation may also be considered as mapping a function $f(x)$ in $x$-space into another function $\hat{F}(\alpha)$ in $\alpha$-space. Figure 2.1-1 depicts the idea behind the application of integral transform. Certain problems can be solved, if at all, in the original coordinates (space). These problems might be solved relatively easily in the transform coordinates. Then, the inverse transform returns the solution from the transform coordinates to the original system.

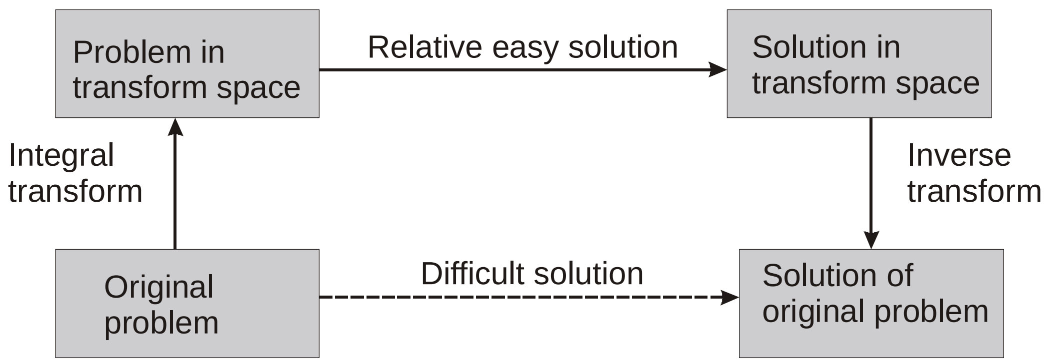


**Figure 2.1-1** Application of integral transform.

Two of the most useful of the infinite number of possible transforms are the Laplace transform,

$$\hat{F}(s) = \int_0^\infty e^{-sx} f(x) dx \tag{2.1-2}$$

and the Fourier transform,

$$\hat{F}(k) = \int_{-\infty}^\infty e^{-ikx} f(x) dx \tag{2.1-3}$$

Integral transform can be used to reduce the number of independent variables in a partial differential equation by one. Thus, the one-dimensional heat equation or wave equation can be transformed into an ordinary differential equation in the transformed function $\hat{F}(\alpha)$. An ordinary differential equation becomes an algebraic equation in the transformed domain. It is usually easier to solve the resultant equation in the transform space than it is to solve the original equation.

## 2.2 The Laplace Transform

Laplace transform is usually used to analyze process dynamic and to design control system. Therefore one of the independent variables is time. The Laplace transform of a function $f(t)$ is defined by

$$\mathscr{L}\{f(t)\} = \hat{F}(s) = \int_0^\infty e^{-st} f(t)dt \qquad (2.2\text{-}1)$$

In this equation $s$ is a parameter that may be complex. However we will mostly consider system with real value of $s$. For the Laplace transform to exist (or the integral (2.2-1) to have a finite value), $s$ must be greater than zero if $s$ is real, or the real part of $s$ must be greater than zero if $s$ is complex. The Laplace transform therefore converts a function of $t$ into a function of $s$. The limits of integration show that the Laplace transform contains information on the function $f(t)$ for positive time only. This is perfectly acceptable for a physical system since nothing can be done about the past (negative time). Some examples of Laplace transform

$$\mathscr{L}\{t^2\} = \hat{F}(s) = \int_0^\infty e^{-st} t^2 dt$$

$$\mathscr{L}\{t^2\} = -t^2 \frac{e^{-st}}{s}\Big|_0^\infty + \frac{2}{s}\int_0^\infty e^{-st} t dt$$

$$\mathscr{L}\{t^2\} = 0 + \frac{2}{s}\left[-t\frac{e^{-st}}{s}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st} dt\right] = -\frac{2}{s^2}\frac{e^{-st}}{s}\Big|_0^\infty$$

$$\mathscr{L}\{t^2\} = \frac{2}{s^3}$$

In general $\mathscr{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$, where $n$ is an integer.

$$\mathscr{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{-(s-a)t} dt = \frac{1}{s-a}$$

The Matlab command Laplace will take the Laplace transform of a function $f(t)$

```
>> syms s t a
>> laplace(t^2)
ans =
2/s^3
>> laplace(exp(a*t))
ans =
1/(s-a)
```

**Example 2.2-1** ---------------------------------------------------------------------------------

Determine the Laplace transform of sin $\omega t$

**Solution** --------------------------------------------------------------------------------------

The sine wave is represented in exponential form by

$$\sin \omega t = \frac{e^{i\omega t} - e^{-i\omega t}}{2i}$$

$$\mathscr{L}\{\sin \omega t\} = \int_0^\infty \frac{e^{i\omega t} - e^{-i\omega t}}{2i} e^{at} dt = \frac{1}{2i}\int_0^\infty \left[e^{-(s-i\omega)t} - e^{-(s+i\omega)t}\right] dt$$

$$\mathscr{L}\{\sin \omega t\} = \frac{1}{2i}\left[-\frac{e^{-(s-i\omega)t}}{s-i\omega} + \frac{e^{-(s+i\omega)t}}{s+i\omega}\right]_0^\infty = \frac{1}{2i}\left[-\frac{0-1}{s-i\omega} + \frac{0-1}{s+i\omega}\right]$$

$$\mathscr{L}\{\sin \omega t\} = \frac{1}{2i}\frac{2i\omega}{s^2+\omega^2} = \frac{\omega}{s^2+\omega^2}$$

Matlab can also take the Laplace transform of sine function.

```
>> syms s t a w
>> laplace(sin(w*t))
ans =
w/(s^2+w^2)
```

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For the integral $\int_0^\infty e^{-st} f(t)dt$ to exit $f(t)$ should be of *exponential order* as $t \to \infty$ or, in another word, cannot grow faster than an exponent. We say that $f(t)$ is of *exponential order* if there exists a constant $\alpha$ such that

$$\lim_{t \to \infty} e^{-\alpha t} f(t) = \text{finite}$$

For example, the functions 1, 4cos 2$t$, 5$t$sin 2$t$, $t^n$ are all of exponential order because $f(t)e^{-bt} \to 0$ as $t \to \infty$ for any $b > 0$. Function $\exp(t^2)$ is not of exponential order since it diverges much faster than $e^{bt}$ for any value of $b$.

The above conditions for the Laplace transform to exist are sufficient, but not necessary. The function $t^{-1/2}$ is not of exponential order, because the function is infinite at $t = 0$. However its Laplace transform exits for all $s$ >0. In this case

$$\mathscr{L}\{t^{-1/2}\} = \hat{F}(s) = \int_0^\infty e^{-st}t^{-1/2}dt = \left(\frac{\pi}{s}\right)^{1/2}$$

## 2.3 Properties of The Laplace Transform

### Linearity

The Laplace transform is a linear operation. This means that if *a* is a constant, then

$$\mathscr{L}\{af(t)\} = a\mathscr{L}\{f(t)\} = a\hat{F}(s) \tag{2.3-1}$$

The distributive property of addition also follows from the linearity property:

$$\mathscr{L}\{af(t) + bg(t)\} = a\hat{F}(s) + b\hat{G}(s) \tag{2.3-2}$$

These properties can be derived form the definition of Laplace transform.

### First Translation Property

$$\text{If } \mathscr{L}\{f(t)\} = \hat{F}(s) \text{ then } \mathscr{L}\{e^{at}f(t)\} = \hat{F}(s - a) \qquad s > a \tag{2.3-3}$$

This property is useful for evaluating transforms of functions that involve exponential functions of time.

### Second Translation Property

This property deals with the translation of a function in the time axis, as shown in Figure 2.3-1.

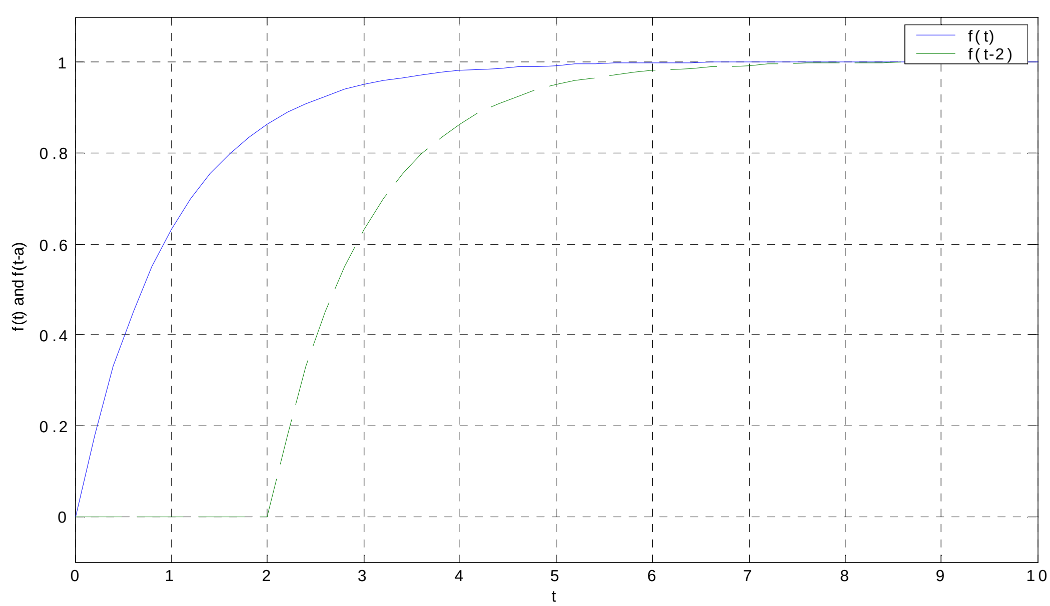


**Figure 2.3-1** Function delayed in 2 unit of time.

The translated function is the original function delayed in time as function defined by

$$g(t) = \begin{cases} 0 & 0 \le t < a \\ f(t-a) & a \le t \end{cases} \qquad (2.3\text{-}4)$$

In this equation $g(t)$ is simply $f(t)$ shifted $a$ units to the right as shown in Figure 2.3-1. The second translation property is given as

$$\mathscr{L}\{f(t-a)\} = e^{-as}\hat{F}(s) \qquad (2.3\text{-}5)$$

This property can be proved as follow

$$\mathscr{L}\{f(t-a)\} = \int_0^\infty e^{-st} f(t-a)dt$$

Let $\tau = t - a$ (or $t = \tau + a$), the above equation becomes

$$\mathscr{L}\{f(t-a)\} = \int_{-a}^\infty e^{-s(\tau+a)} f(\tau)d\tau = \int_0^\infty e^{-s\tau}e^{-as} f(\tau)d\tau$$

$$\mathscr{L}\{f(t-a)\} = e^{-as}\int_0^\infty e^{-s\tau} f(\tau)d\tau = e^{-as}\hat{F}(s)$$

$g(t)$ can be written in term of $f(t - a)$ with the use of unit step function $U(t)$ defined by

$$U(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 < t \end{cases}$$

With this definition $g(t) = U(t - a)f(t - a)$, and equation (2.3-5) can be written as

$$\mathscr{L}\{U(t-a)f(t-a)\} = e^{-as}\hat{F}(s) \qquad (2.3\text{-}6)$$

**Example 2.3-1**[1]. ------------------------------------------------------------------------------

Find the Laplace transform of $e^{-2t}\cos \omega t$

**Solution** ----------------------------------------------------------------------------------------

We have $\mathscr{L}\{\cos \omega t\} = \int_0^\infty e^{-st}\cos \omega t dt = \dfrac{s}{s^2+\omega^2}$

[1] McQuarrie, D.A., *Mathematical Methods for Scientists and Engineers*, University Science Book, 2003, pg. 813

Replacing $s$ by $s$ + 2 gives

$$\mathscr{L}\{e^{-2t}\cos \omega t\} = \frac{s+2}{(s+2)^2+\omega^2}$$

**Example 2.3-2**[2]. ----------------------------------------------------------------------------------

Plot the function defined by $f(t) = 3 - 4(t-1)U(t-1) + 4(t-3)U(t-3)$

**Solution** ------------------------------------------------------------------------------------------

The function can be expressed as

$$f(t) = \begin{cases} 3 & 0 \le t < 1 \\ 3-4(t-1) = 7-4t & 1 \le t < 3 \\ 3-4(t-1)+4(t-3) = -5 & 3 \le t \end{cases}$$

and is plotted in Figure E2.3-2

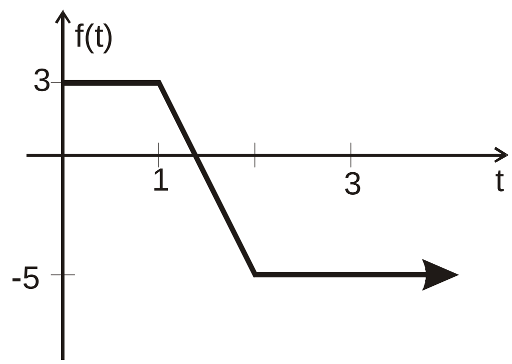


**Figure E2.3-2** Plot of $f(t) = 3 - 4(t-1)U(t-1) + 4(t-3)U(t-3)$

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**Differentiation Property**

This property, which establishes a relationship between the Laplace transform of a function and that of its derivatives, is used to transform ordinary differential equations into algebraic equations.

$$\mathscr{L}\left\{\frac{df(t)}{dt}\right\} = \int_0^\infty \frac{df(t)}{dt} e^{-st}dt = s\hat{F}(s) - f(0) \qquad (2.3\text{-}7)$$

This property can be obtained from the definition of Laplace transform as follows

Integrate by parts

[2] McQuarrie, D.A., *Mathematical Methods for Scientists and Engineers*, University Science Book, 2003, pg. 814

$u$ = e$^{-st}$ $\Rightarrow du = -$ $s$e$^{-st}$

$dv = \dfrac{df(t)}{dt}dt \Rightarrow v = f(t)$

$\mathscr{L}\left\{\dfrac{df(t)}{dt}\right\}$ = e$^{-st}$ $f(t)\Big|_0^\infty - \int_0^\infty f(t)(-se^{-st})dt = -f(0) + s\int_0^\infty f(t)e^{-st}dt$

$\mathscr{L}\left\{\dfrac{df(t)}{dt}\right\} = s\hat{F}(s) - f(0)$

The differentiation property can be extended to higher derivatives

$$\mathscr{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n \hat{F}(s) - s^{n-1}f(0) - s^{n-2}\left.\frac{df(t)}{dt}\right|_{t=0} - \ldots - \left.\frac{d^{n-1}f(t)}{dt^{n-1}}\right|_{t=0} \quad \text{(2.3-8)}$$

**Integration Property**

This property establishes the relationship between the Laplace transform of a function and that of its integral.

$$\mathscr{L}\left\{\int_0^t f(t)dt\right\} = \frac{1}{s}\hat{F}(s) \qquad \text{(2.3-9)}$$

**Final Value Property**

This property can be used to find the final, or steady-state, value of a function from its transform. It is also useful in checking the validity of derived transforms. If the limit of $f(t)$ as $t \to \infty$ exits, then it can be found from its Laplace transform as follows:

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s\hat{F}(s) \qquad \text{(2.3-10)}$$

**Example 2.3-3**[3]. -----------------------------------------------------------------------------

Derive the Laplace transform of the differential equation

$$9\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + y(t) = 2x(t)$$

with initial conditions of zero at steady state: $y(0) = 0$ and $\left.\dfrac{df(t)}{dt}\right|_{t=0} = 0$

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[3] Smith and Corripio., *Principles and Practice of Automatic Process Control*, Wiley, 1997, pg. 19

**Solution** ----------------------------------------------------------------------------------------

Taking the Laplace transform of each term in the equation yields

$$9\mathscr{L}\left\{\frac{d^2y(t)}{dt^2}\right\} + 6\mathscr{L}\left\{\frac{dy(t)}{dt}\right\} + \mathscr{L}\{y(t)\} = 2\mathscr{L}\{x(t)\}$$

We apply the differentiation property to obtain

$$9s^2Y(s) + 6sY(s) + Y(s) = 2X(s)$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{2}{9s^2+6s+1}X(s)$$

**Example 2.3-4**[4]. --------------------------------------------------------------------------------

Obtain the Laplace transform of the equation $c(t) = U(t-3)[1 - e^{-(t-3)/4}]$

**Solution** ----------------------------------------------------------------------------------------

Let $c(t) = f(t-3) = U(t-3)[1 - e^{-(t-3)/4}]$ then $f(t) = U(t) - U(t)e^{-t/4}$

$$\mathscr{L}\{f(t)\} = \hat{F}(s) = \frac{1}{s} - \frac{1}{s+1/4} = \frac{1}{s(4s+1)}$$

Applying the translation property gives

$$\mathscr{L}\{c(t)\} = C(s) = \mathscr{L}\{f(t-3)\} = e^{-3s}\hat{F}(s)$$

Therefore $\qquad C(s) = \dfrac{e^{-3s}}{s(4s+1)}$

We can use the final value property to check the validity of this answer

$$\lim_{t\to\infty} c(t) = \lim_{t\to\infty} U(t-3)[1 - e^{-(t-3)/4}] = 1$$

$$\lim_{s\to 0} s\hat{F}(s) = \lim_{s\to 0} s\frac{e^{-3s}}{s(4s+1)} = 1 \qquad\qquad \textbf{Check!}$$

[4] Smith and Corripio., *Principles and Practice of Automatic Process Control*, Wiley, 1997, pg. 20