Chapter 8

Constrained Optimization

8.1 Introduction

This chapter presents the optimization problems where constraints play a role. The combination of the equality constraints, inequality constraints, and lower and upper bounds defines a feasible region. If the solution also minimizes (or maximizes) the objective function, it is a local optimal solution. Other local optimal solutions may exist in the feasible region, with one or more being a global optimal solution. When the objective function, equality constraints, and inequality constraints are linear with respect to the variables, the problem is called a linear programming (LP) problem. If the objective function, any of the equality constraints, and/or any of the inequality constraints are nonlinear with respect to the variables, the problem is called a nonlinear programming (NLP) problem.

For a function with two independent variables, contours of the objective function can be plotted with one variable against the other as shown in Figure 3-1. The optimization problem is to minimize the objective function:

\[ f(x) = y = 4x_1^2 + 5x_2^2 \]

The unconstrained solutions are shown in Figure 8.1a where the entire region is feasible. The optimum solution is at \( x_1 = 0 \) and \( x_2 = 0 \). Figure 8.1b shows the constrained case with the inequality constraint:

\[ g(x) = 1 - x_1 \leq 0 \text{ or } x_1 \geq 1 \]

Figure 8.1-1 Two-variable optimization: (a) unconstrained; (b): constrained.

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The feasible region is located to the right of the constraint. The optimal point for the constraint case is not located in the feasible region. The minimum value of the objective function occurs at $x_1 = 1$ and $x_2 = 0$, where $y = 4x_1^2 + 5x_2^2 = 4$.

When one optimal solution exists in the feasible region, the objective function is unimodal. When two optimal solutions exist, it is bimodal. For more than two optimal solutions, the objective function is called multimodal. LP problems are unimodal unless the constraints are inconsistent such that no feasible region exists.

### 8.2 Linear Programming

The basic concepts of linear programming can be illustrated by the following example.

**Example 8.2-1**

A gas processing plant receives a fixed amount of raw gas each week. The raw gas is processed into two grades of heating gas, regular and premium quality. Their production involves both time and on-site storage constraints. For example, only one of the grades can be produced at a time, and the facility is open for only 80 hrs/week. There is limited on-site storage for each of the products. All these factors are listed in Table 8.2-1 with a metric ton denoted *tonne* equal to 1000 kg.

<table>
<thead>
<tr>
<th>Resource</th>
<th>Regular</th>
<th>Premium</th>
<th>Availability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Raw gas</td>
<td>7 m$^3$/tonne</td>
<td>11 m$^3$/tonne</td>
<td>77 m$^3$/week</td>
</tr>
<tr>
<td>Production time</td>
<td>10 hr/tonne</td>
<td>8 hr/tonne</td>
<td>80 hr/week</td>
</tr>
<tr>
<td>Storage</td>
<td>9 tonnes</td>
<td>6 tonnes</td>
<td></td>
</tr>
<tr>
<td>Profit</td>
<td>150/tonne</td>
<td>175/tonne</td>
<td></td>
</tr>
</tbody>
</table>

**Solution**

To maximize total profit $y$, we need to maximize the following objective function

$$f(x) = y = 150x_1 + 175x_2$$

In this equation, $x_1$ and $x_2$ are the weekly production of regular and premium heating gas, respectively. The total raw gas used cannot exceed the available supply of 77 m$^3$/week, so the constraint can be represented as

$$7x_1 + 11x_2 \leq 77$$

The remaining constraints can be developed in a similar fashion to produce the following LP formulation

Maximize $y = 150x_1 + 175x_2$

Subject to

---

\[ 7x_1 + 11x_2 \leq 77 \quad \text{(material constraint)} \]
\[ 10x_1 + 8x_2 \leq 80 \quad \text{(time constraint)} \]
\[ x_1 \leq 9 \quad \text{(storage constraint of regular heating gas)} \]
\[ x_2 \leq 6 \quad \text{(storage constraint of premium heating gas)} \]
\[ x_1, x_2 \geq 0 \quad \text{(positive production constraint)} \]

For two-dimensional system, a LP problem can be solved graphically with the solution space defined as a plane with \( x_1 \) measured along the abscissa and \( x_2 \) along the ordinate. The inequality constraints can be plotted as straight lines as follows:

\[
\begin{align*}
x_2 &= -\frac{7}{11} x_1 + 7 \quad \text{(line 1)} \\
x_2 &= -\frac{10}{8} x_1 + 10 \quad \text{(line 2)} \\
x_1 &= 9 \quad \text{(line 3)} \\
x_2 &= 6 \quad \text{(line 4)} \\
x_1 &= 0 \quad \text{(line 5)} \\
x_2 &= 0 \quad \text{(line 6)}
\end{align*}
\]

These lines are plotted in Figure 8.2-2a where the feasible solution space is given by the shaded area. This area satisfies the constraints posed by the LP problem. The constraint 3 (line 3) is redundant since the feasible solution space is unaffected if it were deleted.

The objective function is then plotted:

\[ x_2 = -\frac{150}{175} x_1 + y \]

This equation represents a dashed line as shown in Figure 8.2-2b depending on the value of the total profit \( y \).
As we increase the profit $y$, the dashed line moves away from the origin until it reaches the outer edge of the feasible solution at a value $y = 1400$. At this location, $x_1$ and $x_2$ are equal to approximately 4.9 and 3.9, respectively. This is the optimum solution produced by the graphical method.

8.3 Linear Programming using Software

A. Linear Programming using Excel

The basic concepts of linear programming can be illustrated by the following example\textsuperscript{2}.

**Example 8.3-1**

A gas processing plant receives a fixed amount of raw gas each week. The raw gas is processed into two grades of heating gas, regular and premium quality. Their production involves both time and on-site storage constraints. For example, only one of the grades can be produced at a time, and the facility is open for only 80 hrs/week. There is limited on-site storage for each of the products. All these factors are listed in Table 1-1 with a metric ton denoted *tonne* equal to 1000 kg.

<table>
<thead>
<tr>
<th>Product</th>
<th>Regular</th>
<th>Premium</th>
<th>Availability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Raw gas</td>
<td>7 m\textsuperscript{3}/tonne</td>
<td>11 m\textsuperscript{3}/tonne</td>
<td>77 m\textsuperscript{3}/week</td>
</tr>
<tr>
<td>Production time</td>
<td>10 hr/tonne</td>
<td>8 hr/tonne</td>
<td>80 hr/week</td>
</tr>
<tr>
<td>Storage</td>
<td>9 tonnes</td>
<td>6 tonnes</td>
<td></td>
</tr>
<tr>
<td>Profit</td>
<td>150/tonne</td>
<td>175/tonne</td>
<td></td>
</tr>
</tbody>
</table>

\textsuperscript{2} Chapra and Canale, *Numerical Methods for Engineers*, McGraw Hill, 4\textsuperscript{th} Ed., 2002, pg. 376
**Solution**

To maximize total profit $y$, we need to maximize the following objective function

$$f(x) = y = 150x_1 + 175x_2$$

In this equation, $x_1$ and $x_2$ are the weekly production of regular and premium heating gas, respectively. The total raw gas used cannot exceed the available supply of 77 m$^3$/week, so the constraint can be represented as

$$7x_1 + 11x_2 \leq 77$$

The remaining constraints can be developed in a similar fashion to produce the following LP formulation

Maximize $y = 150x_1 + 175x_2$

Subject to

<table>
<thead>
<tr>
<th>Constraint</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$7x_1 + 11x_2 \leq 77$</td>
<td>(material constraint)</td>
</tr>
<tr>
<td>$10x_1 + 8x_2 \leq 80$</td>
<td>(time constraint)</td>
</tr>
<tr>
<td>$x_1 \leq 9$</td>
<td>(storage constraint of regular heating gas)</td>
</tr>
<tr>
<td>$x_2 \leq 6$</td>
<td>(storage constraint of premium heating gas)</td>
</tr>
<tr>
<td>$x_1, x_2 \geq 0$</td>
<td>(positive production constraint)</td>
</tr>
</tbody>
</table>

An Excel worksheet set up to calculate the weekly production of regular and premium heating gas is shown in Figure 8.3-1. The unshaded cells are those containing numeric and labeling data. The shaded cells involve quantities that are calculated based on other cells. The total profit to be maximized is calculated in cell D12. Cells B4 and C4 give the regular and premium gas. These cells must be varied.

![Figure 8.3-1 Excel spreadsheet set up for linear programming.](image)

The formula for cells in the spreadsheet are given in the following table:
Table 8.3-2

<table>
<thead>
<tr>
<th>Cell</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>D6</td>
<td>=B6<em>B4+C6</em>C4</td>
</tr>
<tr>
<td>D7</td>
<td>=B7<em>B4+C7</em>C4</td>
</tr>
<tr>
<td>D8</td>
<td>=B4</td>
</tr>
<tr>
<td>D9</td>
<td>=C4</td>
</tr>
<tr>
<td>B12</td>
<td>=B4*B11</td>
</tr>
<tr>
<td>C12</td>
<td>=C4*C11</td>
</tr>
<tr>
<td>D12</td>
<td>=B12+C12</td>
</tr>
</tbody>
</table>

Once the spreadsheet is created, **Solver** is chosen from the Tools menu. A dialogue box will be displayed, querying you for pertinent information, which must be filled out as

The constraints must be added one by one by selecting the “Add” button. This will open up a dialogue box that looks like

As shown, the constraint that the total raw gas (cell D6) must be less than or equal to the available supply (E6) can be added as shown. After adding each constraint, the “Add” button must be selected so that the constraint will be listed in the solver parameter dialogue box. The six constraints listed in the solver parameter box are given in order by the following conditions

\[
x_1 \geq 0 \quad \text{(positive production constraint)}
\]
\[
x_2 \geq 0 \quad \text{(positive production constraint)}
\]
\[
7x_1 + 11x_2 \leq 77 \quad \text{(material constraint)}
\]
\[
10x_1 + 8x_2 \leq 80 \quad \text{(time constraint)}
\]
Before execution, the Solver options button should be selected and the box label “Assume linear model” should be check off. This will prevent Excel from using the more general nonlinear solver that will require more calculation time.

Click “OK” to return to the Solver Parameters box. When the Solved button is selected, Excel will solved the problem with a report on the success of the operation as shown:

\[
x_1 \leq 9 \quad \text{(storage constraint of regular heating gas)}
\]
\[
x_2 \leq 6 \quad \text{(storage constraint of premium heating gas)}
\]
Solving set of nonlinear equations with Excel

Use the initial guess \( x = [0.1 \ 0.1 \ -0.1] \) to obtain the solutions to the following equations

\[
\begin{align*}
f_1(x_1, x_2, x_3) &= 3x_1 - \cos(x_2x_3) - \frac{1}{2} = 0 \\
f_2(x_1, x_2, x_3) &= x_1^2 - 81(x_2 + 0.1)^2 + \sin x_3 + 1.06 = 0 \\
f_3(x_1, x_2, x_3) &= e^{-x_1x_2} + 20x_3 + \frac{10\pi - 3}{3} = 0
\end{align*}
\]

Microsoft Excel - Nonlinear.xls

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>x</td>
<td>y</td>
<td>z</td>
<td></td>
</tr>
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<td>0.1</td>
<td>-0.1</td>
</tr>
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<td></td>
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<td></td>
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<td>6</td>
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